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STANDING WAVES FOR A CLASS OF SCHRÖDINGER–POISSON EQUATIONS IN $\mathbf{R}^3$ INVOLVING CRITICAL SOBOLEV EXPONENTS

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Abstract. We are concerned with the following Schrödinger–Poisson equation with critical nonlinearity:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \psi u = \lambda|u|^{p-2}u + |u|^4 u & \text{in } \mathbf{R}^3, \\ -\varepsilon^2\Delta\psi = u^2 & \text{in } \mathbf{R}^3,\ u>0,\ u\in H^1(\mathbf{R}^3), \end{cases}$$

where $\varepsilon > 0$ is a small positive parameter, $\lambda > 0$, $3 < p \le 4$. Under certain assumptions on the potential V, we construct a family of positive solutions $u_\varepsilon \in H^1(\mathbf{R}^3)$ which concentrates around a local minimum of V as $\varepsilon \to 0$. Subcritical growth Schrödinger–Poisson equation

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u + \psi u = f(u) & \text{in } \mathbf{R}^3, \\ -\varepsilon^2\Delta\psi = u^2 & \text{in } \mathbf{R}^3,\ u>0,\ u\in H^1(\mathbf{R}^3), \end{cases}$$

has been studied extensively, where the assumption for $f(u)$ is that $f(u) \sim |u|^{p-2}u$ with $4 < p < 6$ and satisfies the Ambrosetti–Rabinowitz condition which forces the boundedness of any Palais–Smale sequence of the corresponding energy functional of the equation. The more difficult critical case is studied in this paper. As $g(u) := \lambda|u|^{p-2}u + |u|^4u$ with $3 < p \le 4$ does not satisfy the Ambrosetti–Rabinowitz condition ($\exists\mu > 4,\ 0 < \mu\int_0^u g(s)\,ds \le g(u)u$), the boundedness of Palais–Smale sequence becomes a major difficulty in proving the existence of a positive solution. Also, the fact that the function $\frac{g(s)}{s^3}$ is not increasing for $s > 0$ prevents us from using the Nehari manifold directly as usual. The main result we obtained in this paper is new.

1. Introduction and main result

In this paper, we study the following Schrödinger–Poisson equation with critical nonlinearity:

$$(1.1)\qquad \begin{cases} -\varepsilon^2\Delta u + V(x)u + \psi u = \lambda|u|^{p-2}u + |u|^4 u & \text{in } \mathbf{R}^3, \\ -\varepsilon^2\Delta\psi = u^2 & \text{in } \mathbf{R}^3,\ u>0,\ u\in H^1(\mathbf{R}^3), \end{cases}$$

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where $\varepsilon > 0$ is a small positive parameter, $\lambda > 0$, $3 < p \le 4$. We assume that the potential V satisfies:

(V_1) $V \in C(\mathbf{R}^3, \mathbf{R})$ and $\inf_{x \in \mathbf{R}^3} V(x) = \alpha > 0$;

(V_2) There is a bounded domain Λ such that

$$V_0 := \inf_{\Lambda} V < \min_{\partial\Lambda} V.$$

We also set $\mathcal{M} := \{x \in \Lambda \colon V(x) = V_0\}$. Without loss of generality, we may assume that $0 \in \mathcal{M}$.

Problem (1.1) is a variant of the following Schrödinger–Poisson problem

$$\begin{cases} \frac{\hbar^2}{2m}\Delta v - v - \omega\phi v + f(v) = 0 & \text{in } \mathbf{R}^3, \\ \Delta\phi + 4\pi\omega v^2 = 0 & \text{in } \mathbf{R}^3, \\ v, \phi > 0, \ v, \phi \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.2}$$

where $\hbar, m, \omega > 0$, $v, \phi \colon \mathbf{R}^3 \to \mathbf{R}$, $f \colon \mathbf{R} \to \mathbf{R}$. This equation arises in Quantum Mechanics: in 1998, Benci and Fortunato [7] firstly introduced it as a model to describe the interaction of a charged particle with the electrostatic field. In (1.2), m denotes the mass of the particle, ω denotes the electric charge and $\hbar$ is a constant which is known under the name of Planck's constant. The unknowns of the equation are the wave function v associated to the particle and the electric potential ϕ. The presence of the nonlinear term $f(v)$ simulates the interaction effect among many particles.

In the last years, there has been a great deal of works dealing with the Schrödinger–Poisson equations by means of variational tools.

Benci and Fortunato [7] considered the eigenvalue problem for (1.2) of the following form

$$\begin{cases} -\frac{1}{2}\Delta u - \phi u = \omega u & \text{in } \Omega, \\ \Delta\phi = 4\pi u^2 & \text{in } \Omega, \\ u(x) = 0, \ \phi(x) = g & \text{on } \partial\Omega, \ \|u\|_{L^2(\Omega)} = 1, \ \omega > 0, \end{cases} \tag{1.3}$$

where Ω is a bounded set in $\mathbf{R}^3$ and g is a smooth function on the closure $\bar{\Omega}$. They used a constrained minimization argument to show that, there is a sequence (ω_n, u_n, ϕ_n) with $\{\omega_n\} \subset \mathbf{R}$, $\omega_n \to \infty$ and u_n, ϕ_n real functions, solving (1.3).

D'April and Mugnai [18] used a related Pohozaev's identity to show that there does not exist nontrivial solutions of the following Schrödinger–Poisson equation

$$\begin{cases} -\Delta u + u + \lambda\phi u = |u|^{p-2}u & \text{in } \mathbf{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbf{R}^3, \end{cases}$$

for $p \le 2$ or $p \ge 6$.

D'April and Mugnai [17] used Symmetric Mountain-Pass theorem (see [2]) to show that the following Klein–Gordon–Maxwell equation

$$\begin{cases} -\Delta u + [m^2 - (\omega + e\phi)^2]u - |u|^{p-2}u = 0 & \text{in } \mathbf{R}^3, \\ -\Delta\phi + e^2u^2\phi = -e\omega u^2 & \text{in } \mathbf{R}^3 \end{cases}$$

has infinitely many symmetric solutions $(u_n, \phi_n) \in H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$ with $u_n \ne 0$, $\phi_n \ne 0$ under the conditions:

(i) $m > \omega$ and $4 \le p < 6$;
(ii) $m\sqrt{p-2} > \sqrt{2}\omega > 0$ and $2 < p < 4$.

Meanwhile, they used Mountain-Pass theorem (see [2]) to show that the Schrödinger–Poisson equation

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-2}u & \text{in } \mathbf{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbf{R}^3 \end{cases}$$

has at least a radially symmetric solution $(u,\phi) \in H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$ with $u \ne 0$ and $\phi \ne 0$.

Ruiz [43] considered the following Schrödinger–Poisson equation:

$$\begin{cases} -\Delta u + u + \lambda\phi u = u^{p-1} & \text{in } \mathbf{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbf{R}^3, \end{cases} \tag{1.4}$$

where $\lambda > 0$ is a positive parameter and $2 < p < 6$. Ruiz proved that when $2 < p < 3$ (respectively $p = 3$), (1.4) has at least two (respectively one) positive solutions for $\lambda > 0$ small by using the Mountain-Pass theorem (see [2]) and Ekeland's variational principle (see [21]) and (1.4) has no nontrivial solution if $2 < p \le 3$, $\lambda > \frac{1}{4}$. For the case $3 < p < 6$, it was shown in [43] that there is a positive radial nontrivial solution to (1.4) by using the constrained minimization method on a new manifold which is obtained by combining the usual Nehari manifold and the Pohozaev's identity.

Salvatore [45] studied the following Schrödinger–Poisson equation:

$$\begin{cases} -\Delta u + u + \lambda\phi u = |u|^{p-2}u + g(x) & \text{in } \mathbf{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbf{R}^3, \end{cases} \tag{1.5}$$

where $\lambda > 0$, $p \in (4,6)$ and $g(x) = g(|x|) \in L^2(\mathbf{R}^3)$. The author used Three Critical Points theorem to show that (1.5) has at least three radially symmetric solutions for $\|g\|_{L^2}$ small.

Wang and Zhou [49] studied the following problem

$$\begin{cases} -\Delta u + V(x)u + \lambda\phi u = f(x,u) & \text{in } \mathbf{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbf{R}^3,\ \lim\limits_{|x|\to\infty} \phi(x) = 0, \end{cases} \tag{1.6}$$

where $\lambda > 0$, the nonlinearity $f(x,s)$ is asymptotically linear with respect to s at infinity. Under certain assumptions on V and f, they prove that (1.6) has a positive solution for λ small and has no any nontrivial solution for λ large.

Azzollini, D'Avenia and Pomponio [5] used a technique due to Jeanjean ([29] Theorem 1.1) to show that the equation

$$\begin{cases} -\Delta u + q\phi u = g(u) & \text{in } \mathbf{R}^3, \\ -\Delta \phi = qu^2 & \text{in } \mathbf{R}^3 \end{cases}$$

has a nontrivial positive radial solution $(u,\phi) \in H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$ for $q > 0$ small where the nonlinear term g satisfies:

(g_1) $g \in C(\mathbf{R},\mathbf{R})$;
(g_2) $-\infty < \varliminf\limits_{s\to 0^+} g(s)/s \le \varlimsup\limits_{s\to 0^+} g(s)/s = -m < 0$;
(g_3) $-\infty \le \varlimsup\limits_{s\to +\infty} g(s)/s^5 \le 0$;

(g_4) $\exists \xi > 0$ such that

$$G(\xi) := \int_0^\xi g(s)\,ds > 0.$$

Note that the hypotheses on g was firstly introduced by Berestycki and Lions, in their celebrated paper [9].

Mugnai [35] proved that for any $\omega > 0$, there exist $\lambda > 0$ such that the following Schrödinger–Poisson equation

$$\begin{cases} -\Delta u + \omega u - \lambda u\phi + W_u(x,u) = 0 \text{ in } \mathbf{R}^3, \\ -\Delta\phi = u^2 \text{ in } \mathbf{R}^3 \end{cases} \tag{1.7}$$

has a nontrivial radial function $(u,\phi) \in H^1(\mathbf{R}^3) \times D^{1,2}(\mathbf{R}^3)$ by using the minimization argument on an appropriate manifold when the nonlinear term $W\colon \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}$ satisfies:

- (W_1) $W\colon \mathbf{R}^3 \times \mathbf{R} \to [0,\infty)$ is such that the derivative $W_u\colon \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function, $W(x,s) = W(|x|,s)$ for a.e. $x \in \mathbf{R}^3$ and for every $s \in \mathbf{R}$, and $W(x,0) = W_u(x,0) = 0$ for a.e. $x \in \mathbf{R}^3$;
- (W_2) $\exists C_1, C_2 > 0$ and $1 < q < p < 5$ such that $|W_u(x,s)| \le C_1|s|^q + C_2|s|^p$ for every $s \in \mathbf{R}$ and a.e. $x \in \mathbf{R}^3$;
- (W_3) $\exists k \ge 2$ such that $0 \le sW_u(x,s) \le kW(x,s)$ for every $s \in \mathbf{R}$ and a.e. $x \in \mathbf{R}^3$.

Recently, Jiang and Zhou [30] studied the Schrödinger–Poisson equation

$$\begin{cases} -\Delta u + (1+\mu g(x))u + \lambda\phi u = |u|^{p-2}u & \text{in } \mathbf{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbf{R}^3,\ \lim\limits_{|x|\to\infty} \phi(x) = 0, \end{cases} \tag{1.8}$$

where λ, μ are positive parameters, $p \in (2,6)$, $g(x) \in L^\infty(\mathbf{R}^3)$ is nonnegative, $g(x) \equiv 0$ on a bounded domain in $\mathbf{R}^3$ and $\lim_{|x|\to\infty} g(x) = 1$. They used a priori estimate and approximation methods to show that (1.8) with $p \in (2,3)$ has a ground state solution if μ large and λ small. Meanwhile, they also proved that (1.8) with $p \in [4,6)$ has a nontrivial solution for any $\lambda > 0$ and μ large.

As far as we know, there is no result on the existence of positive ground state solutions for (1.4) when the nonlinearity $u^{p-1}(2 < p < 6)$ is replaced by $\lambda|u|^{p-2}u + |u|^4u(3 < p \le 4)$. In this paper, we will fill this gap.

We note that problem (1.2) with $\omega = 0$ and $\frac{\hbar^2}{2m} = 1$ is motivated by the search for standing wave solutions for the nonlinear Schrödinger equation, which is one of the main subjects in nonlinear analysis. Different approaches have been taken to deal with this problem under various hypotheses on the potentials and the nonlinearities (see [9, 10] and so on).

Our motivation to study (1.1) mainly comes from the results of perturbed Schrödinger equations, i.e.,

$$-\varepsilon^2\Delta u + V(x)u = |u|^{q-2}u, \quad x \in \mathbf{R}^N, \tag{1.9}$$

where $2 < q < 2N/(N-2)$, $N \ge 1$.

Many mathematicians proved the existence, concentration and multiplicity of solutions for (1.9).

Floer and Weinstein [23] studied (1.9) in the case where $N = 1$, $q = 4$, $V \in L^\infty$ with $\inf V > 0$. They construct a single peak solution which concentrates around any given non-degenerate critical point of the potential V. Y. G. Oh [36, 37] extended this result in higher dimensions when $2 < q < 2N/(N-2)$ and the potential V belongs to a Kato class which means that V satisfies the following condition:

$(V)_a$ $V \equiv a$ or $V > a$ and $(V-a)^{-\frac{1}{2}} \in \mathrm{Lip}(\mathbf{R}^N)$ for some $a \in \mathbf{R}$.

Furthermore, Oh [38] proved the existence of multi-peak solutions which concentrate around any finite subsets of the non-degenerate critical points of V. The arguments in [23, 36, 37, 38] are mainly based on a Lyapunov–Schmidt reduction.

Rabinowitz [41] studied (1.9) under the conditions:

(V_3) $V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{x\in\mathbf{R}^N} V(x) > 0.$

Rabinowitz proved that (1.9) possesses a positive ground state solution for $\varepsilon > 0$ small by using the Mountain Pass Theorem (see [2]).

The concentration behavior for the family of positive ground state solutions, which was obtained in [41], was proved by Wang [48]. Wang proved that the positive ground state solutions of (1.9) must concentrate at global minima of V as $\varepsilon \to 0$.

Under the same condition (V_3) on $V(x)$, Cingolani and Lazzo [16] proved the multiplicity of positive ground state solutions for (1.9) by using Ljusternik–Schnirelmann theory (see [15], for example).

del Pino and Felmer [39] studied (1.9) with the conditions on V replaced by (V_1) and (V_2). They proved that (1.9) possesses a positive bound state solution for $\varepsilon > 0$ small which concentrates around the local minima of V in Λ as $\varepsilon \to 0$.

Gui [25] studied (1.9) under the conditions (V_1) and

(V_4) There exist k disjoint bounded regions $\Omega_1, \dots, \Omega_k$ such that

$$V_0 := \inf_{\Omega_i} V < \min_{\partial\Omega_i} V, \quad i = 1, \dots, k.$$

Gui showed that (1.9) possesses a positive classial bound state solution for $\varepsilon > 0$ small which exactly possesses k local maximum $P_{\varepsilon,1}, \dots, P_{\varepsilon,k}$ satisfying $P_{\varepsilon,i} \in \Omega_i$ and $\lim_{\varepsilon\to 0} V(P_{\varepsilon,i}) = \inf_{\Omega_i} V$.

D'Aprile and Wei [19] studied (1.2) and extended the method in [23, 36, 37, 38, 38], which was based on Lyapunov–Schmidt reduction, to conclude a similar result in the Schrödinger–Poisson equation (1.2).

Under the same condition (V_3) on $V(x)$, X. He [26] studied (1.1) with the nonlinearity replaced by $f(u)$, where $f \in C^1(\mathbf{R}^+, \mathbf{R}^+)$ and satisfies the Ambrosetti-Rabinowitz condition ((AR) condition in short)

$$\exists \mu > 4, \quad 0 < \mu \int_0^u f(s)\, ds \le f(u)u,$$

$\lim_{s\to 0} \frac{f(s)}{s^3} = 0$, $\lim_{|s|\to\infty} \frac{f(s)}{|s|^q} = 0$ for some $3 < q < 5$ and $\frac{f(s)}{s^3}$ is strictly increasing for $s > 0$. They obtained the existence, concentration and multiplicity of solutions for (1.9) by the same arguments as in [41, 48, 16].

For more results, we can refer to [1, 3, 4, 8, 14, 20, 44, 47] and the references therein.

Our main result is the following:

Theorem 1.1. *Let (V_1), (V_2) hold. There exist $\lambda^* > 0$ and $\varepsilon^* > 0$ such that for each $\lambda \in [\lambda^*, \infty)$ and $\varepsilon \in (0, \varepsilon^*)$, (1.1) possesses a positive solution $u_\varepsilon \in H^1(\mathbf{R}^3)$ such that*

(i) *there exists a maximum point x_ε of u_ε such that*

$$\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathcal{M}) = 0;$$

(ii) $\exists C_1, C_2 > 0$, *such that*

$$u_\varepsilon(x) \le C_1 \exp\Big(-\frac{C_2}{\varepsilon}|x - x_\varepsilon|\Big),$$

where C_1, C_2 are independent of ε.

We note that, to the best of our knowledge, there is no result on the existence and concentration of positive bound state solutions for Schrödinger–Poisson type equation with the nonlinearity $\lambda|u|^{p-2}u + |u|^4u(3 < p \le 4)$.

The proof of Theorem 1.1 is based on variational method. The main difficulties in proving Theorem 1.1 lie in two aspects: (i) The nonlinearity $\lambda|u|^{p-2}u + |u|^4u$ with $p \in (3, 4]$ does not satisfy (AR) condition and the fact that the function $\frac{\lambda u^{p-1}+u^5}{u^3}$ is not increasing for $u > 0$ prevent us from obtaining a bounded Palais–Smale sequence ((PS) sequence in short) and using the Nehari manifold respectively. The arguments in [39] can not be applied in this paper. (ii) The unboundedness of the domain $\mathbf{R}^3$ and the nonlinearity $\lambda|u|^{p-2}u + |u|^4u(3 < p \le 4)$ with the critical Sobolev growth lead to the lack of compactness. As we will see later, the above two aspects prevent us from using the variational method in a standard way.

To overcome these difficulties, inspired by [12, 22], we use a version of quantitative deformation lemma due to Figueiredo, Ikoma and Santos Junior (see Proposition 4.6 below) to construct a special bounded (PS) sequence and recover the compactness by using a penalization method which was firstly introduced in [13].

To complete this section, we sketch our proof.

Firstly, we need to consider the existence of ground state solutions of the associated "limiting problem" of (1.1), which is given as

$$\begin{cases} -\Delta u + au + \phi u = \lambda|u|^{p-2}u + |u|^4u & \text{in } \mathbf{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbf{R}^3,\ u > 0,\ u \in H^1(\mathbf{R}^3), \\ a > 0,\ 3 < p \le 4, \end{cases} \tag{1.10}$$

with the corresponding energy functional

$$\begin{aligned} I_a(u) = {}& \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2 + \frac{a}{2}\int_{\mathbf{R}^3} u^2 + \frac{1}{16\pi}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\,dx\,dy \\ & - \frac{\lambda}{p}\int_{\mathbf{R}^3} (u^+)^p - \frac{1}{6}\int_{\mathbf{R}^3} (u^+)^6, \quad u \in H^1(\mathbf{R}^3). \end{aligned}$$

In [28], Hirata, Ikoma and Tanaka studied the following Schrödinger equation

$$-\Delta u = g(u), \quad u \in H^1(\mathbf{R}^N)$$

with the corresponding energy functional

$$I(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 - \int_{\mathbf{R}^N} G(u), \quad u \in H^1_r(\mathbf{R}^N),$$

where $G(u) = \int_0^u g(s)ds$ and g satisfies the conditions due to the celebrated work by Berestycki and Lions [9]. By studing the behavior of $I(u(e^{-\theta}x))$ for $\theta \in \mathbf{R}$, they constructed a $(\mathrm{PS})_c$ sequence $\{u_n\}_{n=1}^\infty$ with an extra property $P(u_n) \to 0$ as $n \to \infty$ where c is the mountain pass level of I and $P(u) = 0$ is the corresponding

Pohozaev's identity and then proved that the $(\mathrm{PS})_c$ sequence is bounded. But for the Schrödinger–Poisson equation (1.10), one still need something more than $P(u_n) \to 0$ as $n \to \infty$.

For the critical case (1.10), the constrained minimization on a new manifold due to Ruiz [43] seems to be difficult to be applied directly.

Motivated by [28], by studying the behavior of $I_a(e^{2\theta}u(e^\theta x))$ for $\theta \in \mathbf{R}$, we construct a $(\mathrm{PS})_{c_a}$ sequence $\{u_n\}_{n=1}^\infty$ with an extra property $G_a(u_n) \to 0$ as $n \to \infty$ where c_a is the mountain pass level of I_a, $G_a(u) = 2\langle I'_a(u), u\rangle - P_a(u)$ and $P_a(u) = 0$ is the Pohozaev's identity of (1.10) (see Proposition 3.4 below). From this fact, the boundedness of the $(\mathrm{PS})_{c_a}$ sequence is proved easily. Proceeding by the standard arguments, the existence of ground state solution (1.10) follows (see Proposition 3.8 below). Denoting S_a the set of ground state solutions U of (1.10) satisfying $U(0) = \max_{x\in\mathbf{R}^3} U(x)$, we then show that S_a is compact in $H^1(\mathbf{R}^3)$ (see Proposition 3.9 below).

To study (1.1), we will work with the following equivalent equation

$$\begin{cases} -\Delta u + V(\varepsilon x)u + \phi u = \lambda |u|^{p-2}u + |u|^4 u & \text{in } \mathbf{R}^3, \\ -\Delta\phi = u^2 & \text{in } \mathbf{R}^3,\ u > 0,\ u \in H^1(\mathbf{R}^3). \end{cases} \tag{1.11}$$

Note that a solution of (1.11) is in fact a critical point of the following functional

$$\begin{aligned} I_\varepsilon(u) = {} & \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbf{R}^3} V(\varepsilon x)u^2 + \frac{1}{16\pi}\int_{\mathbf{R}^3}\int_{\mathbf{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} \\ & - \frac{\lambda}{p}\int_{\mathbf{R}^3} (u^+)^p - \frac{1}{6}\int_{\mathbf{R}^3} (u^+)^6, \quad u \in H_\varepsilon, \end{aligned}$$

where $H_\varepsilon := \{v \in H^1(\mathbf{R}^3) \mid \int_{\mathbf{R}^3} V(\varepsilon x)v^2 < \infty\}$ endowed with the norm

$$\|v\|_{H_\varepsilon} := \left(\int_{\mathbf{R}^3} |\nabla v|^2 + \int_{\mathbf{R}^3} V(\varepsilon x)v^2\right)^{1/2}.$$

Unlike [26], where the minimum of $V(x)$ is global and the nonlinear term $f(u)$ satisfies the (AR) condition, the Mountain Pass Theorem can be used globally, here in the present paper, the condition (V_2) is local and $3 < p \leq 4$, we need to use a penalization method introduced in [13], which helps us to overcome the obstacle caused by the non-compactness due to the unboundedness of the domain and the lack of (AR) condition. To this end, we should modify the energy functional.

Following [12], we set $J_\varepsilon \colon H_\varepsilon \to \mathbf{R}$ be given by

$$J_\varepsilon(v) = I_\varepsilon(v) + Q_\varepsilon(v),$$

where

$$Q_\varepsilon(v) = \left(\int_{\mathbf{R}^3} \chi_\varepsilon v^2 - 1\right)_+^2$$

and

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \Lambda/\varepsilon, \\ \varepsilon^{-1} & \text{if } x \notin \Lambda/\varepsilon. \end{cases}$$

It will be shown that the functional Q_ε will acts as a penalization to force the concentration phenomena to occur inside Λ (see Lemma 4.3 below).

Using a version of quantitative deformation lemma due to Figueiredo, Ikoma and Santos Junior (see Proposition 4.6 below) to construct a special bounded and convergent (PS) sequence of J_ε in a neighborhood of the compact set S_{V_0} for $\varepsilon > 0$ small, i.e., J_ε possesses a critical point v_ε. To verify the critical point v_ε of J_ε is indeed a solution of the original problem (1.11), we need to establish a uniform estimate on L^∞-norm of v_ε (independent of ε) by using the idea of Brezis–Kato type argument and the Moser iteration technique (see also [31, 53] and Lemma 2.4 below).

Moreover, for the critical case, the existence and concentration phenomenon of problem (1.1) has not been studied so far by variational methods. In the present paper, we will adopt some ideas of Byeon and Jeanjean [12] to study the existence and concentration of positive solutions for equation (1.1) with critical growth. But the method of Byeon and Jeanjean [12] can not be used directly and more careful analysis is needed. For this aspect, we refer to [6, 42, 51].

This paper is organized as follows, in Section 2, we give some preliminary results. In Section 3, we analyze the "limiting problem" (1.10) and show the existence of ground state solutions. In Section 4, we prove the main result Theorem 1.1.

2. Preliminaries

In the following, we recall that by the Lax–Milgram theorem, for each $u \in H^1(\mathbf{R}^3)$, there exists a unique $\phi_u \in D^{1,2}(\mathbf{R}^3)$ such that $-\Delta\phi_u = u^2$. Moreover, ϕ_u can be expressed as

$$\phi_u(x) = \frac{1}{4\pi}\int_{\mathbf{R}^3} \frac{u^2(y)}{|x-y|}\,dy.$$

The function ϕ_u has the following property, see [14] and [43].

Lemma 2.1. *For any $u \in H^1(\mathbf{R}^3)$, we have*

(i) $\|\phi_u\|^2_{D^{1,2}(\mathbf{R}^3)} = \int_{\mathbf{R}^3} \phi_u u^2 \le C\,\|u\|^4_{L^{12/5}(\mathbf{R}^3)} \le C\,\|u\|^4_{H^1(\mathbf{R}^3)}$;

(ii) $\phi_u \ge 0$;

(iii) *If $u_n \rightharpoonup u$ in $H^1(\mathbf{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbf{R}^3)$ and* $\int_{\mathbf{R}^3} \phi_u u^2 \le \varliminf_{n\to\infty} \int_{\mathbf{R}^3} \phi_{u_n} u_n^2$;

(iv) *If $y \in \mathbf{R}^3$ and $\tilde{u}(x) = u(x+y)$, then $\phi_{\tilde{u}}(x) = \phi_u(x+y)$ and* $\int_{\mathbf{R}^3} \phi_{\tilde{u}}\tilde{u}^2 = \int_{\mathbf{R}^3} \phi_u u^2$.

Define $N\colon H^1(\mathbf{R}^3) \to \mathbf{R}$ by

$$N(u) = \int_{\mathbf{R}^3} \phi_u u^2.$$

Then, the functional N and its derivatives N' and N'' possess Brezis–Lieb splitting property, which is similar to the well-known Brezis–Lieb's Lemma (see [11]) and can be stated as the following form (see [17, 52]).

Lemma 2.2. *Let $u_n \rightharpoonup u$ in $H^1(\mathbf{R}^3)$ and $u_n \to u$ a.e. in $\mathbf{R}^3$, then, as $n \to \infty$,*

(i) $N(u_n - u) = N(u_n) - N(u) + o(1)$;

(ii) $N'(u_n - u) = N'(u_n) - N'(u) + o(1)$ *in* $H^{-1}(\mathbf{R}^3)$ *and* $N'\colon H^1(\mathbf{R}^3) \to H^{-1}(\mathbf{R}^3)$ *is weakly sequentially continuous;*

(iii) $N''(u_n - u) = N''(u_n) - N''(u) + o(1)$ *in* $L(H^1(\mathbf{R}^3), H^{-1}(\mathbf{R}^3))$ *and* $N''(u) \in L(H^1(\mathbf{R}^3), H^{-1}(\mathbf{R}^3))$ *is compact for any* $u \in H^1(\mathbf{R}^3)$.

Lemma 2.3. (General Minimax Principle) [50, Theorem 2.8] *Let X be a Banach space. Let M_0 be a closed subspace of the metric space M and $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma := \{\gamma \in C(M, X) \colon \gamma\,|_{M_0} \in \Gamma_0\}.$$

If $\varphi \in C^1(X, \mathbf{R})$ satisfies

$$\infty > c := \inf_{\gamma \in \Gamma} \sup_{u \in M} \varphi(\gamma(u)) > a := \sup_{\gamma_0 \in \Gamma_0} \sup_{u \in M_0} \varphi(\gamma_0(u)),$$

then, for every $\varepsilon \in (0, (c-a)/2)$, $\delta > 0$ and $\gamma \in \Gamma$ such that $\sup_M \varphi \circ \gamma \le c + \varepsilon$, there exists $u \in X$ such that

(a) $c - 2\varepsilon \le \varphi(u) \le c + 2\varepsilon$,
(b) $\mathrm{dist}(u, \gamma(M)) \le 2\delta$,
(c) $\|\varphi'(u)\| \le 8\varepsilon/\delta$.

Consider the following equation

$$-\Delta u + V_n(x) u = f_n(x, u) \quad \text{in } \mathbf{R}^3, \tag{2.1}$$

where $\{V_n\}$ is a sequence of continuous functions satisfying for some positive constant α independent of n such that

$$V_n(x) \ge \alpha > 0 \quad \text{for all } x \in \mathbf{R}^3$$

and $f_n(x, t)$ is a Carathéodory function such that for any $\delta > 0$, there exists $C_\delta > 0$ and

$$|f_n(x, t)| \le \delta |t| + C_\delta |t|^5, \quad \forall (x, t) \in \mathbf{R}^3 \times \mathbf{R},$$

where δ is independent of n.

From the process of proof of Theorem 1 in [53] and Theorem 1.11 in [31], we have the following lemma:

Lemma 2.4. *Assume that $\{v_n\}$ is a sequence of weak solutions to* (2.1) *satisfying $\|v_n\|_{H^1(\mathbf{R}^3)} \le C$ for $n \in \mathbf{N}$.*

(i) *If $\{|v_n|^6\}$ is uniformly integrable in any bounded domain in $\mathbf{R}^3$, then for any $x_0 \in \mathbf{R}^3$, $\exists R_0(x_0) > 0$ such that*

$$\|v_n\|_{L^\infty\left(B_{R_0(x_0)/4}(x_0)\right)} \le C(R_0(x_0)),$$

where $R_0(x_0)$ and $C(R_0(x_0))$ are independent of n.

(ii) *If $\{|v_n|^6\}$ is uniformly integrable near ∞, i.e., $\forall \varepsilon > 0$, $\exists R > 0$, for any $r > R$, $\int_{\mathbf{R}^3 \setminus B_r(0)} |v_n|^6 < \varepsilon$, then*

$$\lim_{|x| \to \infty} v_n(x) = 0 \quad \text{uniformly for } n.$$

Proof. See Lemma 2.10 of [27]. □

Lemma 2.5. [42] *Let R be a positive number and $\{u_n\}$ a bounded sequence in $H^1(\mathbf{R}^N)(N \ge 3)$. If*

$$\lim_{n \to \infty} \sup_{x \in \mathbf{R}^N} \int_{B_R(x)} |u_n|^{2N/(N-2)} = 0,$$

then $u_n \to 0$ in $L^{2N/(N-2)}(\mathbf{R}^N)$ as $n \to \infty$.

Lemma 2.6. [6, Lemma 2.7] *Let* $\{u_n\} \subset H^1_{\mathrm{loc}}(\mathbf{R}^N)$, $N \geq 3$, *be a sequence of functions such that*

$$u_n \rightharpoonup 0 \quad \text{in } H^1(\mathbf{R}^N).$$

Suppose that there exist a bounded open set $Q \subset \mathbf{R}^N$ *and a positive constant* $\gamma > 0$ *such that*

$$\int_Q |\nabla u_n|^2 \geq \gamma > 0, \quad \int_Q |u_n|^{2N/(N-2)} \geq \gamma > 0.$$

Moreover, suppose that

$$\Delta u_n + |u_n|^{4/(N-2)} u_n = \chi_n,$$

where $\chi_n \in H^{-1}(\mathbf{R}^N)$ *and*

$$|\langle \chi_n, \varphi \rangle| \leq \varepsilon_n \|\varphi\|_{H^1(\mathbf{R}^N)}, \quad \forall \varphi \in C_c^\infty(U),$$

where U *is an open neighborhood of* Q *and* $\{\varepsilon_n\}$ *is a sequence of positive numbers converging to* 0. *Then there exist a sequence of points* $\{y_n\} \subset \mathbf{R}^N$ *and a sequence of positive numbers* $\{\sigma_n\}$ *such that*

$$v_n(x) := \sigma_n^{(N-2)/2} u_n(\sigma_n x + y_n)$$

converges weakly in $D^{1,2}(\mathbf{R}^N)$ *to a nontrivial solution* v *of*

$$-\Delta u = |u|^{4/(N-2)} u, \quad u \in D^{1,2}(\mathbf{R}^N).$$

Moreover,

$$y_n \to \bar{y} \in \bar{Q} \quad \text{and} \quad \sigma_n \to 0.$$

The following lemma is a special case of Lemma 8.17 in [24] for Δ.

Lemma 2.7. [24, Lemma 8.17] *Let* Ω *be an open subset of* $\mathbf{R}^N (N \geq 2)$. *Suppose that* $t > N$, $h \in L^{t/2}(\Omega)$ *and* $u \in H^1(\Omega)$ *satisfies* $-\Delta u(x) \leq h(x)$, $x \in \Omega$ *in the weak sense. Then for any ball* $B_{2r}(y) \subset \Omega$,

$$\sup_{B_r(y)} u \leq C\Big(\big\|u^+\big\|_{L^2(B_{2r}(y))} + \|h\|_{L^{t/2}(B_{2r}(y))} \Big),$$

where $C = C(N, t, r)$ *is independent of* y.

3. The limiting problem

The following equation for $a > 0$

$$\text{(3.1)} \qquad \begin{cases} -\Delta u + au + \phi u = \lambda |u|^{p-2} u + |u|^4 u & \text{in } \mathbf{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbf{R}^3,\ u > 0,\ u \in H^1(\mathbf{R}^3) \end{cases}$$

is the limiting equation of (1.1).

We define the energy functional for the limiting problem (3.1) by

$$I_a(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2 + \frac{a}{2}\int_{\mathbf{R}^3} u^2 + \frac{1}{4}\int_{\mathbf{R}^3} \phi_u u^2 - \frac{\lambda}{p}\int_{\mathbf{R}^3} (u^+)^p - \frac{1}{6}\int_{\mathbf{R}^3} (u^+)^6,\ u \in H^1(\mathbf{R}^3).$$

In view of [40], if $u \in H^1(\mathbf{R}^3)$ is a weak solution to problem (3.1), then we have the following Pohozaev's identity:

$$\text{(3.2)} \quad P_a(u) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2 + \frac{3}{2} a \int_{\mathbf{R}^3} u^2 + \frac{5}{4}\int_{\mathbf{R}^3} \phi_u u^2 - \frac{3}{p}\lambda \int_{\mathbf{R}^3} (u^+)^p - \frac{1}{2}\int_{\mathbf{R}^3} (u^+)^6 = 0.$$

As in [43], we introduce the following manifold

$$M_a := \{u \in H^1(\mathbf{R}^3)\backslash\{0\} \mid G_a(u) = 0\},$$

where

$$G_a(u) = \frac{3}{2}\int_{\mathbf{R}^3} |\nabla u|^2 + \frac{1}{2}a\int_{\mathbf{R}^3} u^2 + \frac{3}{4}\int_{\mathbf{R}^3} \phi_u u^2 - \frac{(2p-3)}{p}\lambda\int_{\mathbf{R}^3} (u^+)^p - \frac{3}{2}\int_{\mathbf{R}^3} (u^+)^6.$$

It is clear that

$$G_a(u) = 2\,\langle I'_a(u), u\rangle - P_a(u), \tag{3.3}$$

where $P_a(u)$ is given in (3.2).

Remark 3.1. If $u \in H^1(\mathbf{R}^3)$ is a nontrivial weak solution to (3.1), then by (3.2), (3.3), we see that $u \in M_a$.

Lemma 3.2. *For any $u \in H^1(\mathbf{R}^3)\backslash\{0\}$, there is a unique $\tilde{t} > 0$ such that $u_{\tilde{t}} \in M_a$, where $u_{\tilde{t}}(x) := \tilde{t}^2 u(\tilde{t}x)$. Moreover, $I_a(u_{\tilde{t}}) = \max_{t>0} I_a(u_t)$.*

Proof. For any $u \in H^1(\mathbf{R}^3)\backslash\{0\}$ and $t > 0$, set $u_t(x) := t^2 u(tx)$. Consider

$$\begin{aligned}\gamma(t) := I_a(u_t) = \frac{1}{2}t^3\int_{\mathbf{R}^3} |\nabla u|^2 + \frac{1}{2}at\int_{\mathbf{R}^3} u^2 + \frac{1}{4}t^3\int_{\mathbf{R}^3} \phi_u u^2 \\ - \frac{\lambda}{p}t^{2p-3}\int_{\mathbf{R}^3} (u^+)^p - \frac{1}{6}t^9\int_{\mathbf{R}^3} (u^+)^6.\end{aligned}$$

Since $2p-3 > 3$, by elementary computations, $\gamma(t)$ has a unique critical point $\tilde{t} > 0$ corresponding to its maximum, i.e., $\gamma(\tilde{t}) = \max_{t>0} \gamma(t)$ and $\gamma'(\tilde{t}) = 0$. Hence

$$\frac{3}{2}\tilde{t}^2\int_{\mathbf{R}^3}|\nabla u|^2 + \frac{1}{2}a\int_{\mathbf{R}^3} u^2 + \frac{3}{4}\tilde{t}^2\int_{\mathbf{R}^3} \phi_u u^2 - \frac{(2p-3)}{p}\lambda\tilde{t}^{2p-4}\int_{\mathbf{R}^3} (u^+)^p - \frac{3}{2}\tilde{t}^8\int_{\mathbf{R}^3} (u^+)^6 = 0,$$

then $G_a(u_{\tilde{t}}) = 0$, $u_{\tilde{t}} \in M_a$ and $I_a(u_{\tilde{t}}) = \max_{t>0} I_a(u_t)$. □

Lemma 3.3. *I_a possesses the Mountain-Pass geometry.*

Proof. $\exists \rho, \delta > 0$ small such that

$$\begin{aligned}I_a(u) &= \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2 + \frac{1}{2}a\int_{\mathbf{R}^3} u^2 + \frac{1}{4}\int_{\mathbf{R}^3} \phi_u u^2 - \frac{\lambda}{p}\int_{\mathbf{R}^3} (u^+)^p - \frac{1}{6}\int_{\mathbf{R}^3} (u^+)^6 \\ &\geq \frac{1}{2}\|u\|^2_{H^1(\mathbf{R}^3)} - C\lambda\|u\|^p_{H^1(\mathbf{R}^3)} - C\|u\|^6_{H^1(\mathbf{R}^3)} \\ &\geq \delta > 0 \text{ for } \|u\|_{H^1(\mathbf{R}^3)} = \rho > 0.\end{aligned}$$

Fix $u \in H^1(\mathbf{R}^3)\backslash\{0\}$, set $u_t(x) := t^2 u(tx)$,

$$I_a(u_t) = \frac{1}{2}t^3\int_{\mathbf{R}^3}|\nabla u|^2 + \frac{1}{2}at\int_{\mathbf{R}^3} u^2 + \frac{1}{4}t^3\int_{\mathbf{R}^3}\phi_u u^2 - \frac{\lambda}{p}t^{2p-3}\int_{\mathbf{R}^3}(u^+)^p - \frac{1}{6}t^9\int_{\mathbf{R}^3} (u^+)^6 < 0$$

for $t > 0$ large, then $\exists t_0 > 0$, set $u_0 := u_{t_0}$, $I(u_0) < 0$. □

Hence we can define the Mountain-Pass level of I_a:

$$c_a := \inf_{\gamma\in\Gamma_a} \sup_{t\in[0,1]} I_a(\gamma(t)), \tag{3.4}$$

where the set of paths is defined as

$$\Gamma_a := \left\{\gamma \in C([0,1],\ H^1(\mathbf{R}^3))\colon \gamma(0) = 0 \text{ and } I_a(\gamma(1)) < 0\right\}. \tag{3.5}$$

Next, we will construct a (PS) sequence $\{u_n\}_{n=1}^{\infty}$ for I_a at the level c_a that satisfies $G_a(u_n) \to 0$ as $n \to \infty$, i.e.,

Proposition 3.4. *There exists a sequence $\{u_n\}_{n=1}^{\infty}$ in $H^1(\mathbf{R}^3)$ such that, as $n \to \infty$,*

$$I_a(u_n) \to c_a, \quad I'_a(u_n) \to 0, \quad G_a(u_n) \to 0. \tag{3.6}$$

Proof. We define the map $\Phi\colon \mathbf{R} \times H^1(\mathbf{R}^3) \to H^1(\mathbf{R}^3)$ for $\theta \in \mathbf{R}$, $v \in H^1(\mathbf{R}^3)$ and $x \in \mathbf{R}^3$ by $\Phi(\theta, v) = e^{2\theta} v(e^{\theta}x)$. For every $\theta \in \mathbf{R}$, $v \in H^1(\mathbf{R}^3)$, the functional $I_a \circ \Phi$ is computed as

$$\begin{aligned} I_a \circ \Phi(\theta, v) = {} & \frac{1}{2} e^{3\theta} \int_{\mathbf{R}^3} |\nabla v|^2 + \frac{1}{2} a e^{\theta} \int_{\mathbf{R}^3} v^2 + \frac{1}{4} e^{3\theta} \int_{\mathbf{R}^3} \phi_v v^2 \\ & - \frac{\lambda}{p} e^{(2p-3)\theta} \int_{\mathbf{R}^3} (v^+)^p - \frac{1}{6} e^{9\theta} \int_{\mathbf{R}^3} (v^+)^6. \end{aligned}$$

In view of Lemma 3.3, we can easily check that $I_a \circ \Phi(\theta, v) > 0$ for all (θ, v) with $|\theta|$, $\|v\|_{H^1(\mathbf{R}^3)}$ small and $(I_a \circ \Phi)(0, u_0) < 0$, i.e. $I_a \circ \Phi$ possesses the Mountain-Pass geometry in $\mathbf{R} \times H^1(\mathbf{R}^3)$. Hence we can define the Mountain-Pass level of $I_a \circ \Phi$:

$$\tilde{c}_a := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_a} \sup_{t \in [0,1]} (I_a \circ \Phi)(\tilde{\gamma}(t)), \tag{3.7}$$

where the set of paths is defined as

$$\tilde{\Gamma}_a := \left\{ \tilde{\gamma} \in C([0,1], \mathbf{R} \times H^1(\mathbf{R}^3))\colon \tilde{\gamma}(0) = (0,0) \text{ and } (I_a \circ \Phi)(\tilde{\gamma}(1)) < 0 \right\}. \tag{3.8}$$

As $\Gamma_a = \{\Phi \circ \tilde{\gamma} \colon \tilde{\gamma} \in \tilde{\Gamma}_a\}$, the Mountain-Pass levels of I_a and $I_a \circ \Phi$ coincide, i.e., $c_a = \tilde{c}_a$.

By Lemma 2.3, we see that there exists a sequence $\{(\theta_n, v_n)\}_{n \in \mathbf{N}}$ in $\mathbf{R} \times H^1(\mathbf{R}^3)$ such that as $n \to \infty$,

$$(I_a \circ \Phi)(\theta_n, v_n) \to c_a, \tag{3.9}$$

$$(I_a \circ \Phi)'(\theta_n, v_n) \to 0 \quad \text{in } \mathbf{R} \times H^1(\mathbf{R}^3))^{-1}, \tag{3.10}$$

$$\theta_n \to 0. \tag{3.11}$$

Indeed, set $\varepsilon = \varepsilon_n := \frac{1}{n^2}$, $\delta = \delta_n := \frac{1}{n}$ in Lemma 2.3, (3.9), (3.10) are direct conclusions from (a), (c) of Lemma 2.3, we just need to verify (3.11). In view of (3.4), (3.5), for $\varepsilon = \varepsilon_n := \frac{1}{n^2}$, $\exists \gamma_n \in \Gamma_a$, such that

$$\sup_{t \in [0,1]} I_a(\gamma_n(t)) \le c_a + \frac{1}{n^2}.$$

Set $\tilde{\gamma}_n(t) = (0, \gamma_n(t))$, then

$$\sup_{t \in [0,1]} I_a \circ \Phi(\tilde{\gamma}_n(t)) = \sup_{t \in [0,1]} I_a(\gamma_n(t)) \le c_a + \frac{1}{n^2}.$$

By (b) of Lemma 2.3, there exists $(\theta_n, v_n) \in \mathbf{R} \times H^1(\mathbf{R}^3)$ such that $\operatorname{dist}((\theta_n, v_n), (0, \gamma_n(t))) \le \frac{2}{n}$, then (3.11) holds.

For every $(h, w) \in \mathbf{R} \times H^1(\mathbf{R}^3)$,

$$\langle (I_a \circ \Phi)'(\theta_n, v_n), (h, w) \rangle = \langle I'_a(\Phi(\theta_n, v_n)), \Phi(\theta_n, w) \rangle + G_a(\Phi(\theta_n, v_n))h. \tag{3.12}$$

Taking $h = 1$, $w = 0$ in (3.12), we get

$$G_a(\Phi(\theta_n, v_n)) \to 0 \quad \text{as } n \to \infty.$$

Denote $u_n := \Phi(\theta_n, v_n)$, we have

$$G_a(u_n) \to 0 \quad \text{as } n \to \infty.$$

For any $v \in H^1(\mathbf{R}^3)$, set $w(x) = e^{-2\theta_n} v(e^{-\theta_n} x)$, $h = 0$ in (3.12), we get

$$\langle {I'}_a(u_n), v\rangle = o(1)\big\|e^{-2\theta_n} v(e^{-\theta_n} x)\big\|_{H^1(\mathbf{R}^3)} = o(1)\|v\|_{H^1(\mathbf{R}^3)}$$

for $\theta_n \to 0$ as $n \to \infty$, i.e., ${I'}_a(u_n) \to 0$ in $(H^1(\mathbf{R}^3))^{-1}$ as $n \to \infty$. Hence, we have got a bounded sequence $\{u_n\}_{n=1}^{\infty} \subset H^1(\mathbf{R}^3)$ that satisfies (3.6). □

Moreover, using the same argument as in [41], we can prove

$$c_a = \inf_{u\in H^1(\mathbf{R}^3)\setminus\{0\}} \max_{t>0} I_a(u_t) = \inf_{u\in M_a} I_a(u) > 0. \tag{3.13}$$

For the Mountain-Pass level c_a for I_a, we have the following estimate:

Lemma 3.5.

$$c_a < \frac{1}{3} S^{\frac{3}{2}}$$

for $\lambda > 0$ large, where S is the best Sobolev constant for the embedding $D^{1,2}(\mathbf{R}^3) \hookrightarrow L^6(\mathbf{R}^3)$.

Proof. Let $\varphi \in C_c^\infty(B_2(0))$ satisfying $\varphi \equiv 1$ on $B_1(0)$ and $0 \le \varphi \le 1$ on $B_2(0)$. Given $\delta > 0$, we set $\psi_\delta(x) := \varphi(x) w_\delta(x)$, where

$$w_\delta(x) = (3\delta)^{\frac{1}{4}} \frac{1}{(\delta + |x|^2)^{\frac{1}{2}}}$$

satisfies

$$\int_{\mathbf{R}^3} |\nabla w_\delta|^2 = \int_{\mathbf{R}^3} |w_\delta|^6 = S^{\frac{3}{2}}. \tag{3.14}$$

We see that

$$\int_{\mathbf{R}^3\setminus B_1(0)} |\nabla \psi_\delta|^2 = O(\delta^{1/2}) \quad \text{as } \delta \to 0. \tag{3.15}$$

Let $X_\delta := \int_{\mathbf{R}^3} |\nabla v_\delta|^2$, where $v_\delta := \psi_\delta / (\int_{B_2(0)} |\psi_\delta|^6)^{\frac{1}{6}}$. We find

$$X_\delta \le S + O(\delta^{1/2}) \quad \text{as } \delta \to 0. \tag{3.16}$$

In view of Lemma 3.2, there exists $t_\delta > 0$ such that $\sup_{t\ge 0} I_a((v_\delta)_t) = I_a((v_\delta)_{t_\delta})$. Hence $\frac{dI_a((v_\delta)_t)}{dt}|_{t=t_\delta} = 0$, that is

$$\frac{3}{2} t_\delta^2 \int_{\mathbf{R}^3} |\nabla v_\delta|^2 + \frac{1}{2} a \int_{\mathbf{R}^3} v_\delta^2 + \frac{3}{4} t_\delta^2 \int_{\mathbf{R}^3} \phi_{v_\delta} v_\delta^2 - \frac{(2p-3)}{p} \lambda t_\delta^{2p-5} \int_{\mathbf{R}^3} v_\delta^p - \frac{3}{2} t_\delta^8 \int_{\mathbf{R}^3} v_\delta^6 = 0$$

which implies

$$t_\delta^8 \le t_\delta^2 X_\delta + \frac{1}{3} a \int_{\mathbf{R}^3} v_\delta^2 + \frac{1}{2} t_\delta^2 \int_{\mathbf{R}^3} \phi_{v_\delta} v_\delta^2. \tag{3.17}$$

Direct calculations show that

$$\int_{\mathbf{R}^3} v_\delta^2 = O(\delta^{1/2}), \quad \left(\int_{\mathbf{R}^3} v_\delta^{12/5}\right)^{5/3} = O(\delta). \tag{3.18}$$

(3.16), (3.17), (3.18) and Lemma 2.1 (i) imply that $|t_\delta| \le C_1$, where C_1 is independent of $\delta > 0$ small.

We can assume that there is a positive constant C_2 such that $t_\delta \geq C_2 > 0$ for $\delta > 0$ small. Otherwise, we could find a sequence $\delta_n \to 0$ as $n \to \infty$ such that $t_{\delta_n} \to 0$ as $n \to \infty$. Now, up to a subsequence, we have $(v_{\delta_n})_{t_{\delta_n}} \to 0$ in $H^1(\mathbf{R}^3)$ as $n \to \infty$. Therefore

$$0 < c_a \leq \sup_{t \geq 0} I_a((v_{\delta_n})_t) = I_a((v_{\delta_n})_{t_{\delta_n}}) \to I_a(0) = 0,$$

which is a contradiction.

Denote $g(t) = \frac{t^3}{2}\int_{\mathbf{R}^3} |\nabla v_\delta|^2 - \frac{t^9}{6}\int_{\mathbf{R}^3} v_\delta^6$, it is easy to check that

$$\sup_{t>0} g(t) = \frac{1}{3}\Big(\int_{\mathbf{R}^3} |\nabla v_\delta|^2\Big)^{\frac{3}{2}} \leq \frac{1}{3}\big(S + O(\delta^{1/2})\big)^{3/2} \leq \frac{1}{3}S^{\frac{3}{2}} + O(\delta^{1/2}).$$

Thus

$$\begin{aligned} &I((v_\delta)_{t_\delta}) \\ &= \frac{1}{2}t_\delta^3 \int_{\mathbf{R}^3} |\nabla v_\delta|^2 + \frac{1}{2}t_\delta \int_{\mathbf{R}^3} v_\delta^2 + \frac{1}{4}t_\delta^3 \int_{\mathbf{R}^3} \phi_{v_\delta} v_\delta^2 - \frac{\lambda}{p}t_\delta^{2p-3}\int_{\mathbf{R}^3} v_\delta^p - \frac{1}{6}t_\delta^9 \int_{\mathbf{R}^3} v_\delta^6 \\ &\leq \sup_{t>0} g(t) + C\int_{\mathbf{R}^3} v_\delta^2 + C\Big(\int_{\mathbf{R}^3} v_\delta^{12/5}\Big)^{5/3} - C\lambda\int_{\mathbf{R}^3} v_\delta^p \\ &\leq \frac{1}{3}S^{\frac{3}{2}} + O(\delta^{1/2}) + C\int_{\mathbf{R}^3} v_\delta^2 - C\lambda \int_{\mathbf{R}^3} v_\delta^p, \end{aligned} \tag{3.19}$$

where we have used (3.18).

From (3.19), to complete the proof, it suffices to show that

$$\lim_{\delta\to 0^+} \frac{1}{\delta^{1/2}}\left[C\int_{B_1(0)} v_\delta^2 - C\lambda\int_{B_1(0)} v_\delta^p\right] = -\infty \tag{3.20}$$

and

$$\lim_{\delta\to 0^+} \frac{1}{\delta^{1/2}}\left[C\int_{B_2(0)\backslash B_1(0)} v_\delta^2 - C\lambda\int_{B_2(0)\backslash B_1(0)} v_\delta^p\right] \leq C. \tag{3.21}$$

To this end, we find

$$\begin{aligned} \frac{1}{\delta^{1/2}}C\lambda\int_{B_1(0)} v_\delta^p &\geq \frac{C\lambda}{\delta^{1/2}}\int_{B_1(0)} \frac{\delta^{p/4}}{(\delta + |x|^2)^{p/2}} \\ &\overset{x'=x/\delta^{1/2}}{\geq} \frac{C\lambda}{\delta^{\frac{1}{2}}}\int_{B_{1/\delta^{1/2}}(0)} \frac{\delta^{\frac{p}{4}}}{(\delta + \delta|x'|^2)^{\frac{p}{2}}}\delta^{\frac{3}{2}} \geq C\lambda\delta^{1-\frac{p}{4}}\int_{B_{1/\delta^{1/2}}(0)} \frac{1}{(1+|x'|^2)^{p/2}}. \end{aligned}$$

Since $p \in (3,4]$, choosing $\lambda = 1/\delta$ and combining with (3.18), (3.20) holds.

Since

$$\frac{1}{\delta^{1/2}}\left[\int_{B_2(0)\backslash B_1(0)} v_\delta^2 - C\lambda\int_{B_2(0)\backslash B_1(0)} v_\delta^p\right] \leq \frac{C}{\delta^{1/2}}\int_{B_2(0)\backslash B_1(0)} v_\delta^2 \leq C,$$

where we have used (3.18), then (3.21) holds. □

Lemma 3.6. *Every sequence $\{u_n\}_{n=1}^\infty$ satisfying (3.6) is bounded in $H^1(\mathbf{R}^3)$.*

Proof. By (3.6), we have

$$c_a + o(1) = I_a(u_n) - \frac{1}{2p-3}G_a(u_n) = \frac{p-3}{2p-3}\int_{\mathbf{R}^3}|\nabla u_n|^2 + \frac{p-2}{2p-3}a\int_{\mathbf{R}^3}|u_n|^2 + \frac{p-3}{2(2p-3)}\int_{\mathbf{R}^3}\phi_{u_n}u_n^2 + \frac{6-p}{3(2p-3)}\int_{\mathbf{R}^3}(u_n^+)^6,$$

we get the upper bound of $\|u_n\|_{H^1(\mathbf{R}^3)}$. □

Lemma 3.7. *There is a sequence* $\{x_n\} \subset \mathbf{R}^3$ *and* $R > 0$, $\beta > 0$ *such that*

$$\int_{B_R(x_n)} u_n^2 \geq \beta,$$

where $\{u_n\}$ *is the sequence given in* (3.6).

Proof. Assume the contrary that the lemma does not hold. By the Vanishing Theorem [33, Lemma 1.1], it follows that as $n \to \infty$,

$$\int_{\mathbf{R}^3}|u_n|^s \to 0 \text{ for all } 2 < s < 6 \quad \text{and} \quad \int_{\mathbf{R}^3}\phi_{u_n}u_n^2 \to 0.$$

Using $\langle I'_a(u_n), u_n\rangle = o(1)$, we get

$$\int_{\mathbf{R}^3}|\nabla u_n|^2 + a\int_{\mathbf{R}^3}u_n^2 - \int_{\mathbf{R}^3}(u_n^+)^6 = o(1).$$

By $I_a(u_n) \to c_a$, we have

$$\frac{1}{2}\int_{\mathbf{R}^3}|\nabla u_n|^2 + \frac{1}{2}a\int_{\mathbf{R}^3}u_n^2 - \frac{1}{6}\int_{\mathbf{R}^3}(u_n^+)^6 = c_a + o(1). \tag{3.22}$$

Let $l \geq 0$ be such that

$$\int_{\mathbf{R}^3}|\nabla u_n|^2 + a\int_{\mathbf{R}^3}u_n^2 \to l \tag{3.23}$$

and

$$\int_{\mathbf{R}^3}(u_n^+)^6 \to l. \tag{3.24}$$

It is easy to check that $l > 0$, otherwise $\|u_n\|_{H^1(\mathbf{R}^3)} \to 0$ as $n \to \infty$ which contradicts to $c_a > 0$. From (3.22), (3.23), (3.24), we get $c_a = \frac{1}{3}l$.

Now, using the definition of the constant S, we have

$$\int_{\mathbf{R}^3}|\nabla u_n|^2 + \int_{\mathbf{R}^3}u_n^2 \geq S\Big(\int_{\mathbf{R}^3}(u_n^+)^6\Big)^{\frac{1}{3}}.$$

Letting $n \to \infty$ in the above inequality, we achieve that $l \geq S^{3/2}$. Hence

$$c_a = \frac{1}{3}l \geq \frac{1}{3}S^{\frac{3}{2}},$$

which contradicts to Lemma 3.5. □

We have the following proposition:

Proposition 3.8. (3.1) *has a positive ground state solution* $\tilde{u} \in H^1(\mathbf{R}^3)$.

Proof. Let $\{u_n\}$ be the sequence given in (3.6) and c_a be the Mountain-Pass value for I_a respectively. Denote $\tilde{u}_n(x) = u_n(x + x_n)$, where $\{x_n\}$ is the sequence given in Lemma 3.7. Using standard argument, up to a subsequence, we may assume that there is a $\tilde{u} \in H^1(\mathbf{R}^3)$ such that

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u} & \text{in } H^1(\mathbf{R}^3), \\ \tilde{u}_n \to \tilde{u} & \text{in } L^s_{\text{loc}}(\mathbf{R}^3) \text{ for all } 1 \le s < 6, \\ \tilde{u}_n \to \tilde{u} & \text{a.e. in } \mathbf{R}^3. \end{cases} \tag{3.25}$$

By Lemma 3.7, $\tilde{u}$ is nontrivial. Moreover, $\tilde{u}$ satisfies

$$-\Delta u + au + \phi_u u = \lambda(u^+)^{p-1} + (u^+)^5 \text{ in } \mathbf{R}^3 \tag{3.26}$$

and $G_a(\tilde{u}) = 0$. By (3.13), we have

$$\begin{aligned} c_a \le I_a(\tilde{u}) &= I_a(\tilde{u}) - \frac{1}{3}G_a(\tilde{u}) = \frac{1}{3}a\int_{\mathbf{R}^3} \tilde{u}^2 + \frac{2p-6}{3p}\lambda \int_{\mathbf{R}^3} (\tilde{u}^+)^p + \frac{1}{3}\int_{\mathbf{R}^3} (\tilde{u}^+)^6 \\ &\le \varliminf_{n\to\infty} \frac{1}{3}a\int_{\mathbf{R}^3} \tilde{u}_n^2 + \frac{2p-6}{3p}\lambda\int_{\mathbf{R}^3} (\tilde{u}_n^+)^p + \frac{1}{3}\int_{\mathbf{R}^3} (\tilde{u}_n^+)^6 = \varliminf_{n\to\infty}\Big[I_a(\tilde{u}_n) - \frac{1}{3}G_a(\tilde{u}_n)\Big] \\ &= \varliminf_{n\to\infty}\Big[I_a(u_n) - \frac{1}{3}G_a(u_n)\Big] = c_a. \end{aligned}$$

Hence $I_a(\tilde{u}) = c_a$ and $I'_a(\tilde{u}) = 0$. By the standard elliptic estimate and strong maximum principle, $\tilde{u}(x) > 0$ for all $x \in \mathbf{R}^3$. In view of (3.13), $\tilde{u}$ is in fact a positive ground state solution of (3.1). □

Let S_a the set of ground state solutions U of (3.1) satisfying $U(0) = \max_{x\in\mathbf{R}^3} U(x)$. Then, we obtain the following compactness of S_a.

Proposition 3.9. *For each $a > 0$, S_a is compact in $H^1(\mathbf{R}^3)$.*

Proof. For any $U \in S_a$, we have

$$\begin{aligned} c_a &= I_a(U) - \frac{1}{2p-3}G_a(U) \\ &= \frac{p-3}{2p-3}\int_{\mathbf{R}^3} |\nabla U|^2 + \frac{p-2}{2p-3}a\int_{\mathbf{R}^3} U^2 + \frac{p-3}{2(2p-3)}\int_{\mathbf{R}^3} \phi_U U^2 + \frac{6-p}{3(2p-3)}\int_{\mathbf{R}^3} U^6. \end{aligned}$$

Thus S_a is bounded in $H^1(\mathbf{R}^3)$.

For any sequence $\{U_k\} \subset S_a$, up to a subsequence, we may assume that there is a $U_0 \in H^1(\mathbf{R}^3)$ such that

$$U_k \rightharpoonup U_0 \quad \text{in } H^1(\mathbf{R}^3) \tag{3.27}$$

and U_0 satisfies

$$-\Delta U_0 + aU_0 + \phi_{U_0}U_0 = \lambda U_0^{p-1} + U_0^5 \quad \text{in } \mathbf{R}^3,\ U_0 \ge 0.$$

Next, we will show that U_0 is nontrivial. First, we claim that, up to a subsequence,

$$U_k \to U_0 \quad \text{in } L^6_{\text{loc}}(\mathbf{R}^3). \tag{3.28}$$

Indeed, in view of (3.27), we may assume that

$$|\nabla U_k|^2 \rightharpoonup |\nabla U_0|^2 + \mu \quad \text{and} \quad U_k^6 \rightharpoonup U_0^6 + \nu,$$

where μ and ν are two bounded nonnegative measures on $\mathbf{R}^3$. By the Concentration Compactness Principle II [34, Lemma 1.1], we obtain an at most countable index set Γ, sequence $\{x_i\} \subset \mathbf{R}^3$ and $\{\mu_i\}, \{\nu_i\} \subset (0, \infty)$ such that

$$\mu \geq \sum_{i\in\Gamma} \mu_i \delta_{x_i}, \quad \nu = \sum_{i\in\Gamma} \nu_i \delta_{x_i} \quad \text{and} \quad S(\nu_i)^{\frac{1}{3}} \leq \mu_i. \tag{3.29}$$

It suffices to show that for any bounded domain Ω, $\{x_i\}_{i\in\Gamma} \cap \Omega = \emptyset$. Suppose, by contradiction, that $x_i \in \Omega$ for some $i \in \Gamma$. Define, for $\rho > 0$, the function $\psi_\rho(x) := \psi(\frac{x-x_i}{\rho})$ where ψ is a smooth cut-off function such that $\psi = 1$ on $B_1(0)$, $\psi = 0$ on $\mathbf{R}^3 \backslash B_2(0)$, $0 \leq \psi \leq 1$ and $|\nabla \psi| \leq C$. We suppose that ρ is chosen in such a way that the support of ψ_ρ is contained in Ω. Using $\langle I'_a(U_k), \psi_\rho U_k \rangle = 0$, we see

$$\begin{aligned} &\int_{\mathbf{R}^3} |\nabla U_k|^2 \psi_\rho + \int_{\mathbf{R}^3} (\nabla U_k \cdot \nabla \psi_\rho) U_k + a \int_{\mathbf{R}^3} U_k^2 \psi_\rho + \int_{\mathbf{R}^3} \phi_{U_k} U_k^2 \psi_\rho \\ &= \lambda \int_{\mathbf{R}^3} U_k^p \psi_\rho + \int_{\mathbf{R}^3} U_k^6 \psi_\rho. \end{aligned} \tag{3.30}$$

Since

$$\begin{aligned} &\overline{\lim_{k\to\infty}} \left| \int_{\mathbf{R}^3} (\nabla U_k \cdot \nabla \psi_\rho) U_k \right| \leq \overline{\lim_{k\to\infty}} \Big(\int_{\mathbf{R}^3} |\nabla U_k|^2 \Big)^{\frac{1}{2}} \cdot \Big(\int_{\mathbf{R}^3} U_k^2 |\nabla \psi_\rho|^2 \Big)^{\frac{1}{2}} \\ &\leq C \Big(\int_{\mathbf{R}^3} U_0^2 |\nabla \psi_\rho|^2 \Big)^{\frac{1}{2}} \leq C \Big(\int_{B_{2\rho}(x_i)} U_0^6 \Big)^{\frac{1}{6}} \Big(\int_{B_{2\rho}(x_i)} |\nabla \psi_\rho|^3 \Big)^{\frac{1}{3}} \\ &\leq C \Big(\int_{B_{2\rho}(x_i)} U_0^6 \Big)^{\frac{1}{6}} \to 0 \quad \text{as } \rho \to 0, \end{aligned} \tag{3.31}$$

$$\overline{\lim_{k\to\infty}} \int_{\mathbf{R}^3} |\nabla U_k|^2 \psi_\rho \geq \int_{\mathbf{R}^3} |\nabla U_0|^2 \psi_\rho + \mu_i \to \mu_i \quad \text{as } \rho \to 0, \tag{3.32}$$

$$\overline{\lim_{k\to\infty}} \lambda \int_{\mathbf{R}^3} U_k^p \psi_\rho = \lambda \int_{\mathbf{R}^3} U_0^p \psi_\rho \to 0 \quad \text{as } \rho \to 0, \tag{3.33}$$

and

$$\overline{\lim_{k\to\infty}} \int_{\mathbf{R}^3} U_k^6 \psi_\rho = \int_{\mathbf{R}^3} U_0^6 \psi_\rho + \nu_i \to \nu_i \quad \text{as } \rho \to 0. \tag{3.34}$$

We obtain from (3.30) that $\mu_i \leq \nu_i$. Combining with (3.29), we have $\nu_i \geq S^{3/2}$. On the other hand,

$$c_a = I_a(U_k) - \frac{1}{3} G_a(U_k) = \frac{1}{3} a \int_{\mathbf{R}^3} U_k^2 + \frac{2p-6}{3p} \int_{\mathbf{R}^3} U_k^p + \frac{1}{3} \int_{\mathbf{R}^3} U_k^6 \geq \frac{1}{3} \nu_i \geq \frac{1}{3} S^{\frac{3}{2}},$$

which contradicts to Lemma 3.5, then (3.28) holds.

From (3.28), $\{U_k^6\}$ is uniformly integrable in any bounded domain in $\mathbf{R}^3$. By Lemma 2.4 (i), $\|U_k\|_{L^\infty_{\mathrm{loc}}(\mathbf{R}^3)} \leq C$. In view of [46], $\exists \alpha \in (0,1)$ such that $\|U_k\|_{C^{1,\alpha}_{\mathrm{loc}}(\mathbf{R}^3)} \leq C$, and using Schauder's estimate, we have

$$\|U_k\|_{C^{2,\alpha}_{\mathrm{loc}}(\mathbf{R}^3)} \leq C.$$

By the Arzela–Ascoli's Theorem, we have

$$U_k(0) \to U_0(0) \quad \text{as } k \to \infty.$$

Since $\Delta U_k(0) \le 0$, from (3.1), we can check that $\exists b > 0$ such that $U_k(0) \ge b > 0$, then $U_0(0) \ge b > 0$, this means that U_0 is nontrivial.

Since

$$\begin{aligned}
c_a &\le I_a(U_0) - \frac{1}{2p-3} G_a(U_0) \\
&= \frac{p-3}{2p-3}\int_{\mathbf{R}^3} |\nabla U_0|^2 + \frac{p-2}{2p-3} a \int_{\mathbf{R}^3} U_0^2 + \frac{p-3}{2(2p-3)}\int_{\mathbf{R}^3} \phi_{U_0} U_0^2 + \frac{6-p}{3(2p-3)}\int_{\mathbf{R}^3} U_0^6 \\
&= \lim_{k\to\infty} \frac{p-3}{2p-3}\int_{\mathbf{R}^3} |\nabla U_k|^2 + \frac{p-2}{2p-3} a \int_{\mathbf{R}^3} U_k^2 + \frac{p-3}{2(2p-3)}\int_{\mathbf{R}^3} \phi_{U_k} U_k^2 \\
&\quad + \frac{6-p}{3(2p-3)}\int_{\mathbf{R}^3} U_k^6 = \lim_{k\to\infty}\Big[I_a(U_k) - \frac{1}{2p-3} G_a(U_k)\Big] = c_a,
\end{aligned}$$

which means that $I_a(U_0) = c_a$ and $U_k \to U_0$ in $H^1(\mathbf{R}^3)$. This completes the proof that S_a is compact in $H^1(\mathbf{R}^3)$. □

4. Proof of Theorem 1.1

(1.1) can be rewritten as

$$\text{(4.1)} \qquad \begin{cases} -\Delta v + V(\varepsilon x)v + \phi v = \lambda |v|^{p-2} v + |v|^4 v & \text{in } \mathbf{R}^3, \\ -\Delta \phi = v^2 & \text{in } \mathbf{R}^3,\ v > 0,\ v \in H^1(\mathbf{R}^3), \end{cases}$$

and the corresponding energy functional is

$$I_\varepsilon(v) = \frac{1}{2}\int_{\mathbf{R}^3} |\nabla v|^2 + \frac{1}{2}\int_{\mathbf{R}^3} V(\varepsilon x) v^2 + \frac{1}{4}\int_{\mathbf{R}^3} \phi_v v^2 - \frac{1}{p}\lambda \int_{\mathbf{R}^3} (v^+)^p - \frac{1}{6}\int_{\mathbf{R}^3} (v^+)^6,\ v \in H_\varepsilon,$$

where $H_\varepsilon := \{v \in H^1(\mathbf{R}^3) \mid \int_{\mathbf{R}^3} V(\varepsilon x) v^2 < \infty\}$ endowed with the norm

$$\|v\|_{H_\varepsilon} := \left(\int_{\mathbf{R}^3} |\nabla v|^2 + \int_{\mathbf{R}^3} V(\varepsilon x) v^2\right)^{1/2}.$$

We define

$$\chi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \Lambda/\varepsilon, \\ \varepsilon^{-1} & \text{if } x \notin \Lambda/\varepsilon, \end{cases}$$

and

$$Q_\varepsilon(v) = \left(\int_{\mathbf{R}^3} \chi_\varepsilon v^2 - 1\right)_+^2.$$

Finally, set $J_\varepsilon \colon H_\varepsilon \to \mathbf{R}$ be given by

$$J_\varepsilon(v) = I_\varepsilon(v) + Q_\varepsilon(v).$$

Note that this type of penalization was firstly introduced in [13]. It is standard to show that $J_\varepsilon \in C^1(H_\varepsilon, \mathbf{R})$. To find solutions of (4.1) which concentrate around the local minimum of V in Λ as $\varepsilon \to 0$, we shall search critical points of J_ε for which Q_ε is zero.

Let $c_{V_0} = I_{V_0}(w)$ for $w \in S_{V_0}$ and $10\delta = \operatorname{dist}\{\mathcal{M}, \mathbf{R}^3 \backslash \Lambda\}$, we fix a $\beta \in (0,\delta)$ and a cut-off function $\varphi \in C_c^\infty(\mathbf{R}^3)$ such that $0 \le \varphi \le 1$, $\varphi(x) = 1$ for $|x| \le \beta$, $\varphi(x) = 0$ for $|x| \ge 2\beta$ and $|\nabla\varphi| \le C/\beta$. We will find a solution of (4.1) near the set

$$X_\varepsilon := \left\{\varphi(\varepsilon x - x') w\left(x - \frac{x'}{\varepsilon}\right) : x' \in \mathcal{M}^\beta, w \in S_{V_0}\right\}$$

for sufficiently small $\varepsilon > 0$, where $\mathcal{M}^\beta := \{y \in \mathbf{R}^3\colon \inf_{z\in\mathcal{M}} |y-z| \le \beta\}$. Similarly, for $A \subset H_\varepsilon$, we use the notation

$$A^a := \big\{u \in H_\varepsilon\colon \inf_{v\in A} \|u-v\|_{H_\varepsilon} \le a\big\}.$$

For $U^* \in S_{V_0}$ arbitrary but fixed, we define $W_{\varepsilon,t}(x) := t^2\varphi(\varepsilon x)U^*(tx)$, we will show that J_ε possesses the Mountain-Pass geometry.

Denote $U_t^* := t^2U^*(tx)$, we have

$$\begin{aligned} I_{V_0}(U_t^*) &= \frac{1}{2}t^3\int_{\mathbf{R}^3}|\nabla U^*|^2 + \frac{1}{2}V_0 t\int_{\mathbf{R}^3}(U^*)^2 + \frac{1}{4}t^3\int_{\mathbf{R}^3}\phi_{U^*}(U^*)^2 \\ &\quad - \frac{1}{p}\lambda t^{2p-3}\int_{\mathbf{R}^3}(U^*)^p - \frac{1}{6}t^9\int_{\mathbf{R}^3}(U^*)^6 \to -\infty \quad \text{as } t\to\infty, \end{aligned}$$

then $\exists t_0 > 0$ such that $I_{V_0}(U_{t_0}^*) < -3$.

We can easily check that $Q_\varepsilon(W_{\varepsilon,t_0}) = 0$, then

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,t_0}) = I_\varepsilon(W_{\varepsilon,t_0}) &= \frac{1}{2}\int_{\mathbf{R}^3}|\nabla W_{\varepsilon,t_0}|^2 + \frac{1}{2}\int_{\mathbf{R}^3}V(\varepsilon x)W_{\varepsilon,t_0}^2 \\ &\quad + \frac{1}{4}\int_{\mathbf{R}^3}\phi_{W_{\varepsilon,t_0}}W_{\varepsilon,t_0}^2 - \frac{1}{p}\lambda\int_{\mathbf{R}^3}W_{\varepsilon,t_0}^p - \frac{1}{6}\int_{\mathbf{R}^3}W_{\varepsilon,t_0}^6 \\ &\overset{\tilde{x}=t_0x}{=} \frac{1}{2}t_0^3\int_{\mathbf{R}^3}\Big|\frac{\varepsilon}{t_0}\nabla\varphi\big(\frac{\varepsilon}{t_0}\tilde{x}\big)U^*(\tilde{x}) + \varphi\big(\frac{\varepsilon}{t_0}\tilde{x}\big)\nabla U^*(\tilde{x})\Big|^2 d\tilde{x} \\ &\quad + \frac{1}{2}t_0\int_{\mathbf{R}^3}V\big(\frac{\varepsilon}{t_0}\tilde{x}\big)\varphi^2\big(\frac{\varepsilon}{t_0}\tilde{x}\big)(U^*(\tilde{x}))^2 \\ &\quad + \frac{1}{4}t_0^3\int_{\mathbf{R}^3}\phi_{\varphi(\frac{\varepsilon}{t_0}\tilde{x})U^*(\tilde{x})}\varphi^2\big(\frac{\varepsilon}{t_0}\tilde{x}\big)(U^*(\tilde{x}))^2 \\ &\quad - \frac{1}{p}\lambda t_0^{2p-3}\int_{\mathbf{R}^3}\varphi^p\big(\frac{\varepsilon}{t_0}\tilde{x}\big)(U^*(\tilde{x}))^p - \frac{1}{6}t_0^9\int_{\mathbf{R}^3}\varphi^6\big(\frac{\varepsilon}{t_0}\tilde{x}\big)(U^*(\tilde{x}))^6 \\ &= I_{V_0}(U_{t_0}^*) + o(1) < -2 \quad \text{for } \varepsilon > 0 \text{ small}, \end{aligned} \tag{4.2}$$

where we have used the Dominated Convergence Theorem and Lemma 2.2 (i).

Using the Sobolev's Imbedding Theorem, we have

$$\begin{aligned} J_\varepsilon(u) \ge I_\varepsilon(u) &\ge \frac{1}{2}\|u\|_{H_\varepsilon}^2 - \frac{1}{p}\lambda\int_{\mathbf{R}^3}|u|^p - \frac{1}{6}\int_{\mathbf{R}^3}|u|^6 \\ &\ge \frac{1}{2}\|u\|_{H_\varepsilon}^2 - C\cdot\lambda\|u\|_{H_\varepsilon}^p - C\|u\|_{H_\varepsilon}^6 > 0 \end{aligned}$$

for $\|u\|_{H_\varepsilon}$ small since $p > 2$.

Hence, we can define the Mountain-Pass value of J_ε as follows,

$$c_\varepsilon := \inf_{\gamma\in\Gamma_\varepsilon}\max_{s\in[0,1]} J_\varepsilon(\gamma(s)),$$

where $\Gamma_\varepsilon := \{\gamma \in C([0,1], H_\varepsilon) \mid \gamma(0) = 0,\ \gamma(1) = W_{\varepsilon,t_0}\}$.

Lemma 4.1.

$$\overline{\lim_{\varepsilon\to 0}} c_\varepsilon \le c_{V_0}.$$

Proof. Denote $W_{\varepsilon,0}=\lim_{t\to 0}W_{\varepsilon,t}$ in H_ε sense, then $W_{\varepsilon,0}=0$. Thus, setting $\gamma(s):=W_{\varepsilon,st_0}$ $(0\le s\le 1)$, we have $\gamma(s)\in\Gamma_\varepsilon$, then

$$c_\varepsilon\le\max_{s\in[0,1]}J_\varepsilon(\gamma(s))=\max_{t\in[0,t_0]}J_\varepsilon(W_{\varepsilon,t})$$

and we just need to verify that

$$\overline{\lim_{\varepsilon\to 0}}\max_{t\in[0,t_0]}J_\varepsilon(W_{\varepsilon,t})\le c_{V_0}.$$

Indeed, similar to (4.2), we have

$$\begin{aligned}\max_{t\in[0,t_0]}J_\varepsilon(W_{\varepsilon,t})&=\max_{t\in[0,t_0]}I_{V_0}(U_t^*)+o(1)\le\max_{t\in[0,\infty)}I_{V_0}(U_t^*)+o(1)\\&=I_{V_0}(U^*)+o(1)=c_{V_0}+o(1).\end{aligned}$$

□

Lemma 4.2.

$$\underline{\lim_{\varepsilon\to 0}}\,c_\varepsilon\ge c_{V_0}.$$

Proof. Assuming the contrary that $\underline{\lim}_{\varepsilon\to 0}c_\varepsilon<c_{V_0}$, then, there exist $\delta_0>0$, $\varepsilon_n\to 0$ and $\gamma_n\in\Gamma_{\varepsilon_n}$ satisfying $J_{\varepsilon_n}(\gamma_n(s))<c_{V_0}-\delta_0$ for $s\in[0,1]$. We can fix an ε_n such that

$$\frac{1}{2}V_0\varepsilon_n(1+(1+c_{V_0})^{1/2})<\min\{\delta_0,1\}. \tag{4.3}$$

Since $I_{\varepsilon_n}(\gamma_n(0))=0$ and $I_{\varepsilon_n}(\gamma_n(1))\le J_{\varepsilon_n}(\gamma_n(1))=J_{\varepsilon_n}(W_{\varepsilon_n,t_0})<-2$, we can find an $s_n\in(0,1)$ such that $I_{\varepsilon_n}(\gamma_n(s))\ge -1$ for $s\in[0,s_n]$ and $I_{\varepsilon_n}(\gamma_n(s_n))=-1$. Then, for any $s\in[0,s_n]$,

$$Q_{\varepsilon_n}(\gamma_n(s))=J_{\varepsilon_n}(\gamma_n(s))-I_{\varepsilon_n}(\gamma_n(s))\le 1+c_{V_0}-\delta_0,$$

this implies that

$$\int_{\mathbf{R}^3\setminus(\Lambda/\varepsilon_n)}\gamma_n^2(s)\le\varepsilon_n(1+(1+c_{V_0})^{1/2})\quad\text{for } s\in[0,s_n].$$

Then, for $s\in[0,s_n]$,

$$\begin{aligned}I_{\varepsilon_n}(\gamma_n(s))&=I_{V_0}(\gamma_n(s))+\frac{1}{2}\int_{\mathbf{R}^3}(V(\varepsilon_nx)-V_0)\gamma_n^2(s)\\&\ge I_{V_0}(\gamma_n(s))+\frac{1}{2}\int_{\mathbf{R}^3\setminus(\Lambda/\varepsilon_n)}(V(\varepsilon_nx)-V_0)\gamma_n^2(s)\\&\ge I_{V_0}(\gamma_n(s))-\frac{1}{2}V_0\varepsilon_n(1+(1+c_{V_0})^{1/2}),\end{aligned}$$

then

$$\begin{aligned}I_{V_0}(\gamma_n(s_n))&\le I_{\varepsilon_n}(\gamma_n(s_n))+\frac{1}{2}V_0\varepsilon_n(1+(1+c_{V_0})^{1/2})\\&=-1+\frac{1}{2}V_0\varepsilon_n(1+(1+c_{V_0})^{1/2})<0\end{aligned}$$

and recalling (3.4), we have

$$\max_{s\in[0,s_n]}I_{V_0}(\gamma_n(s))\ge c_{V_0}.$$

Hence, we deduce that

$$\begin{aligned} c_{V_0} - \delta_0 &\geq \max_{s\in[0,1]} J_{\varepsilon_n}(\gamma_n(s)) \geq \max_{s\in[0,1]} I_{\varepsilon_n}(\gamma_n(s)) \geq \max_{s\in[0,s_n]} I_{\varepsilon_n}(\gamma_n(s)) \\ &\geq \max_{s\in[0,s_n]} I_{V_0}(\gamma_n(s)) - \frac{1}{2}V_0\varepsilon_n(1+(1+c_{V_0})^{1/2}), \end{aligned}$$

i.e., $0 < \delta_0 \leq \frac{1}{2}V_0\varepsilon_n(1+(1+c_{V_0})^{1/2})$, which contradicts to (4.3). □

Lemma 4.1 and Lemma 4.2 imply that

$$\lim_{\varepsilon\to 0}\big(\max_{s\in[0,1]} J_\varepsilon(\gamma_\varepsilon(s)) - c_\varepsilon\big) = 0,$$

where $\gamma_\varepsilon(s) = W_{\varepsilon,st_0}$ for $s \in [0,1]$. Denote

$$\tilde{c}_\varepsilon := \max_{s\in[0,1]} J_\varepsilon(\gamma_\varepsilon(s)),$$

we see that $c_\varepsilon \leq \tilde{c}_\varepsilon$ and $\lim_{\varepsilon\to 0} c_\varepsilon = \lim_{\varepsilon\to 0}\tilde{c}_\varepsilon = c_{V_0}$.

In order to state the next lemma, we need some notations. For each $R > 0$, we regard $H_0^1(B_R(0))$ as a subspace of H_ε. Namely, for any $u \in H_0^1(B_R(0))$, we extend u by defining $u(x) = 0$ for $|x| > R$, then $\|\cdot\|_{H_\varepsilon}$ is equivalent to the standard norm of $H_0^1(B_R(0))$ for each $R > 0$, $\varepsilon > 0$. Using $\|\cdot\|_{H_\varepsilon}$, for each $T \in (H_0^1(B_R(0)))^{-1}$, we define

$$\|T\|_{*,\varepsilon,R} := \sup\left\{Tu\colon u \in H_0^1(B_R(0)),\ \|u\|_{H_\varepsilon} \leq 1\right\}.$$

Note also that $\|\cdot\|_{*,\varepsilon,R}$ is equivalent to the standard norm of $(H_0^1(B_R(0)))^{-1}$.

We use the notation

$$J_\varepsilon^\alpha := \{u \in H_\varepsilon\colon J_\varepsilon(u) \leq \alpha\}$$

and fix a $R_0 > 0$ such that $B_{R_0}(0) \supset \Lambda$.

Inspired by [51], we have the following lemma and this lemma is a key for the proof of Theorem 1.1:

Lemma 4.3. (i) *There exists a $d_0 > 0$ such that for any $\{\varepsilon_i\}_{i=1}^\infty$, $\{R_{\varepsilon_i}\}$, $\{u_{\varepsilon_i}\}$ with*

$$\begin{cases} \lim\limits_{i\to\infty}\varepsilon_i = 0,\ R_{\varepsilon_i} \geq R_0/\varepsilon_i,\ u_{\varepsilon_i} \in X_{\varepsilon_i}^{d_0} \cap H_0^1(B_{R_{\varepsilon_i}}(0)), \\ \lim\limits_{i\to\infty} J_{\varepsilon_i}(u_{\varepsilon_i}) \leq c_{V_0} \text{ and } \lim\limits_{i\to\infty}\|J'_{\varepsilon_i}(u_{\varepsilon_i})\|_{*,\varepsilon_i,R_{\varepsilon_i}} = 0, \end{cases} \tag{4.4}$$

then there exists, up to a subsequence, $\{y_i\}_{i=1}^\infty \subset \mathbf{R}^3$, $x_0 \in \mathcal{M}$, $U \in S_{V_0}$ such that

$$\lim_{i\to\infty}|\varepsilon_i y_i - x_0| = 0 \text{ and } \lim_{i\to\infty}\|u_{\varepsilon_i} - \varphi(\varepsilon_i x - \varepsilon_i y_i)U(x-y_i)\|_{H_{\varepsilon_i}} = 0.$$

(ii) *If we drop $\{R_{\varepsilon_i}\}$ and replace (4.4) by*

$$\lim_{i\to\infty}\varepsilon_i = 0,\ u_{\varepsilon_i} \in X_{\varepsilon_i}^{d_0},\ \lim_{i\to\infty} J_{\varepsilon_i}(u_{\varepsilon_i}) \leq c_{V_0} \text{ and } \lim_{i\to\infty}\|J'_{\varepsilon_i}(u_{\varepsilon_i})\|_{(H_{\varepsilon_i})^{-1}} = 0, \tag{4.5}$$

then the same conclusion holds.

Proof. We only prove (i). The proof of (ii) is similar. For notational brevity, we write ε for ε_i, and still use ε after taking a subsequence. By the definition of $X_\varepsilon^{d_0}$, there exist $\{U_\varepsilon\} \subset S_{V_0}$ and $\{x_\varepsilon\} \subset \mathcal{M}^\beta$ such that

$$\Big\|u_\varepsilon - \varphi(\varepsilon x - x_\varepsilon)U_\varepsilon\Big(x - \frac{x_\varepsilon}{\varepsilon}\Big)\Big\|_{H_\varepsilon} \leq \frac{3}{2}d_0.$$

Since S_{V_0} and $\mathcal{M}^\beta$ are compact, there exist $U_0 \in S_{V_0}$, $x_0 \in \mathcal{M}^\beta$ such that $U_\varepsilon \to U_0$ in $H^1(\mathbf{R}^3)$ and $x_\varepsilon \to x_0$ as $\varepsilon \to 0$. Thus, for $\varepsilon > 0$ small,

$$\Big\| u_\varepsilon - \varphi(\varepsilon x - x_\varepsilon) U_0\Big(x - \frac{x_\varepsilon}{\varepsilon}\Big)\Big\|_{H_\varepsilon} \le 2d_0. \tag{4.6}$$

Step 1. We claim that

$$\lim_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^6 = 0, \tag{4.7}$$

where $A_\varepsilon = B_{3\beta/\varepsilon}(x_\varepsilon/\varepsilon)\backslash B_{\beta/2\varepsilon}(x_\varepsilon/\varepsilon)$. If the claim is true, by Lemma 2.5, we see that

$$\lim_{\varepsilon\to 0} \int_{B_\varepsilon} |u_\varepsilon|^6 = 0, \tag{4.8}$$

where $B_\varepsilon = B_{2\beta/\varepsilon}(x_\varepsilon/\varepsilon)\backslash B_{\beta/\varepsilon}(x_\varepsilon/\varepsilon)$. Indeed, since

$$\sup_{y\in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^6 \ge \sup_{y\in \mathbf{R}^3} \int_{B_1(y)} |u_\varepsilon \cdot \chi_{A_\varepsilon^1}|^6,$$

where $A_\varepsilon^1 = B_{(3\beta/\varepsilon)-1}(x_\varepsilon/\varepsilon)\backslash B_{(\beta/2\varepsilon)+1}(x_\varepsilon/\varepsilon)$, then

$$\lim_{\varepsilon\to 0} \sup_{y\in \mathbf{R}^3} \int_{B_1(y)} |u_\varepsilon \cdot \chi_{A_\varepsilon^1}|^6 = 0.$$

By Lemma 2.5, we have

$$\int_{\mathbf{R}^3} |u_\varepsilon \cdot \chi_{A_\varepsilon^1}|^6 \to 0 \quad \text{as } \varepsilon \to 0.$$

Since $A_\varepsilon^1 \supset B_\varepsilon$ for $\varepsilon > 0$ small, (4.8) holds.

Next, we will prove (4.7). Assuming the contrary, there exists $r > 0$ such that

$$\underline{\lim}_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B_1(y)} |u_\varepsilon|^6 = 2r > 0,$$

then there exists $y_\varepsilon \in A_\varepsilon$ such that for $\varepsilon > 0$ small, $\int_{B_1(y_\varepsilon)} |u_\varepsilon|^6 \ge r > 0$. Note also that $y_\varepsilon \in A_\varepsilon$, there exists $x^* \in \mathcal{M}^{4\beta} \subset \Lambda$ such that $\varepsilon y_\varepsilon \to x^*$ as $\varepsilon \to 0$. Set $v_\varepsilon(x) := u_\varepsilon(x + y_\varepsilon)$, then, for $\varepsilon > 0$ small,

$$\int_{B_1(0)} |v_\varepsilon|^6 \ge r > 0, \tag{4.9}$$

up to a subsequence, $v_\varepsilon \rightharpoonup v$ in $H^1(\mathbf{R}^3)$ and v satisfies

$$-\Delta v + V(x^*)v + \phi_v v = \lambda v^{p-1} + v^5 \quad \text{in } \mathbf{R}^3,\ v \ge 0.$$

Case 1. If $v \ne 0$, then

$$c_{V(x^*)} \le I_{V(x^*)}(v) - \frac{1}{3} G_{V(x^*)}(v) = \frac{1}{3} V(x^*) \int_{\mathbf{R}^3} v^2 + \frac{2p-6}{3p}\lambda \int_{\mathbf{R}^3} v^p + \frac{1}{3}\int_{\mathbf{R}^3} v^6,$$

we have

$$\begin{aligned} &\|V\|_{L^\infty(\bar\Lambda)} \int_{\mathbf{R}^3} v^2 + \frac{2p-6}{p}\lambda \int_{\mathbf{R}^3} v^p + \int_{\mathbf{R}^3} v^6 \\ &\ge V(x^*) \int_{\mathbf{R}^3} v^2 + \frac{2p-6}{p}\lambda \int_{\mathbf{R}^3} v^p + \int_{\mathbf{R}^3} v^6 \ge 3c_{V(x^*)} \ge 3c_{V_0}. \end{aligned}$$

Hence, for sufficiently large R,

$$\begin{aligned}
&\varliminf_{\varepsilon\to 0}\left[\|V\|_{L^\infty(\bar\Lambda)}\int_{B_R(y_\varepsilon)} u_\varepsilon^2+\frac{2p-6}{p}\lambda\int_{B_R(y_\varepsilon)} u_\varepsilon^p+\int_{B_R(y_\varepsilon)} u_\varepsilon^6\right]\\
&=\varliminf_{\varepsilon\to 0}\left[\|V\|_{L^\infty(\bar\Lambda)}\int_{B_R(0)} v_\varepsilon^2+\frac{2p-6}{p}\lambda\int_{B_R(0)} v_\varepsilon^p+\int_{B_R(0)} v_\varepsilon^6\right]\\
&\ge\left[\|V\|_{L^\infty(\bar\Lambda)}\int_{B_R(0)} v^2+\frac{2p-6}{p}\lambda\int_{B_R(0)} v^p+\int_{B_R(0)} v^6\right]\\
&\ge\frac12\left[\|V\|_{L^\infty(\bar\Lambda)}\int_{\mathbf{R}^3} v^2+\frac{2p-6}{p}\lambda\int_{\mathbf{R}^3} v^p+\int_{\mathbf{R}^3} v^6\right]\ge\frac32 c_{V_0}>0.
\end{aligned}$$

On the other hand, by the Sobolev's Imbedding Theorem and (4.6),

$$\begin{aligned}
&\|V\|_{L^\infty(\bar\Lambda)}\int_{B_R(y_\varepsilon)} u_\varepsilon^2+\frac{2p-6}{p}\lambda\int_{B_R(y_\varepsilon)} u_\varepsilon^p+\int_{B_R(y_\varepsilon)} u_\varepsilon^6\\
&\le Cd_0+C\int_{B_R(y_\varepsilon)}\left|\varphi(\varepsilon x-x_\varepsilon)U_0\left(x-\frac{x_\varepsilon}{\varepsilon}\right)\right|^2\\
&\quad+C\lambda\int_{B_R(y_\varepsilon)}\left|\varphi(\varepsilon x-x_\varepsilon)U_0\left(x-\frac{x_\varepsilon}{\varepsilon}\right)\right|^p\\
&\quad+C\int_{B_R(y_\varepsilon)}\left|\varphi(\varepsilon x-x_\varepsilon)U_0\big(x-\frac{x_\varepsilon}{\varepsilon}\big)\right|^6\\
&\le Cd_0+C\int_{B_R(y_\varepsilon-\frac{x_\varepsilon}{\varepsilon})}|U_0(x)|^2+C\lambda\int_{B_R(y_\varepsilon-\frac{x_\varepsilon}{\varepsilon})}|U_0(x)|^p\\
&\quad+C\int_{B_R(y_\varepsilon-\frac{x_\varepsilon}{\varepsilon})}|U_0(x)|^6=Cd_0+o(1),
\end{aligned}\tag{4.10}$$

where $o(1)\to 0$ as $\varepsilon\to 0$, and we have used the fact that $|y_\varepsilon-\frac{x_\varepsilon}{\varepsilon}|\ge\beta/2\varepsilon$. This leads to a contradiction if d_0 is small enough.

Case 2. If $v=0$, i.e., $v_\varepsilon\rightharpoonup 0$ in $H^1(\mathbf{R}^3)$, then $v_\varepsilon\to 0$ in $L^s_{\rm loc}(\mathbf{R}^3)$ for $s\in[1,6)$. Thus, by (4.9) and the Sobolev's Imbedding $H^1_{\rm loc}(\mathbf{R}^3)\hookrightarrow L^s_{\rm loc}(\mathbf{R}^3)$, $\exists C>0$ (independent of ε) such that, for $\varepsilon>0$ small,

$$\int_{B_1(0)}|\nabla v_\varepsilon|^2\ge Cr^{1/3}>0.\tag{4.11}$$

Now we claim that

$$\lim_{\varepsilon\to 0}\ \sup_{\varphi\in C_c^\infty(B_2(0)),\|\varphi\|_{H^1(\mathbf{R}^3)}=1}|\langle\rho_\varepsilon,\varphi\rangle|=0,\tag{4.12}$$

where $\rho_\varepsilon=\Delta v_\varepsilon+(v_\varepsilon^+)^5\in(H^1(\mathbf{R}^3))^{-1}$. It is easy to check that for $\varepsilon>0$ small, $\int_{\mathbf{R}^3}\chi_\varepsilon(x)u_\varepsilon(x)\varphi(x-y_\varepsilon)\equiv 0$ uniformly for any $\varphi\in C_c^\infty(B_2(0))$. Thus for any $\varphi\in C_c^\infty(B_2(0))$ with $\|\varphi\|_{H^1(\mathbf{R}^3)}=1$,

$$\begin{aligned}
\langle\rho_\varepsilon,\varphi\rangle=&-\langle J'(u_\varepsilon),\varphi(x-y_\varepsilon)\rangle+\int_{\mathbf{R}^3}V(\varepsilon x)u_\varepsilon(x)\varphi(x-y_\varepsilon)+\int_{\mathbf{R}^3}\phi_{u_\varepsilon}(x)u_\varepsilon(x)\varphi(x-y_\varepsilon)\\
&-\lambda\int_{\mathbf{R}^3}(u_\varepsilon^+)^{p-1}(x)\varphi(x-y_\varepsilon)=J_1+J_2+J_3+J_4.
\end{aligned}$$

In view of the facts that $\|J'_\varepsilon(u_\varepsilon)\|_{*,\varepsilon,R_\varepsilon} \to 0$, $\text{supp}\varphi \subset B_2(0)$, $\sup_{x\in B_2(0)} V(\varepsilon x+\varepsilon y_\varepsilon) \le C$ uniformly for all $\varepsilon > 0$ small, $v_\varepsilon \to 0$ in $L^s_{\text{loc}}(\mathbf{R}^3)$ for $s \in [1,6)$ and Lemma 2.1, we have

$$\begin{aligned}
|J_1| &\le \|J'_\varepsilon(u_\varepsilon)\|_{*,\varepsilon,R_\varepsilon}\|\varphi(x-y_\varepsilon)\|_{H_\varepsilon} = o(1)\|\varphi(x-y_\varepsilon)\|_{H_\varepsilon} \\
&\le o(1)\|\varphi(x-y_\varepsilon)\|_{H^1(\mathbf{R}^3)} \to 0, \\
|J_2| &\le \sup_{x\in B_2(0)} V(\varepsilon x+\varepsilon y_\varepsilon)\Big(\int_{B_2(0)} |v_\varepsilon|^2\Big)^{1/2}\Big(\int_{B_2(0)} \varphi^2\Big)^{1/2} \to 0, \\
|J_3| &= \Big|\int_{\mathbf{R}^3} \phi_{v_\varepsilon} v_\varepsilon \varphi\Big| \le \Big(\int_{\mathbf{R}^3} |\phi_{v_\varepsilon}|^6\Big)^{1/6}\Big(\int_{B_2(0)} |v_\varepsilon|^3\Big)^{1/3}\Big(\int_{B_2(0)} \varphi^2\Big)^{1/2} \\
&\le C\,\|v_\varepsilon\|^2_{L^{12/5}(\mathbf{R}^3)}\Big(\int_{B_2(0)} |v_\varepsilon|^3\Big)^{1/3}\Big(\int_{B_2(0)} \varphi^2\Big)^{1/2} \to 0
\end{aligned}$$

and

$$|J_4| = \lambda\Big|\int_{\mathbf{R}^3} (v_\varepsilon^+)^{p-1}\varphi\Big| \le \lambda\Big(\int_{B_2(0)} |v_\varepsilon|^p\Big)^{(p-1)/p}\Big(\int_{B_2(0)} |\varphi|^p\Big)^{1/p} \to 0$$

as $\varepsilon \to 0$ uniformly for all $\varphi \in C_c^\infty(B_2(0))$ with $\|\varphi\|_{H^1(\mathbf{R}^3)} = 1$, i.e., (4.12) holds.

In view of Lemma 2.6, we see from (4.9), (4.11) and (4.12) that, there exist $\tilde{y}_\varepsilon \in \mathbf{R}^3$ and $\sigma_\varepsilon > 0$ with $\tilde{y}_\varepsilon \to \tilde{y} \in \overline{B_1(0)}$, $\sigma_\varepsilon \to 0$ as $\varepsilon \to 0$ such that

$$w_\varepsilon(x) := \sigma_\varepsilon^{1/2} v_\varepsilon(\sigma_\varepsilon x + \tilde{y}_\varepsilon) \rightharpoonup w \quad \text{in } D^{1,2}(\mathbf{R}^3)$$

and $w \ge 0$ is a nontrivial solution of

$$-\Delta u = u^5, \quad u \in D^{1,2}(\mathbf{R}^3). \tag{4.13}$$

It is well known that

$$w(x) = \frac{3^{1/4}\delta^{1/2}}{(\delta^2 + |x-x_0|^2)^{1/2}}$$

for some $\delta > 0$, $x_0 \in \mathbf{R}^3$ and

$$\int_{\mathbf{R}^3} |\nabla w|^2 = \int_{\mathbf{R}^3} w^6 = S^{3/2}, \tag{4.14}$$

then $\exists R > 0$ such that

$$\int_{B_R(0)} w^6 \ge \frac{1}{2}\int_{\mathbf{R}^3} w^6 = \frac{1}{2}S^{3/2} > 0.$$

On the other hand,

$$\int_{B_R(0)} w^6 \le \varliminf_{\varepsilon\to 0}\int_{B_R(0)} w_\varepsilon^6 = \varliminf_{\varepsilon\to 0}\int_{B_{\sigma_\varepsilon R}(\tilde{y}_\varepsilon)} v_\varepsilon^6 = \varliminf_{\varepsilon\to 0}\int_{B_{\sigma_\varepsilon R}(\tilde{y}_\varepsilon + y_\varepsilon)} u_\varepsilon^6 \le \varliminf_{\varepsilon\to 0}\int_{B_2(y_\varepsilon)} u_\varepsilon^6, \tag{4.15}$$

where we have used the facts that $\sigma_\varepsilon \to 0$ and $\tilde{y}_\varepsilon \to \tilde{y} \in \overline{B_1(0)}$ as $\varepsilon \to 0$.

Similar to (4.10), we can check that (4.15) leads to a contradiction for $d_0 > 0$ small. Hence (4.7) holds.

For any $s \in (2,6)$, using the Interpolation Inequality for L^p norms and (4.8), we have

(4.16)
$$\int_{B_\varepsilon} |u_\varepsilon|^s \le \Big(\int_{B_\varepsilon} |u_\varepsilon|^2\Big)^{\frac{3}{2}-\frac{s}{4}} \Big(\int_{B_\varepsilon} |u_\varepsilon|^6\Big)^{\frac{s}{4}-\frac{1}{2}} \le C\Big(\int_{B_\varepsilon} |u_\varepsilon|^6\Big)^{\frac{s}{4}-\frac{1}{2}} \to 0 \ \text{ as } \varepsilon \to 0.$$

It follows that

$$\lim_{\varepsilon\to 0} \int_{B_\varepsilon} |u_\varepsilon|^s = 0 \ \text{ for all } s \in (2,6]. \tag{4.17}$$

Step 2. Let $u_{\varepsilon,1}(x) = \varphi(\varepsilon x - x_\varepsilon)u_\varepsilon(x)$, $u_{\varepsilon,2}(x) = (1-\varphi(\varepsilon x - x_\varepsilon))u_\varepsilon(x)$. By (4.17) and direct computations, we can check that

$$\begin{aligned}
\int_{\mathbf{R}^3} (u_\varepsilon^+)^s &= \int_{\mathbf{R}^3} ((u_{\varepsilon,1})^+)^s + \int_{\mathbf{R}^3} ((u_{\varepsilon,2})^+)^s + o(1), \quad s \in (2,6],\\
\int_{\mathbf{R}^3} |\nabla u_\varepsilon|^2 &\ge \int_{\mathbf{R}^3} |\nabla u_{\varepsilon,1}|^2 + \int_{\mathbf{R}^3} |\nabla u_{\varepsilon,2}|^2 + o(1),\\
\int_{\mathbf{R}^3} V(\varepsilon x)|u_\varepsilon|^2 &\ge \int_{\mathbf{R}^3} V(\varepsilon x)|u_{\varepsilon,1}|^2 + \int_{\mathbf{R}^3} V(\varepsilon x)|u_{\varepsilon,2}|^2,\\
\int_{\mathbf{R}^3} \phi_{u_\varepsilon}(u_\varepsilon)^2 &\ge \int_{\mathbf{R}^3} \phi_{u_{\varepsilon,1}}(u_{\varepsilon,1})^2 + \int_{\mathbf{R}^3} \phi_{u_{\varepsilon,2}}(u_{\varepsilon,2})^2,\\
Q_\varepsilon(u_{\varepsilon,1}) &= 0, \quad Q_\varepsilon(u_{\varepsilon,2}) = Q_\varepsilon(u_\varepsilon) \ge 0.
\end{aligned}$$

Hence we get,

$$J_\varepsilon(u_\varepsilon) \ge I_\varepsilon(u_{\varepsilon,1}) + I_\varepsilon(u_{\varepsilon,2}) + o(1). \tag{4.18}$$

Next, we claim that $\|u_{\varepsilon,2}\|_{H_\varepsilon} \to 0$ as $\varepsilon \to 0$. By (4.6), we have

$$\begin{aligned}
\|u_{\varepsilon,2}\|_{H_\varepsilon} &\le \Big\|u_{\varepsilon,1} - \varphi(\varepsilon x - x_\varepsilon)U_0\big(x - \frac{x_\varepsilon}{\varepsilon}\big)\Big\|_{H_\varepsilon} + 2d_0\\
&= \Big\|u_{\varepsilon,1} - \varphi(\varepsilon x - x_\varepsilon)U_0\big(x - \frac{x_\varepsilon}{\varepsilon}\big)\Big\|_{H_\varepsilon(B_{2\beta/\varepsilon}(x_\varepsilon/\varepsilon))} + 2d_0\\
&\le \|u_{\varepsilon,2}\|_{H_\varepsilon(B_{2\beta/\varepsilon}(x_\varepsilon/\varepsilon))} + 4d_0\\
&= \|u_{\varepsilon,2}\|_{H_\varepsilon(B_{2\beta/\varepsilon}(x_\varepsilon/\varepsilon)\setminus B_{\beta/\varepsilon}(x_\varepsilon/\varepsilon))} + 4d_0\\
&\le C\|u_\varepsilon\|_{H_\varepsilon(B_{2\beta/\varepsilon}(x_\varepsilon/\varepsilon)\setminus B_{\beta/\varepsilon}(x_\varepsilon/\varepsilon))} + 4d_0\\
&\le C\Big\|\varphi(\varepsilon x - x_\varepsilon)U_0\big(x - \frac{x_\varepsilon}{\varepsilon}\big)\Big\|_{H_\varepsilon(B_{2\beta/\varepsilon}(x_\varepsilon/\varepsilon)\setminus B_{\beta/\varepsilon}(x_\varepsilon/\varepsilon))} + Cd_0\\
&\le C\Big\|U_0\big(x - \frac{x_\varepsilon}{\varepsilon}\big)\Big\|_{H^1(B_{2\beta/\varepsilon}(x_\varepsilon/\varepsilon)\setminus B_{\beta/\varepsilon}(x_\varepsilon/\varepsilon))} + Cd_0\\
&\le C\|U_0\|_{H^1(B_{2\beta/\varepsilon}(0)\setminus B_{\beta/\varepsilon}(0))} + Cd_0 = Cd_0 + o(1),
\end{aligned} \tag{4.19}$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Hence we have $\overline{\lim\limits_{\varepsilon\to 0}} \|u_{\varepsilon,2}\|_{H_\varepsilon} \le Cd_0$.

By (4.17) and the facts that $\langle {J'}_\varepsilon(u_\varepsilon), u_{\varepsilon,2}\rangle \to 0$ as $\varepsilon \to 0$ and $\langle {Q'}_\varepsilon(u_\varepsilon), u_{\varepsilon,2}\rangle = \langle {Q'}_\varepsilon(u_{\varepsilon,2}), u_{\varepsilon,2}\rangle \geq 0$, we get

$$\begin{aligned}&\int_{\mathbf{R}^3} \nabla u_\varepsilon \cdot \nabla u_{\varepsilon,2} + \int_{\mathbf{R}^3} V(\varepsilon x) u_\varepsilon u_{\varepsilon,2} + \int_{\mathbf{R}^3} \phi_{u_\varepsilon} u_\varepsilon u_{\varepsilon,2} + \langle {Q'}_\varepsilon(u_{\varepsilon,2}), u_{\varepsilon,2}\rangle \\ &= \lambda \int_{\mathbf{R}^3} (u_\varepsilon^+)^{p-1} u_{\varepsilon,2} + \int_{\mathbf{R}^3} (u_\varepsilon^+)^5 u_{\varepsilon,2} + o(1),\end{aligned}$$

then

$$\begin{aligned}\|u_{\varepsilon,2}\|_{H_\varepsilon}^2 &\leq \lambda \int_{\mathbf{R}^3} |u_{\varepsilon,2}|^p + \int_{\mathbf{R}^3} |u_{\varepsilon,2}|^6 + o(1) \\ &\leq C\lambda \|u_{\varepsilon,2}\|_{H_\varepsilon}^p + C\|u_{\varepsilon,2}\|_{H_\varepsilon}^6 + o(1) \leq \frac{1}{2}\|u_{\varepsilon,2}\|_{H_\varepsilon}^2 + C\|u_{\varepsilon,2}\|_{H_\varepsilon}^6 + o(1),\end{aligned}$$

i.e., $\|u_{\varepsilon,2}\|_{H_\varepsilon}^2 \leq C\|u_{\varepsilon,2}\|_{H_\varepsilon}^6 + o(1)$.

Taking $d_0 > 0$ small, we have $\|u_{\varepsilon,2}\|_{H_\varepsilon} = o(1)$. From (4.18), it holds that

$$J_\varepsilon(u_\varepsilon) \geq I_\varepsilon(u_{\varepsilon,1}) + o(1). \tag{4.20}$$

Step 3. Let $\tilde{w}_\varepsilon(x) = u_{\varepsilon,1}\big(x + \frac{x_\varepsilon}{\varepsilon}\big) = \varphi(\varepsilon x) u_\varepsilon\big(x + \frac{x_\varepsilon}{\varepsilon}\big)$, up to a subsequence, $\exists \tilde{w} \in H^1(\mathbf{R}^3)$ such that

$$\tilde{w}_\varepsilon \rightharpoonup \tilde{w} \ \text{ in } H^1(\mathbf{R}^3) \tag{4.21}$$

and

$$\tilde{w}_\varepsilon \to \tilde{w} \ \text{ a.e. in } \mathbf{R}^3. \tag{4.22}$$

We claim that

$$\tilde{w}_\varepsilon \to \tilde{w} \ \text{ in } L^6(\mathbf{R}^3). \tag{4.23}$$

In view of Lemma 2.5, assuming the contrary that $\exists r > 0$ such that

$$\varliminf_{\varepsilon \to 0} \sup_{z \in \mathbf{R}^3} \int_{B_1(z)} |\tilde{w}_\varepsilon - \tilde{w}|^6 = 2r > 0.$$

Then, for $\varepsilon > 0$ small, there exists $z_\varepsilon \in \mathbf{R}^3$ such that

$$\int_{B_1(z_\varepsilon)} |\tilde{w}_\varepsilon - \tilde{w}|^6 \geq r > 0. \tag{4.24}$$

Case 1. $\{z_\varepsilon\}$ is bounded, i.e., $|z_\varepsilon| \leq \alpha$ for some $\alpha > 0$, then for $\varepsilon > 0$ small,

$$\int_{B_{\alpha+1}(0)} |\tilde{v}_\varepsilon|^6 \geq r > 0, \tag{4.25}$$

where $\tilde{v}_\varepsilon = \tilde{w}_\varepsilon - \tilde{w}$ and $\tilde{v}_\varepsilon \rightharpoonup 0$ in $H^1(\mathbf{R}^3)$. Similarly as in Step 1, $\exists C > 0$ (independent of ε), such that for $\varepsilon > 0$ small,

$$\int_{B_{\alpha+1}(0)} |\nabla \tilde{v}_\varepsilon|^2 \geq Cr^{1/3} > 0. \tag{4.26}$$

Now, we claim that

$$\lim_{\varepsilon \to 0} \sup_{\tilde{\varphi} \in C_c^\infty(B_{\alpha+2}(0)), \|\tilde{\varphi}\|_{H^1(\mathbf{R}^3)} = 1} |\langle \tilde{\rho}_\varepsilon, \tilde{\varphi}\rangle| = 0, \tag{4.27}$$

where $\tilde{\rho}_\varepsilon = \Delta\tilde{v}_\varepsilon + (\tilde{v}_\varepsilon^+)^5 \in (H^1(\mathbf{R}^3))^{-1}$. It is easy to check that for $\varepsilon > 0$ small, $\int_{\mathbf{R}^3} \chi_\varepsilon(x)u_\varepsilon(x)\tilde{\varphi}\Big(x - \frac{x_\varepsilon}{\varepsilon}\Big) \equiv 0$ uniformly for all $\tilde{\varphi} \in C_c^\infty(B_{\alpha+2}(0))$. Hence, we have

$$
\begin{aligned}
o(1) &= \left\langle J'_\varepsilon(u_\varepsilon), \tilde{\varphi}\left(x - \frac{x_\varepsilon}{\varepsilon}\right)\right\rangle = \int_{\mathbf{R}^3} \nabla u_\varepsilon\left(x + \frac{x_\varepsilon}{\varepsilon}\right)\cdot\nabla\tilde{\varphi} \\
&\quad + \int_{\mathbf{R}^3} V(\varepsilon x + x_\varepsilon)u_\varepsilon\left(x + \frac{x_\varepsilon}{\varepsilon}\right)\tilde{\varphi} + \int_{\mathbf{R}^3} \phi_{u_\varepsilon(x+\frac{x_\varepsilon}{\varepsilon})}u_\varepsilon\left(x + \frac{x_\varepsilon}{\varepsilon}\right)\tilde{\varphi} \\
&\quad - \lambda\int_{\mathbf{R}^3}\left(u_\varepsilon^+\left(x + \frac{x_\varepsilon}{\varepsilon}\right)\right)^{p-1}\tilde{\varphi} - \lambda\int_{\mathbf{R}^3}\left(u_\varepsilon^+\left(x + \frac{x_\varepsilon}{\varepsilon}\right)\right)^5\tilde{\varphi} \\
&= \int_{\mathbf{R}^3}\nabla\tilde{w}_\varepsilon\cdot\nabla\tilde{\varphi} + \int_{\mathbf{R}^3} V(\varepsilon x + x_\varepsilon)\tilde{w}_\varepsilon\tilde{\varphi} + \int_{\mathbf{R}^3}\phi_{\tilde{w}_\varepsilon}\tilde{w}_\varepsilon\tilde{\varphi} \\
&\quad - \lambda\int_{\mathbf{R}^3}(\tilde{w}_\varepsilon^+)^{p-1}\tilde{\varphi} - \lambda\int_{\mathbf{R}^3}(\tilde{w}_\varepsilon^+)^5\tilde{\varphi} + o(1),
\end{aligned} \tag{4.28}
$$

where we have used the fact that $\|u_{\varepsilon,2}\|_{H_\varepsilon} \to 0$ as $\varepsilon \to 0$ and note that $o(1) \to 0$ as $\varepsilon \to 0$ uniformly for all $\tilde{\varphi} \in C_c^\infty(B_{\alpha+2}(0))$ with $\|\tilde{\varphi}\|_{H^1(\mathbf{R}^3)} = 1$.

By (4.28) and the fact that $x_\varepsilon \to x_0 \in \mathcal{M}^\beta$ as $\varepsilon \to 0$, we see that $\tilde{w} \geq 0$ and satisfies

$$
-\Delta\tilde{w} + V(x_0)\tilde{w} + \phi_{\tilde{w}}\tilde{w} = \lambda\tilde{w}^{p-1} + \tilde{w}^5 \quad \text{in } \mathbf{R}^3. \tag{4.29}
$$

By Lemma 2.2(ii) and direct computations, we can check that the following Brezis–Lieb splitting properties hold, as $\varepsilon \to 0$,

$$
\begin{cases}
\int_{\mathbf{R}^3}(\tilde{w}_\varepsilon^+)^5\tilde{\varphi} - (\tilde{v}_\varepsilon^+)^5\tilde{\varphi} - (\tilde{w})^5\tilde{\varphi} \to 0, \\
\int_{\mathbf{R}^3}(\tilde{w}_\varepsilon^+)^{p-1}\tilde{\varphi} - (\tilde{v}_\varepsilon^+)^{p-1}\tilde{\varphi} - (\tilde{w})^{p-1}\tilde{\varphi} \to 0, \\
\int_{\mathbf{R}^3}\phi_{\tilde{w}_\varepsilon}\tilde{w}_\varepsilon\tilde{\varphi} - \phi_{\tilde{v}_\varepsilon}\tilde{v}_\varepsilon\tilde{\varphi} - \phi_{\tilde{w}}\tilde{w}\tilde{\varphi} \to 0, \\
\int_{\mathbf{R}^3}\nabla\tilde{w}_\varepsilon\cdot\nabla\tilde{\varphi} - \nabla\tilde{v}_\varepsilon\cdot\nabla\tilde{\varphi} - \nabla\tilde{w}\cdot\nabla\tilde{\varphi} = 0
\end{cases} \tag{4.30}
$$

and

$$
\int_{\mathbf{R}^3}(V(\varepsilon x + x_\varepsilon)\tilde{w}_\varepsilon - V(x_0)\tilde{w})\tilde{\varphi} \to 0 \tag{4.31}
$$

uniformly for all $\tilde{\varphi} \in C_c^\infty(B_{\alpha+2}(0))$ with $\|\tilde{\varphi}\|_{H^1(\mathbf{R}^3)} = 1$. From (4.28), (4.29), (4.30) and (4.31), we can verify (4.27).

By Lemma 2.6, we see from (4.25), (4.26) and (4.27) that, there exist $\tilde{z}_\varepsilon \in \mathbf{R}^3$ and $\delta_\varepsilon > 0$ such that $\tilde{z}_\varepsilon \to \tilde{z} \in B_{\alpha+1}(0)$, $\delta_\varepsilon \to 0$ and

$$
\hat{w}_\varepsilon(x) := \delta_\varepsilon^{1/2}\tilde{v}_\varepsilon(\delta_\varepsilon x + \tilde{z}_\varepsilon) \rightharpoonup \hat{w}(x) \quad \text{in } D^{1,2}(\mathbf{R}^3),
$$

where $\hat{w} \geq 0$ is a nontrivial solution of (4.13) and satisfies (4.14).

Since

$$
\begin{aligned}
\int_{\mathbf{R}^3}|\hat{w}|^6 &\leq \varliminf_{\varepsilon\to0}\int_{\mathbf{R}^3}|\hat{w}_\varepsilon|^6 = \varliminf_{\varepsilon\to0}\int_{\mathbf{R}^3}|\tilde{v}_\varepsilon|^6 \\
&= \varliminf_{\varepsilon\to0}\int_{\mathbf{R}^3}|\tilde{w}_\varepsilon|^6 - \int_{\mathbf{R}^3}|\tilde{w}|^6 \leq \varliminf_{\varepsilon\to0}\int_{\mathbf{R}^3}|u_\varepsilon|^6,
\end{aligned} \tag{4.32}
$$

then by (4.6) and the Sobolev's Imbedding Theorem, we get

$$\int_{\mathbf{R}^3} |u_\varepsilon|^6 \le Cd_0 + \int_{\mathbf{R}^3} \left|\varphi(\varepsilon x - x_\varepsilon) U_0\big(x - \frac{x_\varepsilon}{\varepsilon}\big)\right|^6 \le Cd_0 + \int_{\mathbf{R}^3} U_0^6,$$

and combining with (4.32), it holds that

$$\int_{\mathbf{R}^3} |\hat{w}|^6 \le Cd_0 + \int_{\mathbf{R}^3} U_0^6. \tag{4.33}$$

Thus

$$\begin{aligned} c_{V_0} = I_{V_0}(U_0) - \frac{1}{3} G_{V_0}(U_0) &= \frac{1}{3}\int_{\mathbf{R}^3} U_0^2 + \frac{2p-6}{3p}\lambda \int_{\mathbf{R}^3} U_0^p + \frac{1}{3}\int_{\mathbf{R}^3} U_0^6 \\ &\ge \frac{1}{3}\int_{\mathbf{R}^3} |\hat{w}|^6 - Cd_0 \ge \frac{1}{3} S^{\frac{3}{2}} - Cd_0, \end{aligned}$$

where we have used (4.14) and (4.33). Letting $d_0 \to 0$, we have

$$c_{V_0} \ge \frac{1}{3} S^{\frac{3}{2}},$$

which contradicts to Lemma 3.5.

Case 2. $\{z_\varepsilon\}$ is unbounded. Without loss of generality, $\lim_{\varepsilon\to 0} |z_\varepsilon| = \infty$. Then, by (4.24),

$$\varliminf_{\varepsilon\to 0} \int_{B_1(z_\varepsilon)} |\tilde{w}_\varepsilon|^6 \ge r > 0, \tag{4.34}$$

i.e.,

$$\varliminf_{\varepsilon\to 0} \int_{B_1(z_\varepsilon)} \left|\varphi(\varepsilon x) u_\varepsilon\left(x + \frac{x_\varepsilon}{\varepsilon}\right)\right|^6 \ge r > 0.$$

Since $\varphi(x) = 0$ for $|x| \ge 2\beta$, we see that $|z_\varepsilon| \le 3\beta/\varepsilon$ for $\varepsilon > 0$ small. If $|z_\varepsilon| \ge \beta/2\varepsilon$, then $z_\varepsilon \in B_{3\beta/\varepsilon}(0) \backslash B_{\beta/2\varepsilon}(0)$ and by Step 1, we get

$$\varliminf_{\varepsilon\to 0} \int_{B_1(z_\varepsilon)} |\tilde{w}_\varepsilon|^6 \le \varliminf_{\varepsilon\to 0} \sup_{z\in B_{3\beta/\varepsilon}(0)\backslash B_{\beta/2\varepsilon}(0)} \int_{B_1(z)} \left|u_\varepsilon\left(x + \frac{x_\varepsilon}{\varepsilon}\right)\right|^6 = \varliminf_{\varepsilon\to 0} \sup_{z\in A_\varepsilon} \int_{B_1(z)} |u_\varepsilon|^6 = 0,$$

which contradicts to (4.34). Thus $|z_\varepsilon| \le \beta/2\varepsilon$ for $\varepsilon > 0$ small. Assume that $\varepsilon z_\varepsilon \to z_0 \in \overline{B_{\beta/2}(0)}$ and $\bar{w}_\varepsilon(x) := \tilde{w}_\varepsilon(x + z_\varepsilon) \rightharpoonup \bar{w}(x)$ in $H^1(\mathbf{R}^3)$. If $\bar{w} \ne 0$, we see that $\bar{w}$ satisfies

$$-\Delta \bar{w} + V(x_0 + z_0)\bar{w} + \phi_{\bar{w}}\bar{w} = \lambda \bar{w}^{p-1} + \bar{w}^5 \ \text{ in } \mathbf{R}^3,\ \bar{w} \ge 0.$$

Similarly as in Step 1 (4.10), we get a contradiction if $d_0 > 0$ is small enough. Thus $\bar{w} \equiv 0$, i.e.,

$$\bar{w}_\varepsilon \rightharpoonup 0 \ \text{ in } H^1(\mathbf{R}^3).$$

By (4.34), we have

$$\varliminf_{\varepsilon\to 0} \int_{B_1(0)} |\bar{w}_\varepsilon|^6 \ge r > 0 \tag{4.35}$$

and similar as in Step 1, we can check that $\exists C > 0$ (independent of ε) such that for $\varepsilon > 0$ small,

$$\int_{B_1(0)} |\nabla \bar{w}_\varepsilon|^2 \ge Cr^{1/3} > 0 \tag{4.36}$$

and

$$\lim_{\varepsilon\to 0}\sup_{\bar\varphi\in C_c^\infty(B_2(0)),\|\bar\varphi\|_{H^1(\mathbf{R}^3)}=1}|\langle\bar\rho_\varepsilon,\bar\varphi\rangle|=0, \tag{4.37}$$

where $\bar\rho_\varepsilon=\Delta\bar w_\varepsilon+(\bar w_\varepsilon^+)^5\in(H^1(\mathbf{R}^3))^{-1}$. By Lemma 2.6 again, we see from (4.35), (4.36) and (4.37) that $\exists\tilde x_\varepsilon\in\mathbf{R}^3$ and $\gamma_\varepsilon>0$ such that $\tilde x_\varepsilon\to\tilde x\in\overline{B_1(0)}$, $\gamma_\varepsilon\to 0$ as $\varepsilon\to 0$ and

$$w_\varepsilon^*(x):=\gamma_\varepsilon^{1/2}\bar w_\varepsilon(\gamma_\varepsilon x+\tilde x_\varepsilon)\rightharpoonup w^*(x)\ \text{ in } D^{1,2}(\mathbf{R}^3),$$

where $w^*\geq 0$ is a nontrivial solution of (4.13) and satisfies (4.14). Thus, $\exists R>0$ such that

$$\int_{B_R(0)}|w^*|^6\geq\frac12\int_{\mathbf{R}^3}|w^*|^6=\frac12 S^{\frac32}>0.$$

On the other hand,

$$\begin{aligned}\frac12 S^{\frac32}&\leq\int_{B_R(0)}|w^*|^6\leq\varliminf_{\varepsilon\to 0}\int_{B_R(0)}|w_\varepsilon^*|^6=\varliminf_{\varepsilon\to 0}\int_{B_{\gamma_\varepsilon R}(\tilde x_\varepsilon)}|\bar w_\varepsilon|^6\\&\leq\varliminf_{\varepsilon\to 0}\int_{B_{\gamma_\varepsilon R}(\tilde x_\varepsilon+z_\varepsilon+\frac{x_\varepsilon}{\varepsilon})}|u_\varepsilon|^6\leq\varliminf_{\varepsilon\to 0}\int_{B_2(z_\varepsilon+\frac{x_\varepsilon}{\varepsilon})}|u_\varepsilon|^6,\end{aligned}$$

which contradicts to (4.6) for $d_0>0$ small. Therefore

$$\lim_{\varepsilon\to 0}\sup_{z\in\mathbf{R}^3}\int_{B_1(z)}|\tilde w_\varepsilon-\tilde w|^6=0.$$

By Lemma 2.5, (4.23) holds. Similar to (4.16), using the Interpolation Inequality for L^p norms, we have

$$\tilde w_\varepsilon\to\tilde w\ \text{ in } L^s(\mathbf{R}^3),\quad s\in(2,6]. \tag{4.38}$$

In view of (4.20) and recall that $\tilde w_\varepsilon(x)=u_{\varepsilon,1}\big(x+\frac{x_\varepsilon}{\varepsilon}\big)$, we have

$$\begin{aligned}&\frac12\int_{\mathbf{R}^3}|\nabla\tilde w_\varepsilon|^2+\frac12\int_{\mathbf{R}^3}V(\varepsilon x+x_\varepsilon)\tilde w_\varepsilon^2+\frac14\int_{\mathbf{R}^3}\phi_{\tilde w_\varepsilon}\tilde w_\varepsilon^2-\frac1p\lambda\int_{\mathbf{R}^3}(\tilde w_\varepsilon^+)^p-\frac16\int_{\mathbf{R}^3}(\tilde w_\varepsilon^+)^6\\&\leq c_{V_0}+o(1).\end{aligned}$$

By Lemma 2.1(iii), (4.21), (4.22) and (4.38), we get

$$\frac12\int_{\mathbf{R}^3}|\nabla\tilde w|^2+\frac12\int_{\mathbf{R}^3}V(x_0)\tilde w^2+\frac14\int_{\mathbf{R}^3}\phi_{\tilde w}\tilde w^2-\frac1p\lambda\int_{\mathbf{R}^3}(\tilde w^+)^p-\frac16\int_{\mathbf{R}^3}(\tilde w^+)^6\leq c_{V_0},$$

i.e.,

$$I_{V(x_0)}(\tilde w)\leq c_{V_0}. \tag{4.39}$$

Since $\langle J'_\varepsilon(u_\varepsilon),u_{\varepsilon,1}\rangle\to 0$, $\|u_{\varepsilon,2}\|_{H_\varepsilon}\to 0$ as $\varepsilon\to 0$ and $\langle Q'_\varepsilon(u_\varepsilon),u_{\varepsilon,1}\rangle\equiv 0$ and together with the fact that $\tilde w_\varepsilon(x)=u_{\varepsilon,1}\big(x+\frac{x_\varepsilon}{\varepsilon}\big)$, we get

$$\int_{\mathbf{R}^3}|\nabla\tilde w_\varepsilon|^2+\int_{\mathbf{R}^3}V(\varepsilon x+x_\varepsilon)\tilde w_\varepsilon^2+\int_{\mathbf{R}^3}\phi_{\tilde w_\varepsilon}\tilde w_\varepsilon^2=\lambda\int_{\mathbf{R}^3}(\tilde w_\varepsilon^+)^p+\int_{\mathbf{R}^3}(\tilde w_\varepsilon^+)^6+o(1),$$

then by (4.29), we have

$$\begin{aligned}&\int_{\mathbf{R}^3}|\nabla\tilde{w}|^2+\int_{\mathbf{R}^3}V(x_0)\tilde{w}^2+\int_{\mathbf{R}^3}\phi_{\tilde{w}}\tilde{w}^2\\&\le\varliminf_{\varepsilon\to0}\int_{\mathbf{R}^3}|\nabla\tilde{w}_\varepsilon|^2+\int_{\mathbf{R}^3}V(\varepsilon x+x_\varepsilon)\tilde{w}_\varepsilon^2+\int_{\mathbf{R}^3}\phi_{\tilde{w}_\varepsilon}\tilde{w}_\varepsilon^2\\&=\varliminf_{\varepsilon\to0}\lambda\int_{\mathbf{R}^3}(\tilde{w}_\varepsilon^+)^p+\int_{\mathbf{R}^3}(\tilde{w}_\varepsilon^+)^6=\lambda\int_{\mathbf{R}^3}(\tilde{w}^+)^p+\int_{\mathbf{R}^3}(\tilde{w}^+)^6\\&=\int_{\mathbf{R}^3}|\nabla\tilde{w}|^2+\int_{\mathbf{R}^3}V(x_0)\tilde{w}^2+\int_{\mathbf{R}^3}\phi_{\tilde{w}}\tilde{w}^2,\end{aligned}$$

hence as $\varepsilon\to0$,

$$\int_{\mathbf{R}^3}V(\varepsilon x+x_\varepsilon)\tilde{w}_\varepsilon^2\to\int_{\mathbf{R}^3}V(x_0)\tilde{w}^2 \tag{4.40}$$

and

$$\int_{\mathbf{R}^3}|\nabla\tilde{w}_\varepsilon|^2\to\int_{\mathbf{R}^3}|\nabla\tilde{w}|^2. \tag{4.41}$$

In view of (4.6), (4.38) and the fact that $\|u_{\varepsilon,2}\|_{H_\varepsilon}\to0$ as $\varepsilon\to0$, taking $d_0>0$ small, we can check that $\tilde{w}\neq0$. By (4.29), we have

$$I_{V(x_0)}(\tilde{w})\ge c_{V(x_0)}. \tag{4.42}$$

Since $x_0\in\mathcal{M}^\beta\subset\Lambda$, (4.39) and (4.42) imply that $V(x_0)=V_0$ and $x_0\in\mathcal{M}$. At this point, it is clear that $\exists U\in S_{V_0}$ and $z_0\in\mathbf{R}^3$ such that $\tilde{w}(x)=U(x-z_0)$. Since

$$\int_{\mathbf{R}^3}V(x_0)\tilde{w}_\varepsilon^2\le\int_{\mathbf{R}^3}V(\varepsilon x+x_\varepsilon)\tilde{w}_\varepsilon^2,$$

by (4.40) and (4.41), we have

$$\tilde{w}_\varepsilon\to\tilde{w}\ \text{ in } H^1(\mathbf{R}^3),$$

which implies that

$$\Big\|u_\varepsilon-\varphi(\varepsilon x-(x_\varepsilon+\varepsilon z_0))U\Big(x-\big(\frac{x_\varepsilon}{\varepsilon}+z_0\big)\Big)\Big\|_{H_\varepsilon}\to0\ \text{ as } \varepsilon\to0.$$

And we recall that $x_\varepsilon\to x_0\in\mathcal{M}$ as $\varepsilon\to0$, this completes the proof. □

Lemma 4.4. *Let d_0 be the number given in Lemma 4.3, then for any $d\in(0,d_0)$, there exist $\varepsilon_d>0$, $\rho_d>0$ and $\omega_d>0$ such that*

$$\|J'_\varepsilon(u)\|_{*,\varepsilon,R}\ge\omega_d>0$$

for all $u\in J_\varepsilon^{c_{V_0}+\rho_d}\cap(X_\varepsilon^{d_0}\backslash X_\varepsilon^d)\cap H_0^1(B_R(0))$ with $\varepsilon\in(0,\varepsilon_d)$ and $R\ge R_0/\varepsilon$.

Proof. If the lemma does not hold, there exist $d\in(0,d_0)$, $\{\varepsilon_i\}$, $\{\rho_i\}$ with $\varepsilon_i,\ \rho_i\to0$, $R_{\varepsilon_i}\ge R_0/\varepsilon_i$ and $u_i\in J_{\varepsilon_i}^{c_{V_0}+\rho_i}\cap(X_{\varepsilon_i}^{d_0}\backslash X_{\varepsilon_i}^d)\cap H_0^1(B_{R_{\varepsilon_i}}(0))$ such that

$$\|J'_{\varepsilon_i}(u_i)\|_{*,\varepsilon_i,R_{\varepsilon_i}}\to0\ \text{ as } i\to\infty.$$

By Lemma 4.3(i), we can find $\{y_i\}_{i=1}^\infty\subset\mathbf{R}^3$, $x_0\in\mathcal{M}$, $U\in S_{V_0}$ such that

$$\lim_{i\to\infty}|\varepsilon_iy_i-x_0|=0 \text{ and } \lim_{i\to\infty}\|u_i-\varphi(\varepsilon_ix-\varepsilon_iy_i)U(x-y_i)\|_{H_{\varepsilon_i}}=0,$$

which implies that $u_i\in X_{\varepsilon_i}^d$ for sufficiently large i. This contradicts that $u_i\notin X_{\varepsilon_i}^d$. □

Lemma 4.5. *There exists $T_0 > 0$ with the following property: for any $\delta > 0$ small, there exist $\alpha_\delta > 0$ and $\varepsilon_\delta > 0$ such that if $J_\varepsilon(\gamma_\varepsilon(s)) \ge c_{V_0} - \alpha_\delta$ and $\varepsilon \in (0, \varepsilon_\delta)$, then $\gamma_\varepsilon(s) \in X_\varepsilon^{T_0\delta}$, where $\gamma_\varepsilon(s) := W_{\varepsilon,st_0}$, $s \in [0,1]$.*

Proof. First, we may find a $T_0 > 0$ such that for any $u \in H^1(\mathbf{R}^3)$,

$$\|\varphi(\varepsilon x)u(x)\|_{H_\varepsilon} \le T_0\|u(x)\|_{H^1(\mathbf{R}^3)}. \tag{4.43}$$

Define

$$\alpha_\delta = \frac{1}{4}\min\Big\{c_{V_0} - I_{V_0}(s^2t_0^2U^*(st_0x))\colon s \in [0,1], \big\|s^2t_0^2U^*(st_0x) - U^*(x)\big\|_{H^1(\mathbf{R}^3)} \ge \delta\Big\} > 0,$$

we have

$$I_{V_0}(s^2t_0^2U^*(st_0x)) \ge c_{V_0} - 2\alpha_\delta \ \text{ implies } \ \big\|s^2t_0^2U^*(st_0x) - U^*(x)\big\|_{H^1(\mathbf{R}^3)} \le \delta. \tag{4.44}$$

Similar as in the proof of (4.2), we have

$$\max_{0\le s\le 1}|J_\varepsilon(\gamma_\varepsilon(s)) - I_{V_0}(s^2t_0^2U^*(st_0x))| \le \alpha_\delta \tag{4.45}$$

for all $\varepsilon \in (0, \varepsilon_\delta)$. Thus if $\varepsilon \in (0, \varepsilon_\delta)$ and $J_\varepsilon(\gamma_\varepsilon(s)) \ge c_{V_0} - \alpha_\delta$, by (4.44) and (4.45), we have $\|s^2t_0^2U^*(st_0x) - U^*(x)\|_{H^1(\mathbf{R}^3)} \le \delta$, then by (4.43), we have

$$\begin{aligned}\|W_{\varepsilon,st_0}(x) - \varphi(\varepsilon x)U^*(x)\|_{H_\varepsilon} &= \big\|\varphi(\varepsilon x)s^2t_0^2U^*(st_0x) - \varphi(\varepsilon x)U^*(x)\big\|_{H_\varepsilon}\\ &\le T_0\big\|s^2t_0^2U^*(st_0x) - U^*(x)\big\|_{H^1(\mathbf{R}^3)} \le T_0\delta.\end{aligned}$$

Recall that $0 \in \mathcal{M}$, we have $\gamma_\varepsilon(s) := W_{\varepsilon,st_0} \in X_\varepsilon^{T_0\delta}$. □

For each $R > R_0/\varepsilon$, we have

$$\gamma_\varepsilon(s) := W_{\varepsilon,st_0} \in H_0^1(B_R(0)) \ \text{ for each } s \in [0,1],\ X_\varepsilon \subset H_0^1(B_R(0)).$$

Define

$$c_{\varepsilon,R} := \inf_{\gamma\in\Gamma_{\varepsilon,R}}\max_{0\le t\le 1} J_\varepsilon(\gamma(t)),$$

where

$$\Gamma_{\varepsilon,R} := \big\{\gamma \in C([0,1],\ H_0^1(B_R(0)))\colon \gamma(0) = 0,\ \gamma(1) = \gamma_\varepsilon(1) = W_{\varepsilon,t_0}\big\}.$$

Remark that $\gamma_\varepsilon(s) := W_{\varepsilon,st_0} \in \Gamma_{\varepsilon,R}$, $c_\varepsilon \le c_{\varepsilon,R} \le \tilde{c}_\varepsilon$ and $J_\varepsilon^{\tilde{c}_\varepsilon} \cap X_\varepsilon \cap H_0^1(B_R(0)) \ne \emptyset$.

Choosing $\delta_1 > 0$ such that $T_0\delta_1 < d_0/4$ in Lemma 4.5 and fixing $d = d_0/4 := d_1$ in Lemma 4.4. The next Lemma comes from [22], for reader's convenience, we give a detailed proof.

Lemma 4.6. *$\exists\bar{\varepsilon} > 0$ such that for each $\varepsilon \in (0, \bar{\varepsilon}]$ and $R > R_0/\varepsilon$, there exists a sequence $\{v_{n,\varepsilon}^R\}_{n=1}^\infty \subset J_\varepsilon^{\tilde{c}_\varepsilon+\varepsilon} \cap X_\varepsilon^{d_0} \cap H_0^1(B_R(0))$ such that $J'_\varepsilon(v_{n,\varepsilon}^R) \to 0$ in $(H_0^1(B_R(0)))^{-1}$ as $n \to \infty$.*

Proof. Since $J_\varepsilon(\gamma_\varepsilon(1)) \to I_{V_0}(U_{t_0}^*) < -3$ as $\varepsilon \to 0$, we choose $0 < \bar{\varepsilon} \le \min\{\varepsilon_{d_1}, \varepsilon_{\delta_1}\}$ such that for each $\varepsilon \in (0, \bar{\varepsilon}]$,

$$\tilde{c}_\varepsilon + \varepsilon \le c_{V_0} + \rho_{d_1},\ \tilde{c}_\varepsilon - c_\varepsilon < \frac{1}{8}\omega_{d_1}d_0,\ c_{V_0} - \frac{1}{2}\alpha_{\delta_1} < c_\varepsilon,\ J_\varepsilon(\gamma_\varepsilon(1)) < 0. \tag{4.46}$$

Assuming the contrary that for some $\varepsilon^* \in (0, \bar{\varepsilon}]$ and $R^* > R_0/\varepsilon^*$, there exists a $\gamma(\varepsilon^*, R^*) > 0$ such that

$$\|J'_{\varepsilon^*}(u)\|_{*,\varepsilon^*,R^*} \ge \gamma(\varepsilon^*, R^*) > 0 \tag{4.47}$$

for all $u \in J_{\varepsilon^*}^{\tilde{c}_{\varepsilon^*}+\varepsilon^*} \cap X_{\varepsilon^*}^{d_0} \cap H_0^1(B_{R^*}(0))$.

Let Y be a pseudo-gradient vector field for J'_{ε^*} in $H_0^1(B_{R^*}(0))$, i.e., $Y\colon J_{\varepsilon^*}^{\tilde{c}_{\varepsilon^*}+\varepsilon^*} \cap X_{\varepsilon^*}^{d_0} \cap H_0^1(B_{R^*}(0)) \to H_0^1(B_{R^*}(0))$ is a locally Lipschitz continuous vector field such that for every $u \in J_{\varepsilon^*}^{\tilde{c}_{\varepsilon^*}+\varepsilon^*} \cap X_{\varepsilon^*}^{d_0} \cap H_0^1(B_{R^*}(0))$,

$$\|Y(u)\|_{H_{\varepsilon^*}} \le 2\|J'_{\varepsilon^*}(u)\|_{*,\varepsilon^*,R^*}, \tag{4.48}$$

$$\langle J'_{\varepsilon^*}(u), Y(u)\rangle \ge \|J'_{\varepsilon^*}(u)\|^2_{*,\varepsilon^*,R^*}. \tag{4.49}$$

Let ψ_1, ψ_2 be locally Lipschitz continuous functions in $H_0^1(B_{R^*}(0))$ such that $0 \le \psi_1, \psi_2 \le 1$ and

$$\psi_1(u) = \begin{cases} 1 & \text{if } c_{V_0} - \alpha_{\delta_1} \le J_{\varepsilon^*}(u) \le \tilde{c}_{\varepsilon^*}, \\ 0 & \text{if } J_{\varepsilon^*}(u) \le c_{V_0} - 2\alpha_{\delta_1} \text{ or } \tilde{c}_{\varepsilon^*} + \varepsilon^* \le J_{\varepsilon^*}(u), \end{cases}$$

$$\psi_2(u) = \begin{cases} 1 & \text{if } \|u - X_{\varepsilon^*}\|_{H_{\varepsilon^*}} \le \frac{3}{4}d_0, \\ 0 & \text{if } \|u - X_{\varepsilon^*}\|_{H_{\varepsilon^*}} \ge d_0. \end{cases}$$

Consider the following ordinary differential equations:

$$\begin{cases} \frac{d}{ds}\eta(s,u) = -\dfrac{Y(\eta(s,u))}{\|Y(\eta(s,u))\|_{H_{\varepsilon^*}}}\psi_1(\eta(s,u))\psi_2(\eta(s,u)), \\ \eta(0,u) = u. \end{cases}$$

By (4.48) and (4.49), we have

$$\begin{aligned} \frac{d}{ds}J_{\varepsilon^*}(\eta(s,u)) &= \Big\langle J'_{\varepsilon^*}(\eta(s,u)), \frac{d}{ds}\eta(s,u)\Big\rangle \\ &= \Big\langle J'_{\varepsilon^*}(\eta(s,u)), -\frac{Y(\eta(s,u))}{\|Y(\eta(s,u))\|_{H_{\varepsilon^*}}}\psi_1(\eta(s,u))\psi_2(\eta(s,u))\Big\rangle \\ &\le -\frac{\psi_1(\eta(s,u))\psi_2(\eta(s,u))}{\|Y(\eta(s,u))\|_{H_{\varepsilon^*}}}\|J'_{\varepsilon^*}(\eta(s,u))\|^2_{*,\varepsilon^*,R^*} \\ &\le -\frac{1}{2}\psi_1(\eta(s,u))\psi_2(\eta(s,u))\|J'_{\varepsilon^*}(\eta(s,u))\|_{*,\varepsilon^*,R^*} \end{aligned}$$

and combining with (4.46), (4.47) and Lemma 4.4, it is standard to show that $\eta \in C([0,\infty) \times H_0^1(B_{R^*}(0)), H_0^1(B_{R^*}(0)))$ and satisfies

(i) $\frac{d}{ds}J_{\varepsilon^*}(\eta(s,u)) \le 0$ for each $s \in [0,\infty)$ and $u \in H_0^1(B_{R^*}(0))$;

(ii) $\frac{d}{ds}J_{\varepsilon^*}(\eta(s,u)) \le -\omega_{d_1}/2$ if $\eta(s,u) \in \overline{J_{\varepsilon^*}^{\tilde{c}_{\varepsilon^*}} \backslash J_{\varepsilon^*}^{c_{V_0}-\alpha_{\delta_1}}} \cap \overline{X_{\varepsilon^*}^{3d_0/4} \backslash X_{\varepsilon^*}^{d_0/4}}$;

(iii) $\frac{d}{ds}J_{\varepsilon^*}(\eta(s,u)) \le -\gamma(\varepsilon^*, R^*)/2$ if $\eta(s,u) \in \overline{J_{\varepsilon^*}^{\tilde{c}_{\varepsilon^*}} \backslash J_{\varepsilon^*}^{c_{V_0}-\alpha_{\delta_1}}} \cap X_{\varepsilon^*}^{3d_0/4}$;

(iv) $\eta(s,u) = u$ if $J_{\varepsilon^*}(u) \le 0$.

Set $s_1 := \omega_{d_1} d_0 (\gamma(\varepsilon^*, R^*))^{-1}$ and $\xi(t) := \eta(s_1, \gamma_{\varepsilon^*}(t))$, by (4.46) and (iv), we have $\xi(t) \in \Gamma_{\varepsilon^*,R^*}$. In view of (4.46) and (i), we may find a $t_1 \in (0,1)$ such that

$$c_{V_0} - \alpha_{\delta_1}/2 \le c_{\varepsilon^*} \le c_{\varepsilon^*,R^*} \le J_{\varepsilon^*}(\xi(t_1)) \le J_{\varepsilon^*}(\gamma_{\varepsilon^*}(t_1)) \le \tilde{c}_{\varepsilon^*}. \tag{4.50}$$

Hence, Lemma 4.5 yields

$$\gamma_{\varepsilon^*}(t_1) \in X_{\varepsilon^*}^{d_0/4} \cap \overline{J_{\varepsilon^*}^{\tilde{c}_{\varepsilon^*}} \backslash J_{\varepsilon^*}^{c_{V_0}-\alpha_{\delta_1}}}.$$

Now, we have two cases:

Case 1. $\eta(s, \gamma_{\varepsilon^*}(t_1)) \notin X_{\varepsilon^*}^{3d_0/4}$ for some $s \in [0, s_1]$;

Case 2. $\eta(s,\gamma_{\varepsilon^*}(t_1)) \in X_{\varepsilon^*}^{3d_0/4}$ for all $s\in[0,s_1]$.

In Case 1, denote

$$s_2 := \inf\{s\in[0,s_1] | \eta(s,\gamma_{\varepsilon^*}(t_1)) \notin X_{\varepsilon^*}^{3d_0/4}\}$$

and

$$s_3 := \sup\{s\in[0,s_2] | \eta(s,\gamma_{\varepsilon^*}(t_1)) \in X_{\varepsilon^*}^{d_0/4}\},$$

then

$$s_2-s_3\ge \frac{1}{2}d_0,\ \ \eta(s,\gamma_{\varepsilon^*}(t_1))\in \overline{X_{\varepsilon^*}^{3d_0/4}\backslash X_{\varepsilon^*}^{d_0/4}}\ \ \text{for every } s\in[s_3,s_2].$$

By (i) and (4.50), for all $s\in[0,s_1]$,

$$\begin{aligned} c_{V_0}-\frac{1}{2}\alpha_{\delta_1} &\le J_{\varepsilon^*}(\eta(s_1,\gamma_{\varepsilon^*}(t_1)))\le J_{\varepsilon^*}(\eta(s,\gamma_{\varepsilon^*}(t_1)))\\ &\le J_{\varepsilon^*}(\eta(0,\gamma_{\varepsilon^*}(t_1))) = J_{\varepsilon^*}(\gamma_{\varepsilon^*}(t_1))\le \tilde c_{\varepsilon^*},\end{aligned}$$

then by (4.46) and (ii), we obtain

$$\begin{aligned} J_{\varepsilon^*}(\xi(t_1)) &= J_{\varepsilon^*}(\gamma_{\varepsilon^*}(t_1)) + \int_0^{s_1}\frac{d}{ds}J_{\varepsilon^*}(\eta(s,\gamma_{\varepsilon^*}(t_1)))\,ds\\ &\le \tilde c_{\varepsilon^*} + \int_{s_3}^{s_2}\frac{d}{ds}J_{\varepsilon^*}(\eta(s,\gamma_{\varepsilon^*}(t_1)))\,ds \le \tilde c_{\varepsilon^*}-\frac{1}{4}\omega_{d_1}d_0 < c_{\varepsilon^*},\end{aligned}$$

which contradicts to (4.50).

In Case 2, by (4.46), (iii) and the definition of s_1, we have

$$J_{\varepsilon^*}(\xi(t_1)) \le \tilde c_{\varepsilon^*}-\frac{1}{2}\gamma(\varepsilon^*,R^*)s_1 = \tilde c_{\varepsilon^*}-\frac{1}{2}\omega_{d_1}d_0 < c_{\varepsilon^*},$$

which contradicts to (4.50). The lemma is proved. □

Proof of Theorem 1.1. *Step 1.* By Lemma 4.6, $\exists\bar\varepsilon>0$ such that for each $\varepsilon\in(0,\bar\varepsilon]$ and $R>R_0/\varepsilon$, there exists a sequence $\{v_{n,\varepsilon}^R\}_{n=1}^\infty\subset J_\varepsilon^{\tilde c_\varepsilon+\varepsilon}\cap X_\varepsilon^{d_0}\cap H_0^1(B_R(0))$ such that $J'_\varepsilon(v_{n,\varepsilon}^R)\to 0$ in $(H_0^1(B_R(0)))^{-1}$ as $n\to\infty$.

Since $\{v_{n,\varepsilon}^R\}$ is bounded in $H_0^1(B_R(0))$, up to a subsequence, as $n\to\infty$, we have

$$\begin{cases} v_{n,\varepsilon}^R \rightharpoonup v_\varepsilon^R & \text{in } H_0^1(B_R(0)),\\ v_{n,\varepsilon}^R \to v_\varepsilon^R & \text{in } L^s(B_R(0)),\ s\in[1,6),\\ v_{n,\varepsilon}^R \to v_\varepsilon^R & \text{a.e. in } B_R(0).\end{cases}\tag{4.51}$$

By standard argument, we can check that $v_\varepsilon^R\ge 0$ and satisfies

$$\begin{cases} -\Delta v_\varepsilon^R + V(\varepsilon x)v_\varepsilon^R + \phi_{v_\varepsilon^R}v_\varepsilon^R + 4\Big(\int_{\mathbf{R}^3}\chi_\varepsilon (v_\varepsilon^R)^2dx-1\Big)_+\chi_\varepsilon v_\varepsilon^R\\ = \lambda(v_\varepsilon^R)^{p-1}+(v_\varepsilon^R)^5 \ \text{ in } B_R(0),\\ v_\varepsilon^R = 0 \text{ on } \partial B_R(0),\end{cases}\tag{4.52}$$

and we will show that $v_\varepsilon^R\in J_\varepsilon^{\tilde c_\varepsilon+\varepsilon}\cap X_\varepsilon^{d_0}$ for $d_0>0$ small.

Indeed, we write that $v_{n,\varepsilon}^R = u_{n,\varepsilon}^R + w_{n,\varepsilon}^R$ with $u_{n,\varepsilon}^R\in X_\varepsilon$ and $\|w_{n,\varepsilon}^R\|_{H_\varepsilon}\le d_0$. Since S_{V_0} is compact in $H^1(\mathbf{R}^3)$, up to a subsequence, we can assume that $u_{n,\varepsilon}^R\to u_\varepsilon^R$ in $H_0^1(B_R(0))$ and $w_{n,\varepsilon}^R\rightharpoonup w_\varepsilon^R$ in $H_0^1(B_R(0))$ as $n\to\infty$. Then we have $v_\varepsilon^R = u_\varepsilon^R + w_\varepsilon^R$ with $u_\varepsilon^R\in X_\varepsilon$ and $\|w_\varepsilon^R\|_{H_\varepsilon}\le d_0$, i.e., $v_\varepsilon^R\in X_\varepsilon^{d_0}$.

By Brezis–Lieb's Lemma [11, Theorem 1], Lemma 2.1(i), Lemma 2.2(i) and (4.51), we have

$$\begin{aligned}
\tilde{c}_\varepsilon + \varepsilon \geq J_\varepsilon(v_{n,\varepsilon}^R) &= J_\varepsilon(v_\varepsilon^R) + \frac{1}{2}\left\|v_{n,\varepsilon}^R - v_\varepsilon^R\right\|_{H_\varepsilon}^2 - \frac{1}{6}\left\|v_{n,\varepsilon}^R - v_\varepsilon^R\right\|_{L^6(\mathbf{R}^3)}^6 + o(1)\\
&= J_\varepsilon(v_\varepsilon^R) + \frac{1}{2}\left\|w_{n,\varepsilon}^R - w_\varepsilon^R\right\|_{H_\varepsilon}^2 - \frac{1}{6}\left\|w_{n,\varepsilon}^R - w_\varepsilon^R\right\|_{L^6(\mathbf{R}^3)}^6 + o(1)\\
&\geq J_\varepsilon(v_\varepsilon^R) + \frac{1}{2}\left\|w_{n,\varepsilon}^R - w_\varepsilon^R\right\|_{H_\varepsilon}^2 - \frac{1}{6}S^{-3}\left\|w_{n,\varepsilon}^R - w_\varepsilon^R\right\|_{H_\varepsilon}^6 + o(1)\\
&= J_\varepsilon(v_\varepsilon^R) + \left\|w_{n,\varepsilon}^R - w_\varepsilon^R\right\|_{H_\varepsilon}^2 \left(\frac{1}{2} - \frac{1}{6}S^{-3}\left\|w_{n,\varepsilon}^R - w_\varepsilon^R\right\|_{H_\varepsilon}^4\right) + o(1)\\
&\geq J_\varepsilon(v_\varepsilon^R) + o(1) \quad \text{for } d_0 > 0 \text{ small.}
\end{aligned}$$

Letting $n \to \infty$, we have $J_\varepsilon(v_\varepsilon^R) \leq \tilde{c}_\varepsilon + \varepsilon$, that is $v_\varepsilon^R \in J_\varepsilon^{\tilde{c}_\varepsilon+\varepsilon}$.

Step 2. We claim that $\exists \bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$ and $R > R_0/\varepsilon$,

$$\left\|v_\varepsilon^R\right\|_{L^\infty(\mathbf{R}^3)} \leq C. \tag{4.53}$$

Otherwise, $\exists \varepsilon_j \to 0$, $R_j > R_0/\varepsilon_j$ such that $\left\|v_{\varepsilon_j}^{R_j}\right\|_{L^\infty(\mathbf{R}^3)} \to \infty$ as $j \to \infty$. By Lemma 4.3(i), there exist, up to a subsequence, $\{y_j\}_{i=j}^\infty \subset \mathbf{R}^3$, $x_0 \in \mathcal{M}$, $U \in S_{V_0}$ such that

$$\lim_{j\to\infty} |\varepsilon_j y_j - x_0| = 0 \quad \text{and} \quad \lim_{j\to\infty} \left\|v_{\varepsilon_j}^{R_j}(x) - \varphi(\varepsilon_j x - \varepsilon_j y_j)U(x - y_j)\right\|_{H_{\varepsilon_j}} = 0,$$

then

$$\lim_{j\to\infty} \left\|v_{\varepsilon_j}^{R_j}(x + y_j) - \varphi(\varepsilon_j x)U(x)\right\|_{L^6(\mathbf{R}^3)} = 0,$$

which implies that as $j \to \infty$,

$$v_{\varepsilon_j}^{R_j}(x + y_j) \to U(x) \quad \text{in } L^6(\mathbf{R}^3).$$

Using the Brezis–Kato type argument (see also Lemma 2.4), we have

$$\left\|v_{\varepsilon_j}^{R_j}(x + y_j)\right\|_{L^\infty(\mathbf{R}^3)} \leq C,$$

which leads to a contradiction.

Step 3. Next, we claim that $v_\varepsilon^R \to v_\varepsilon \in H_\varepsilon \cap X_\varepsilon^{d_0} \cap J_\varepsilon^{\tilde{c}_\varepsilon+\varepsilon}$ as $R \to \infty$ in H_ε sense for $\varepsilon > 0$ small but fixed.

Since $Q_\varepsilon(v_\varepsilon^R)$ is uniformly bounded for all $\varepsilon > 0$ small and $R > R_0/\varepsilon$, we have

$$\int_{\mathbf{R}^3\setminus(\Lambda/\varepsilon)} (v_\varepsilon^R)^2 \leq C\varepsilon. \tag{4.54}$$

By (4.52), we have that for any $\delta > 0$,

$$-\Delta v_\varepsilon^R + V(\varepsilon x)v_\varepsilon^R \leq \delta v_\varepsilon^R + C_\delta (v_\varepsilon^R)^5,$$

taking $\delta = \inf\limits_{x\in\mathbf{R}^3} V(x) > 0$ and combining with (4.53), it holds that

$$-\Delta v_\varepsilon^R \leq C(v_\varepsilon^R)^5 \leq C(v_\varepsilon^R)^{2/3},$$

in the weak sense. Letting $t = 6$ in Lemma 2.7, we have

$$\sup_{B_1(y)} v_\varepsilon^R \leq C\left(\left\|v_\varepsilon^R\right\|_{L^2(B_2(y))} + \left\|v_\varepsilon^R\right\|_{L^2(B_2(y))}^{2/3}\right), \quad y \in \mathbf{R}^3.$$

By (4.54), we see that

$$v_\varepsilon^R(x) \le C(\varepsilon^{1/2} + \varepsilon^{1/3}) \quad \text{for all } |x| \ge R_0/\varepsilon + 2 \quad \text{and} \quad R > R_0/\varepsilon.$$

Hence, for $\varepsilon > 0$ small but fixed, we have

$$\lambda(v_\varepsilon^R)^{p-1} + (v_\varepsilon^R)^5 \le \frac{1}{2} V(\varepsilon x) v_\varepsilon^R \text{ for all } |x| \ge R_0/\varepsilon + 2 \quad \text{and} \quad R > R_0/\varepsilon.$$

By the Maximum Principle (see also [32]), we have

$$0 \le v_\varepsilon^R(x) \le C_1(\varepsilon) e^{-C_2(\varepsilon)|x|} \quad \text{for all } |x| \ge R_0/\varepsilon + 2 \quad \text{and} \quad R > R_0/\varepsilon, \tag{4.55}$$

where $C_1(\varepsilon)$ and $C_2(\varepsilon)$ are independent of R.

Choosing a cut-off function $\varphi_A \in C^\infty(\mathbf{R}^3)$ such that $0 \le \varphi_A \le 1$, $\varphi_A = 0$ for $|x| \le A$, $\varphi_A = 1$ for $|x| \ge 2A$ and $|\nabla \varphi_A| \le C/A$. It follows from $\langle J'_\varepsilon(v_\varepsilon^R), \varphi_A v_\varepsilon^R \rangle = 0$ and (4.55) that

$$\begin{aligned}
&\int_{\mathbf{R}^3 \setminus B_{2A}(0)} |\nabla v_\varepsilon^R|^2 + V(\varepsilon x)|v_\varepsilon^R|^2 \\
&\le \frac{C}{A} \int_{\mathbf{R}^3 \setminus B_A(0)} |\nabla v_\varepsilon^R|^2 + |v_\varepsilon^R|^2 + \int_{\mathbf{R}^3 \setminus B_A(0)} \lambda (v_\varepsilon^R)^p + (v_\varepsilon^R)^6 \\
&\le \frac{C}{A} \int_{\mathbf{R}^3} |\nabla v_\varepsilon^R|^2 + |v_\varepsilon^R|^2 + C(\varepsilon) \int_{\mathbf{R}^3 \setminus B_A(0)} e^{-C(\varepsilon)|x|} \to 0 \quad \text{as } A \to \infty,
\end{aligned}$$

i.e., for $\varepsilon > 0$ small but fixed,

$$\lim_{A \to \infty} \int_{\mathbf{R}^3 \setminus B_{2A}(0)} |\nabla v_\varepsilon^R|^2 + V(\varepsilon x)|v_\varepsilon^R|^2 = 0. \tag{4.56}$$

Since $\{v_\varepsilon^R\}$ is bounded in H_ε, we can assume that as $R \to \infty$,

$$\begin{cases} v_\varepsilon^R \rightharpoonup v_\varepsilon & \text{in } H_\varepsilon, \\ v_\varepsilon^R \to v_\varepsilon & \text{in } L^s_{\mathrm{loc}}(\mathbf{R}^3), \ s \in [1,6), \\ v_\varepsilon^R \to v_\varepsilon & \text{a.e.} \end{cases}$$

By (4.56) and Sobolev's Imbedding Theorem, we get

$$v_\varepsilon^R \to v_\varepsilon \text{ in } L^s(\mathbf{R}^3), \quad s \in [2,6) \quad \text{as } R \to \infty.$$

By (4.53), we have

$$v_\varepsilon^R \to v_\varepsilon \text{ in } L^s(\mathbf{R}^3), \quad s \in [2,6] \quad \text{as } R \to \infty.$$

Using standard argument, we can prove the claim.

Hence, $v_\varepsilon \in H_\varepsilon \cap X_\varepsilon^{d_0} \cap J_\varepsilon^{\tilde{c}_\varepsilon + \varepsilon}$ is a nontrivial solution of

$$-\Delta u + V(\varepsilon x)u + \phi_u u + 4\left(\int_{\mathbf{R}^3} \chi_\varepsilon u^2\,dx - 1\right)_+ \chi_\varepsilon u = \lambda u^{p-1} + u^5 \quad \text{in } \mathbf{R}^3.$$

Since S_{V_0} is compact in $H^1(\mathbf{R}^3)$, it is easy to see that $0 \notin X_\varepsilon^{d_0}$ for $d_0 > 0$ small. Thus $v_\varepsilon \ne 0$.

Step 4. For any sequence $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$, by Lemma 4.3(ii), there exist, up to a subsequence, $\{y_j\}_{i=j}^\infty \subset \mathbf{R}^3$, $x_0 \in \mathcal{M}$, $U \in S_{V_0}$ such that

$$\lim_{j\to\infty} |\varepsilon_j y_j - x_0| = 0 \quad \text{and} \quad \lim_{j \to \infty} \left\| v_{\varepsilon_j}(x) - \varphi(\varepsilon_j x - \varepsilon_j y_j) U(x - y_j) \right\|_{H_{\varepsilon_j}} = 0, \tag{4.57}$$

which implies that as $j \to \infty$,

$$w_{\varepsilon_j}(x) := v_{\varepsilon_j}(x + y_j) \to U(x) \text{ in } L^6(\mathbf{R}^3).$$

By Lemma 2.4 (ii), we get

$$\lim_{|x|\to\infty} w_{\varepsilon_j}(x) = 0 \text{ uniformly for all } \varepsilon_j. \tag{4.58}$$

Proceeding as in [32], we get

$$w_{\varepsilon_j}(x) \le C_1 e^{-C_2|x|}, \quad x \in \mathbf{R}^3,$$

where C_1 and C_2 are independent of ε_j.

Thus

$$\varepsilon_j^{-1} \int_{\mathbf{R}^3\setminus(\Lambda/\varepsilon_j)} v_{\varepsilon_j}^2(x) = \varepsilon_j^{-1} \int_{\mathbf{R}^3\setminus(\Lambda/\varepsilon_j - y_j)} w_{\varepsilon_j}^2(x) \le \varepsilon_j^{-1} \int_{\mathbf{R}^3\setminus B_{\beta/\varepsilon_j}(0)} (C_1)^2 e^{-2C_2|x|} \to 0,$$

as $j \to \infty$, i.e., $Q_{\varepsilon_j}(v_{\varepsilon_j}) = 0$ for ε_j small. Therefore v_{ε_j} is a solution of (4.1). Set $u_\varepsilon(x) = v_\varepsilon(\frac{x}{\varepsilon})$, u_{ε_j} is a solution of (1.1).

Let P_j be a maximum point of w_{ε_j}, similar to the arguments in Proposition 3.9, we can check that $\exists b > 0$ such that $w_{\varepsilon_j}(P_j) > b$, then by (4.58), $\{P_j\}$ must be bounded.

Since $u_{\varepsilon_j}(x) = w_{\varepsilon_j}(\frac{x}{\varepsilon_j} - y_j)$, $x_j := \varepsilon_j P_j + \varepsilon_j y_j$ is a maximum point of u_{ε_j}. From (4.57), $x_j \to x_0 \in \mathcal{M}$ as $j \to \infty$. Since the sequence $\{\varepsilon_j\}$ is arbitrary, we have obtained the existence and concentration results in Theorem 1.1.

To complete the proof, we only need to prove the exponential decay of u_ε. Since the proof is standard (see [26, 32] for example), we omit it here. □

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