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DYNAMICAL CONVERGENCE OF A CERTAIN POLYNOMIAL FAMILY TO $f_a(z) = z + e^z + a$

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Abstract. A transcendental entire function $f_a(z) = z + e^z + a$ may have a Baker domain or a wandering domain, which never appear in the dynamics of polynomials. We consider a sequence of polynomials $P_{a,d}(z) = (1 + a/d)z + (1 + z/d)^{d+1} + a$, which converges uniformly on compact sets to f_a as $d \to \infty$. We show its dynamical convergence under a certain assumption, even though f_a has a Baker domain or a wandering domain. We also investigate the parameter spaces of f_a and $P_{a,d}$.

1. Introduction

Let X be the complex plane $\mathbf{C}$, the complex sphere $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$ or the punctured plane $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. We consider iterates of analytic self-maps of X. Fundamental facts of iteration theory can be found in [2, 3, 9, 24]. Let f be an analytic self-map of X. If $X = \widehat{\mathbf{C}}$, then f is rational and if $X = \mathbf{C}$ and f cannot be continuously extended to $\widehat{\mathbf{C}}$, then f is transcendental entire. The maximal open subset of X where the family $\{f^n\}$ is normal is called the Fatou set of f and denoted by $F(f)$. The complement of $F(f)$ in X is called the Julia set of f and denoted by $J(f)$. Fatou sets and Julia sets are completely invariant. A connected component of $F(f)$ is called a Fatou component. A Fatou component U is called periodic if $f^p(U) \subset U$ holds for some $p \in \mathbf{N}$. Periodic components are well understood and are completely classified into five cases. A component named a Baker domain is a periodic one where the limit function of $\{f^n\}$ is not contained in X. By definition, rational functions do not have Baker domains. Furthermore, if a transcendental entire function has a Baker domain, then the limit function defined there is infinity. See [26] for a survey on Baker domains. Singular values play an important role in the study of complex dynamics. Here, singular values are critical values, asymptotic values or points in the closure of the set of critical and asymptotic values. If a function has only finitely many singular values, then it is called of finite type. Every type of periodic components except Baker domains has a relationship with singular values which is useful to estimate the number of the non-repelling cycles. We call a component U of $F(f)$ is wandering if $f^n(U) \neq f^m(U)$ for all n and m ($n \neq m$). Sullivan [27] showed that rational functions do not have wandering domains. As similar results on Baker domains and wandering domains of rational functions, if a transcendental entire function is of finite type, then it has neither Baker domains nor wandering domains (see, for example, [9, 13]). The

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possibility of existence of Baker domains or wandering domains is a great difference between dynamics of rational functions and that of transcendental entire functions.

For an entire function f, the escaping set of f is defined by

$$I(f) = \{z \mid f^n(z) \to \infty \quad \text{as} \ \ n \to \infty\}.$$

If f is a polynomial, then infinity is a super-attracting fixed point and thus $I(f)$ is its immediate basin, which is contained in $F(f)$. For a general transcendental entire function f, Eremenko [8] studied it and proved that $I(f) \neq \emptyset$, $J(f) = \partial I(f)$ and $I(f) \cap J(f) \neq \emptyset$. Obviously, Baker domains and wandering domains tending to infinity are contained in escaping sets. Before his study, Devaney and Krych [6] showed that the Julia set of λe^z contains uncountable many curves tending to infinity for some λ. Each point of the curves except the end points tends to infinity under the iterate of λe^z. Hence every curve without its end point is contained in the escaping set. Each curve is a so-called hair and the union of hairs is a so-called Cantor bouquet.

One approach to investigate the dynamics of a transcendental entire function is to consider some suitable sequence of polynomials which converges uniformly on compact sets to it. Bodelén et al. [5] considered the exponential family $E_\lambda(z) = \lambda e^z$ and families of polynomial maps $Q_{\lambda,d}(z) = \lambda(1+z/d)^d$. For a fixed λ, $Q_{\lambda,d}$ converges uniformly on compact sets to E_λ as $d \to \infty$. One of the important facts is that E_λ has only one singular value and so do $Q_{\lambda,d}$ in $\mathbf{C}$. This implies that E_λ and $Q_{\lambda,d}$ have at most one non-repelling cycle and that they have neither Baker domains nor wandering domains. Hence we obtain bifurcation sets for E_λ and $Q_{\lambda,d}$ just like defining the Mandelbrot set in the case of quadratic polynomials. They showed the hyperbolic components of the parameter planes of $Q_{\lambda,d}$ converge to those of E_λ as $d \to \infty$. They also showed that for some parameters λ, hairs defined for $Q_{\lambda,d}$ converge point wise to the corresponding hairs defined for E_λ as $d \to \infty$. We note that every hair for $Q_{\lambda,d}$ is contained in $F(Q_{\lambda,d})$ except its endpoint and that every hair for E_λ is contained in $J(E_\lambda)$. In this context, Krauskopf [15] considered how the Julia set $J(Q_{\lambda,d})$ tends to $J(E_\lambda)$ as $d \to \infty$. By definition $J(f)$ is contained in $\mathbf{C}$ if f is transcendental entire. We denote $J(f) \cup \{\infty\}$ by $\widehat{J(f)}$. We note that it is a compact set in $\widehat{\mathbf{C}}$. He showed that if E_λ has an attracting cycle, then $J(Q_{\lambda,d})$ converges to $\widehat{J(E_\lambda)}$ in the Hausdorff metric. Kisaka [14] extended this result as follows: Assume a sequence of polynomials P_n converges uniformly on compact sets to a transcendental entire function f as $n \to \infty$. If $F(f)$ contains all the singular values and consists only of basins of attracting cycles, then $J(P_n)$ converges to $\widehat{J(f)}$ in the Hausdorff metric (see also [16] as remark). Krauskopf and Kriete [18] proved the similar results for meromorphic functions. Moreover, the same authors [17] considered convergence of hyperbolic components in a parameter plane in more general case. However, they treated a family of entire functions of constant finite type, that is, there exists a finite constant that equals the number of the singular values of each function.

In this paper, we consider a one-parameter family of transcendental entire function $f_a(z) = z + e^z + a$. It has infinitely many singular values $(2n+1)\pi i + a - 1$ ($n \in \mathbf{Z}$). It is easy to check that f_{-1} has a Baker domain by the similar argument that shows Fatou's first example of a Baker domain (see [10]). Furthermore, for some parameters, f_a has wandering domains, where a limit function is always infinity. It is clear that $P_{a,d}(z) = (1+a/d)z + (1+z/d)^{d+1} + a$ converges uniformly on compact sets to f_a. Recall that $P_{a,d}$ has neither Baker domains nor wandering domains. Therefore, we are interested in a dynamical approximation of f_a by $P_{a,d}$. We show that $J(P_{a,d})$ converges to $\widehat{J(f_a)}$ in the Hausdorff metric under the assumption that $\exp f_a(z)$ has an attracting cycle, even though f_a has a Baker domain or a wandering domain (Theorem 13). Roughly speaking, in this case, a Baker domain is a limit of a sequence of attracting immediate basins growing bigger. Analogously a wandering domain is a limit of a sequence of periodic components whose periods tend to infinity. In [23], we see the results for $a = -1$ as a case of a Baker domain and for $a = 2\pi i$ as a case of wandering domains with rough sketch of proofs. In this note, the proof for the case of wandering domains is quite different from that in [23]. We also remark that Garfias [12] considered the convergence to Baker domains of functions $z \mapsto z - 1 + \lambda z e^z$.

This paper is organized as follows. In Section 2, as a preliminary, we define two convergences, the Hausdorff convergence and the Carathéodory convergence. The relationship between the convergences is considered from the view point of the uniform convergence on compact sets of a sequence of polynomials. Section 3 deals with f_a. To understand the dynamics of f_a, we introduce the idea of logarithmic lifts. The bifurcation set in the parameter space of f_a is defined and its components are investigated. Section 4 deals with $P_{a,d}$. We consider hyperbolic components of its bifurcation set. We also see some sequences of hyperbolic components of $P_{a,d}$ converge to components in the bifurcation set of f_a functions corresponding to which have wandering domains. In Section 5, we are concerned with the Hausdorff convergence of $J(P_{a,d})$ to $\widehat{J(f_a)}$.

2. The Carathéodory convergence and the Hausdorff convergence

We introduce two ideas of convergence. The first one is a convergence of compact sets in $\widehat{\mathbf{C}}$. Let ρ be the spherical metric on $\widehat{\mathbf{C}}$. We denote the ε-neighborhood of a set A in $\widehat{\mathbf{C}}$ by $U_\varepsilon(A)$. The Hausdorff distance between two non-empty compact sets A and B is defined by

$$d(A, B) = \inf\{\varepsilon > 0 \mid A \subset U_\varepsilon(B), \quad B \subset U_\varepsilon(A)\}.$$

This distance defines the Hausdorff metric on the set of all the non-empty compact sets in $\widehat{\mathbf{C}}$. Let K and K_n ($n \in \mathbf{N}$) be non-empty compact sets in $\widehat{\mathbf{C}}$. We say that K_n converges to K in the Hausdorff metric, if $d(K_n, K) \to 0$ as $n \to \infty$.

The second one is a convergence of open sets in $\widehat{\mathbf{C}}$. Let O and O_n ($n \in \mathbf{N}$) be open sets in $\widehat{\mathbf{C}}$. We say that O_n converges to O in the sense of Carathéodory, if the following two conditions hold:

(1) for an arbitrary compact set $K \subset O$, there exists $N \in \mathbf{N}$ such that $K \subset O_n$ for all $n > N$, and
(2) if an open set U is contained in O_n for infinitely many n, then $U \subset O$.

Two ideas of the convergence defined above have the following relationship.

Lemma 1. *Non-empty closed set K_n converges to K in the Hausdorff metric if and only if the complement of K_n converges to the complement of K in the sense of Carathéodory.*

These concepts are formerly used, for example, in a study of Kleinian groups: the convergence of limit sets, which are compact sets in $\widehat{\mathbf{C}}$ and the convergence of ordinary sets, which are complements of limit sets. The proof of Lemma 1 is straightforward. We find an outline of the proof, for example, in [21].

To consider the Hausdorff convergence of Julia sets, we deal with $\widehat{J(f)} = J(f) \cup \{\infty\}$ instead of $J(f)$, if f is transcendental entire. Douady [7] showed that the Hausdorff convergence of Julia sets of polynomials is lower semicontinuous. This can be proved by the density of repelling periodic points in Julia sets and the Hurwitz theorem. Hence we easily extend the result as follows.

Proposition 2. *Let f be a transcendental entire function and P_n be polynomials. If P_n converges uniformly on compact sets to f, then, for an arbitrary $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that*

$$\widehat{J(f)} \subset U_\varepsilon(J(P_n))$$

for all $n > N$.

From the lemma above, Proposition 2 is rephrased as follows.

Proposition 3. *Let f and P_n be as in Proposition 2. If there exists an open set U such that $U \subset F(P_n)$ for infinitely many n, then $U \subset F(f)$.*

From the proposition above, to prove the Carathéodory convergence of Fatou sets, we only need to show that the condition (1) is satisfied.

3. Functions $f_a(z) = z + e^z + a$

3.1. To understand the dynamics of f_a, we introduce the idea of logarithmic lifts. Let g be an analytic self-map on $\mathbf{C}^*$. Then there exists an entire function f satisfying

$$\exp f(z) = g(e^z).$$

We call it a logarithmic lift of g. The difference of arbitrary two logarithmic lifts of g is a multiple of $2\pi i$. Bergweiler [4] showed that $\exp^{-1} J(g) = J(f)$ if f is neither linear nor constant. For a set $A \subset \mathbf{C}$ and a constant $a \in \mathbf{C}$, $\{z + a \mid z \in A\}$ is written by $A + a$. A logarithmic lift f satisfies $F(f) = F(f) + 2\pi i$. From this property, examples of functions which have wandering domains can be constructed (see [1]).

We consider following two families of functions:

$$g_\lambda(z) = \lambda z e^z \quad \text{and} \quad f_a(z) = z + e^z + a,$$

where $\lambda \in \mathbf{C}^*$ and $a \in \mathbf{C}$. Since $\exp f_a(z) = g_\lambda(e^z)$ for $\lambda = e^a$, $f_a(z)$ is a logarithmic lift of $g_\lambda(z)$. Hence $f_{a+2\pi ki}(z)$ also is a logarithmic lift of $g_\lambda(z)$ for all $k \in \mathbf{Z}$. We see that, for $k \in \mathbf{Z}$,

$$f_a(z + 2\pi ki) = f_a(z) + 2\pi ki \quad \text{and} \quad f_{a+2\pi ki}(z) = f_a(z) + 2\pi ki$$

and by induction we have

$$f_a^n(z + 2\pi ki) = f_a^n(z) + 2\pi ki \quad \text{and} \quad f_{a+2\pi ki}^n(z) = f_a^n(z) + 2\pi kni$$

for $n \in \mathbf{N}$ (see [4]).

Here we show a rough sketch of the reason why f_a has wandering domains for some a. It might help readers intuitively understand the proof of Theorem 13. For $A \subset \mathbf{C}^*$, we call $\{z \mid e^z \in A\}$ the logarithmic lift of A. Choosing η so that $|1+\eta| < 1$, we see that $g_{e^{-\eta}}(z)$ has an attracting fixed point η. As a logarithmic lift of $g_{e^{-\eta}}(z)$, we consider $f_{-\eta}(z) = z + e^z - \eta$. The logarithmic lift of $\{\eta\}$ is denoted by Q. Every point of Q is an attracting fixed point of $f_{-\eta}$. Take one point of Q and denote it by ζ. Then points of Q are written by $\zeta + 2\pi ki$, where $k \in \mathbf{Z}$. Every Fatou component containing $\zeta + 2\pi ki$, which we denote by D_k, is disjoint from the others. Since $F(f_{-\eta}) = F(f_{-\eta+2\pi i})$, D_k is a component of $F(f_{-\eta+2\pi i})$, too. We see that

$$f^n_{-\eta+2\pi i}(D_k) = D_k + 2\pi ni = D_{k+n}.$$

Therefore D_k is a wandering domain of $f_{-\eta+2\pi i}$. Furthermore, the argument above also implies that the limit function of every wandering domain is always infinity.

3.2. We briefly look at $g_\lambda(z) = \lambda z e^z$ for $\lambda \in \mathbf{C}^*$. This family was considered in [11, 19, 22]. Each $g_\lambda(z)$ has two singular values. One is a critical value $g_\lambda(-1) = -\lambda/e$ and the other is the asymptotic value 0. The finiteness of the number of singular values implies that g_λ has neither Baker domains nor wandering domains. The asymptotic value 0 is always a fixed point of g_λ. Assume that 0 is an attracting fixed point and let A be its immediate basin. Choose a sufficiently small neighborhood U of 0 in A so that U does not contain the critical value. Hence $g^{-1}(U)$ consists of two components, say U_1 and U_2, both of which are contained in A, because g_λ is not univalent on the attracting immediate basin. Suppose U_1 contains 0. There exist two components of $g_\lambda^{-n}(U_1)$, say U_1^n and U_2^n, satisfying $U_1 \subset U_1^n$ and $U_2 \subset U_2^n$ for every $n \in \mathbf{N}$. Since $A = \bigcup_n (U_1^n \cup U_2^n)$, there exists $n \in \mathbf{N}$ such that $U_1^n \cap U_2^n \neq \emptyset$. This shows that $U_1^n \cap U_2^n$ contains the critical point and thus so does A. Therefore g_λ has at most one non-repelling cycle for every $\lambda \in \mathbf{C}^*$ and the behavior of the orbit of the critical value determines the dynamics. The bifurcation set $\widetilde{M}$ of g_λ is defined as

$$\widetilde{M} = \{\lambda \in \mathbf{C}^* \mid \{g_\lambda^n(-\lambda/e)\} \text{ is bounded}\}.$$

Let $\widetilde{H}$ be the set

$$\widetilde{H} = \{\lambda \in \mathbf{C}^* \mid g_\lambda \text{ has an attracting cycle}\}.$$

We call a connected component of $\widetilde{H}$ an a-component of $\widetilde{M}$. We define the sets

$$V_0 = \{\lambda \mid 0 < |\lambda| < 1\} \quad \text{and} \quad V_1 = \{\lambda = e^{1-\mu} \mid |\mu| < 1\}.$$

Note that V_0 is doubly connected. Kremer [19] showed the following proposition.

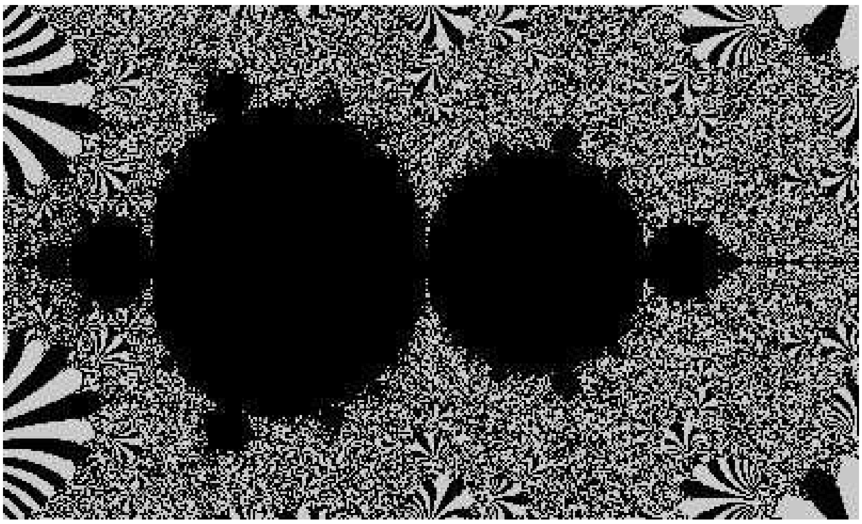

Figure 1. The bifurcation set of g_λ. Range: $-2.5 \leq \operatorname{Re}\lambda \leq 17.5$, $-4 \leq \operatorname{Im}\lambda \leq 4$.

Proposition 4. *Each of V_0 and V_1 is an a-component of $\widetilde{M}$ corresponding to an attracting fixed point. Conversely, if g_λ has an attracting fixed point, then λ belongs to V_0 or V_1. If λ belongs to V_0, then 0 is an attracting fixed point and $F(g_\lambda)$ consists of exactly one component. Each a-component of $\widetilde{M}$ except V_0 is open and simply connected.*

Remark. In general, an entire function is called hyperbolic if each singular value is contained in the Fatou set and is attracted by an attracting cycle. For example, in the parameter space of the set of quadratic polynomials $\{z \mapsto z^2 + c \mid c \in \mathbf{C}\}$, the subset of all the hyperbolic functions is open. Its connected components are called hyperbolic components. Each component corresponds to a period of an attracting cycle. Hyperbolic components are one of the most important objects in the study of complex dynamics. In the case of our family, if $|\lambda| > 1$, then the asymptotic value 0 is a repelling fixed point. Hence g_λ is not hyperbolic even though λ is contained in $\widetilde{H}$. However, since 0 is a fixed point, g_λ has a nice property like the hyperbolicity for $\lambda \in \widetilde{H}$ (see, for example, [22]). Thus we are interested in connected components of $\widetilde{H}$.

3.3. Since f_a may have a Baker domain or wandering domains where a limit function is infinity, we define the bifurcation set M of f_a by the logarithmic lift of $\widetilde{M}$, that is,

$$M = \{a \in \mathbf{C} \mid e^a \in \widetilde{M}\}.$$

The logarithmic lift of V_0 is $\{a \mid \operatorname{Re} a < 0\}$, which we denote by B. Summarizing Lauber's results in [20] what we need in this paper, we state the following theorem.

Theorem 5. *For $a \in B$, f_a has a Baker domain which is the only component of $F(f_a)$. Conversely, if f_a has a Baker domain, then $a \in \overline{B}$.*

The logarithmic lift of each a-component of $\widetilde{M}$ except V_0 consists of infinitely many components. For any two of these, say U_1 and U_2, there exists $k \in \mathbf{Z}$ such that $U_2 = U_1 + 2\pi k i$.

Theorem 6. *Let U be a component of the logarithmic lift of some a-component of $\widetilde{M}$ except V_0. Then, for $a \in U$, $F(f_a)$ only consists of either attracting basins or wandering domains. If f_a has wandering domains, then $f_{a'}$ has wandering domains for every $a' \in U$.*

Proof. Theorem 5 shows that f_a has no Baker domain. Suppose f_a has a non-repelling periodic point of period p, say ζ. We write $\lambda = e^a$. Since $g_\lambda^p(e^\zeta) = e^\zeta$ and $(g_\lambda^p)'(e^\zeta) = (f_a^p)'(\zeta)$, e^ζ is a non-repelling periodic point of g_λ. Hence e^ζ is an attracting periodic point and thus so are $\zeta + 2\pi k i$ for $k \in \mathbf{Z}$. Let D be a component of $F(f_a)$ and denote $\exp(D)$ by E. Since E is a component of $F(g_\lambda)$, there exists $n \in \mathbf{N}$ such that $g_\lambda^n(E)$ contains e^ζ. Therefore $f_a^n(D)$ contains $\zeta + 2\pi k i$ for some $k \in \mathbf{Z}$. This gives that $F(f_a)$ consists of only attracting basins.

Assume that f_{a_0} has wandering domains for $a_0 \in U$. We write $\lambda(a_0) = e^{a_0}$. Let $\zeta(a_0)$ be an attracting periodic point of $g_{\lambda(a_0)}$ of period p. Take a point $w(a_0)$ of the logarithmic lift of $\zeta(a_0)$. As it was seen in § 3.1, there exists $k \in \mathbf{Z}^* = \mathbf{Z} \setminus \{0\}$ such that $(f_{a_0}^p(w(a_0)) - w(a_0))/2\pi i = k$. From the continuity of $(f_a^p)'(z)$ with respect to z and a and the Hurwitz theorem, there exists $\varepsilon > 0$ such that f_a has an attracting periodic point $\zeta(a)$ of period p for all a satisfying $|a - a_0| < \varepsilon$. Due to the continuity of $\zeta(a)$, we can take a point $w(a)$ of the logarithmic lift of $\zeta(a)$ so that it is continuous with respect to a. Since $(f_{a_0}(w(a_0)) - w(a_0))/2\pi i$ takes only integers and is continuous with respect to a, it is constant in $\{a \mid |a - a_0| < \varepsilon\}$. The component U being open, we see that every f_a has wandering domains for $a \in U$. $\square$

A component U of the logarithmic lift of an a-component of $\widetilde{M}$ is called an a-component of M if f_a has an attracting periodic point for some and therefore for all $a \in U$. A component U of the logarithmic lift of an a-component of $\widetilde{M}$ is called an w-component of M if f_a has wandering domains for some and therefore for all $a \in U$.

Theorem 7. *The logarithmic lift of every a-component of* $\widetilde{M}$ *has at most one a-component of* M.

Proof. It is clear if an a-component is V_0. Assume that the logarithmic lift of an a-component except V_0 has an a-component, say W. Any other component of the logarithmic lift is given by $W + 2\pi ki$ for $k \in \mathbf{Z}^*$. Take $a \in W$ and let D be an attracting periodic component of f_a. Then D again is a Fatou component of $f_{a+2\pi ki}$ and satisfies

$$f^p_{a+2\pi ki}(D) = D + 2\pi kpi.$$

Hence D is a wandering domain of $f_{a+2\pi ki}$. $\square$

Remark. There exist a-components of $\widetilde{M}$ whose logarithmic lift only consists of w-components of M. For example, let λ_0 be the negative real root of $\lambda^2 = e^{\lambda e^{-1}+1}$, which is approximately $-1.29844\ldots$. The function g_{λ_0} has an attracting two cycle, which is $\{-1, -\lambda_0/e\}$. Since every logarithmic lift of $g_{\lambda_0}(z)$ is of the form $f_{\log|\lambda_0|+(2k+1)\pi i}(z) = z + e^z + \log|\lambda_0| + (2k+1)\pi i$ for some $k \in \mathbf{Z}$, we see $f^2_{\log|\lambda_0|+(2k+1)\pi i}(\pi i) = (4k+3)\pi i$. Hence every component of the logarithmic lift of the a-component of $\widetilde{M}$ containing λ_0 is a w-component of M.

One component of the logarithmic lift of V_1 is given by $\{a \mid |1 - a| < 1\}$, which we denote by A_0. Any other components of the logarithmic lift are given by $\{a \mid |1 + 2\pi ki - a| < 1\}$ for $k \in \mathbf{Z}^*$, which we denote by W_k.

Proposition 8. A_0 *is an a-component of* M *and* W_k*'s* ($k \in \mathbf{Z}^*$) *are w-components of* M.

Proof. We denote the principal branch of logarithm by $\operatorname{Log} z = \log|z| + i\arg z$, where $\arg z$ satisfies $-\pi < \arg z \leq \pi$. Every fixed point of f_a is given by $z_k = \operatorname{Log}(-a) + 2\pi ki$ for $k \in \mathbf{Z}$. They are attracting if and only if $|f'_a(z_k)| = |1 - a| < 1$. $\square$

4. Functions $P_{a,d}(z) = \left(1 + \frac{a}{d}\right) z + \left(1 + \frac{z}{d}\right)^{d+1} + a$

Every polynomial

$$P_{a,d}(z) = \left(1 + \frac{a}{d}\right) z + \left(1 + \frac{z}{d}\right)^{d+1} + a$$

has d critical points in $\mathbf{C}$

$$c^k_{a,d} = -d + d\sqrt[d]{\left|\frac{d+a}{d+1}\right|}e^{i(\theta/d + 2\pi k/d)}$$

for $k = 0, 1, \dots d-1$, where $\theta = \arg(-(d+a)/(d+1))$ satisfying $-\pi < \theta \le \pi$. We note that the all are on the circle $\{z \mid |z+d| = d\sqrt[d]{|d+a/d+1|}\}$ and divide it into d arcs of same length. The set of all the critical points of $P_{a,d}$ is denoted by $C_{a,d}$.

We define auxiliary functions

$$\varphi_d(z) = -d + (z+d)e^{i2\pi/d} \quad \text{and} \quad \psi_d(z) = -d + (z+d)e^{i2\pi/d^2}$$

for $d \in \mathbf{N}$. These are rotations around $-d$ of angle $2\pi/d$ and of angle $2\pi/d^2$, respectively.

Proposition 9. *The action of $P_{a,d}$ has rotation symmetry around $-d$ of angle $2\pi/d$, that is , $P_{a,d}(\varphi_d(z)) = \varphi_d(P_{a,d}(z))$ holds. In particular, $\varphi_d(F(P_{a,d})) = F(P_{a,d})$ and $\varphi_d(J(P_{a,d})) = J(P_{a,d})$ hold. Assume $P_{a,d}$ has an attracting cycle. If it has another non-repelling cycles, then they are attracting of the same period.*

Proof. A straightforward calculation gives $P_{a,d}(\varphi_d(z)) = \varphi_d(P_{a,d}(z))$. By induction, we have $P^n_{a,d}(\varphi_d(z)) = \varphi_d(P^n_{a,d}(z))$ for all $n \in \mathbf{N}$. It immediately follows that $\varphi_d(F(P_{a,d})) = F(P_{a,d})$ and $\varphi_d(J(P_{a,d})) = J(P_{a,d})$ from the definitions of Fatou sets and Julia sets.

Assume $P_{a,d}$ has an attracting cycle. Its immediate basin contains at least one critical point. The number of all the accumulation points of its orbit is finite, which is the period. For any critical point, the number of those is the same as above. The claim is obtained. □

Proposition 9 shows that the dynamics of $P_{a,d}$ is essentially determined by the behavior of the orbit of one critical point since all the critical points are equally distributed on the circle centered at $-d$. Fixing d, we denote one of the critical points of $P_{a,d}$ by c_a. We define the bifurcation set of $P_{a,d}$ as

$$M_d = \{a \in \mathbf{C} \mid \{P^n_{a,d}(c_a)\} \text{ is bounded}\}.$$

In addition, we call a component of

$$\{a \in \mathbf{C} \mid P_{a,d} \text{ has an attracting cycle}\}$$

a hyperbolic component of M_d according to the standard definition.

Theorem 10. *The bifurcation set M_d has a rotation symmetry around $-d$ of angle $2\pi/d$, that is, $\varphi_d(M_d) = M_d$ holds.*

Proof. The definition of φ_d and a simple calculation show that

$$P_{\varphi_d(a),d}(z) = \left(1+\frac{a}{d}\right)e^{i2\pi/d}z + \left(1+\frac{z}{d}\right)^{d+1} - d + (d+a)e^{i2\pi/d}.$$

Hence we have

$$\begin{aligned} P_{\varphi_d(a),d}(\psi_d(z)) &= e^{i2\pi/d}e^{i2\pi/d^2}\left(\left(1+\frac{a}{d}\right)z + \left(1+\frac{z}{d}\right)^{d+1} + a\right) - d + de^{i2\pi/d}e^{i2\pi/d^2} \\ &= \psi_d \circ \varphi_d \circ P_{a,d}(z). \end{aligned}$$

Fix a and abbreviate $\varphi_d(a)$ to a'. It is easy to see that $C_{a',d} = \psi_d(C_{a,d})$. Hence, for $c^k_{a',d} \in C_{a',d}$, there exists $k(0) \in \{0, 1, \dots, d-1\}$ such that $c^k_{a',d} = \psi_d(c^{k(0)}_{a,d})$. From Proposition 9 and the formula above, we have

$$P_{a',d}(c^k_{a',d}) = P_{a',d}(\psi_d(c^{k(0)}_{a,d})) = \psi_d \circ \varphi_d \circ P_{a,d}(c^{k(0)}_{a,d}) = \psi_d \circ P_{a,d} \circ \varphi_d(c^{k(0)}_{a,d}).$$

Since $\varphi_d(c_{a,d}^{k(0)})$ is also contained in $C_{a,d}$, there exists $k(1) \in \{0, 1, \dots, d-1\}$ such that $c_{a,d}^{k(1)} = \varphi_d(c_{a,d}^{k(0)})$. It follows

$$P_{a',d}^2(c_{a',d}^k) = P_{a',d} \circ \psi_d \circ P_{a,d} \circ \varphi_d(c_{a,d}^{k(0)}) = P_{a',d} \circ \psi_d \circ P_{a,d}(c_{a,d}^{k(1)}) = \psi_d \circ P_{a,d}^2 \circ \varphi_d(c_{a,d}^{k(1)}).$$

Iterating this procedure, we obtain

$$P_{a',d}^n(c_{a',d}^k) = \psi_d \circ P_{a,d}^n(c_{a,d}^{k(n)})$$

for some $k(n) \in \{0, 1, \dots, d-1\}$. If $a \in M_d$, then $\{P_{a,d}^n(c_{a,d}^k)\}_{n=0}^\infty$ is bounded for all k, and so is $\psi_d(\cup_{k=1}^d \{P_{a,d}^n(c_{a,d}^k)\}_{n=0}^\infty)$. We conclude that $a' \in M_d$. $\square$

Every $P_{a,d}$ has fixed points $-d$ and

$$-d + d\sqrt[d]{|a|}e^{i(\nu/d + 2\pi k/d)},$$

for $k = 0, 1, \dots d-1$, where $\nu = \arg(-a)$ satisfying $-\pi < \nu \le \pi$. We define sets

$$A_d = \{a \mid |1-a| < 1\} \quad \text{and} \quad B_d = \{a \mid |a+d| < d\}.$$

Proposition 11. *A_d and B_d both are hyperbolic components corresponding to attracting fixed points of $P_{a,d}$. Conversely, if $P_{a,d}$ has an attracting fixed point, then a belongs to A_d or B_d. If a belongs to B_d, then $F(P_{a,d})$ consists of two components.*

Proof. The point $-d$ is an attracting fixed point if and only if $|P'_{a,d}(-d)| = |1 + a/d| < 1$. The point $-d + d\sqrt[d]{|a|}e^{i\nu/d}$ is an attracting fixed point if and only if $|P'_{a,d}(-d + d\sqrt[d]{|a|}e^{i\nu/d})| = |1 - a| < 1$. From Proposition 9, we see that if one of $-d + d\sqrt[d]{|a|}e^{i(\nu/d + 2\pi k/d)}$ $(k = 0, 1, \dots, d-1)$ is an attracting fixed point, so are all.

If $a \in B_d$, then $-d$ is an attracting fixed point. Since infinity is a super-attracting fixed point and its immediate basin is completely invariant, we only need to show that the immediate basin of $-d$ is completely invariant. The immediate basin of $-d$ contains at least one critical point and hence contains all the critical points in **C** from Proposition 9. Consequently, the immediate basin of $-d$ is completely invariant. $\square$

We denote $\varphi_d^k(A_d)$ by W_d^k for $k \neq 0$ satisfying $-d/2 < k \le d/2$ if d is even and $-[d/2] \le k \le [d/2]$ if d is odd.

Theorem 12. *For $a \in W_d^k$, $P_{a,d}$ has an attracting cycle whose period is greater than 1. For fixed k, the period corresponding to W_d^k tends to infinity as $d \to \infty$. Furthermore, W_d^k converges to W^k in the sense of Carathéodory and A_d also converges to A_0 in the sense of Carathéodory.*

Proof. Choose $a \in A_d$ and $k \neq 0$. We write $a' = \varphi_d^k(a)$. Let w be an attracting fixed point of $P_{a,d}$. The formula in the proof of Theorem 10 gives

$$P_{a',d}(\psi_d(w)) = \psi_d \circ \varphi_d^k \circ P_{a,d}(w) = \psi_d \circ \varphi_d^k(w).$$

Iterating this, we conclude $\psi_d(w)$ is a periodic point of $P_{a',d}$ of period d/ℓ, where ℓ is the greatest common divisor of d and k. As ψ_d and φ_d are rotations, we have

$$|P'_{a',d}(\psi_d(z))| = |(\psi_d \circ \varphi_d^k \circ P_{a,d})'(z)| = |(P_{a,d})'(z)|$$

for $z \in \mathbf{C}$. Since $\varphi_d^t(w)$ is an attracting fixed point of $P_{a,d}$ for t $(0 \le t \le d-1)$, it follows that

$$|P'_{a',d}(\psi_d \circ \varphi_d^t(w))| = |(P_{a,d})'(\varphi_d^t(w))| < 1.$$

Therefore $\psi_d(w)$ is an attracting periodic point of $P_{a',d}$. Because of $\ell < k$, the period d/ℓ tends to infinity as $d \to \infty$. Every W_d^k and every W^k is an open disk of radius 1. Fix $k \in \mathbf{Z}^*$. The center of W_d^k, $\varphi_d^k(0) = -d + de^{i2\pi k/d}$ converges to $2\pi ki$ as $d \to \infty$, which is the center of W^k. Hence it is immediate to show the convergence by definition. $\square$

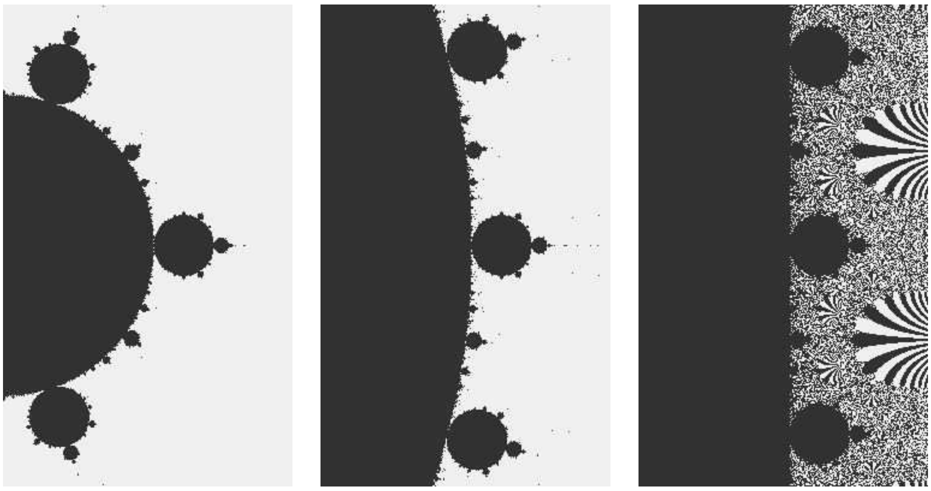

Figure 2. Left: the bifurcation set of $P_{a,5}$, middle: the bifurcation set of $P_{a,25}$, right: the bifurcation set of f_a. Range: $-5 \le \operatorname{Re} a \le 4.6$, $-8 \le \operatorname{Im} a \le 8$.

5. The Hausdorff convergence of $J(P_{a,d})$ to $\widehat{J(f_a)}$

We show the main theorem in this paper.

Theorem 13. *If e^a belongs to an a-component of $\widetilde{M}$, then $J(P_{a,d})$ converges to $\widehat{J(f_a)}$ in the Hausdorff metric.*

Proof. In the case that a belongs to an a-component of M, the claim is shown by the similar argument in [14].

If $a \in B$, then $\operatorname{Re} a < 0$ by definition. We recall that the Baker domain equals $F(f_a)$. Let $F_a = \{z \mid \operatorname{Re} z < \log(|\operatorname{Re} a|/2)\}$. We have

$$\operatorname{Re} f_a(z) < \operatorname{Re} z + \operatorname{Re} a/2$$

for $z \in F_a$. It follows that F_a is contained in the Baker domain. Furthermore, for every compact set K in $F(f_a)$, there exists $n \in \mathbf{N}$ such that $f_a^n(K) \subset F_a$ from Theorem 5.

Let ζ be an attracting periodic point of a holomorphic map h of period p. We say that $D = \{z \mid |z - \zeta| < r\}$ is an absorbing disk of ζ if $\overline{h^p(D)} \subset D$. Hence every absorbing disk is contained in the immediate basin of the attracting cycle. Certainly, every attracting periodic point has absorbing disks.

Choose $d > |a|^2/|\operatorname{Re} a|$. This implies $|a|/d + \operatorname{Re} a/|a| < 0$. Hence $-d$ is an attracting fixed point of $P_{a,d}$ for all $d > |a|^2/|\operatorname{Re} a|$, because we have

$$|P'_{a,d}(-d)|^2 = \left|1 + \frac{a}{d}\right|^2 = 1 + 2\frac{\operatorname{Re} a}{d} + \frac{|a|^2}{d^2} < 1 + \frac{|a|}{d}\left(\frac{|a|}{d} + \frac{\operatorname{Re} a}{|a|}\right) < 1.$$

From the above $a \in B_d$ and thus $F(P_{a,d})$ consists of two components by Proposition 11. The immediate basin of the attracting fixed point $-d$ of $P_{a,d}$ is denoted by

O_d. We also have

$$1 - \left|1 + \frac{a}{d}\right| > \frac{1 - \left|1 + \frac{a}{d}\right|^2}{2} = \frac{|a|}{2d}\left|\frac{|a|}{d} + 2\frac{\operatorname{Re} a}{|a|}\right| > \frac{|\operatorname{Re} a|}{2d}.$$

Moreover, let $D_d = \{z \mid |z + d| < d\sqrt[d]{|\operatorname{Re} a|/2}\}$. An elementary calculation yields, for $z \in D_d$,

$$\begin{aligned}|P_{a,d}(z) + d| &= \left|(z + d)\left(\left(1 + \frac{a}{d}\right) + \frac{1}{d}\left(1 + \frac{z}{d}\right)^d\right)\right| \\ &\le |(z + d)|\left(\left|1 + \frac{a}{d}\right| + \frac{1}{d}\left|\frac{d + z}{d}\right|^d\right) < |z + d|\left(\left|1 + \frac{a}{d}\right| + \frac{|\operatorname{Re} a|}{2d}\right).\end{aligned}$$

We conclude that D_d is an absorbing disk of $-d$. Hence D_d is contained in O_d. The sequence $\{-d + d\sqrt[d]{|\operatorname{Re} a|/2}\}$ is monotonically decreasing and tends to $\log|\operatorname{Re} a|/2$ as $d \to \infty$. Hence it is easy to check that D_d tends to F_a in the sense of Carathéodory. Take an arbitrary compact set K contained in the Baker domain and an arbitrary $\varepsilon > 0$. There exists $n \in \mathbf{N}$ such that $f_a^n(K) \subset F_a - 2\varepsilon$. Since $P_{a,d}$ converges uniformly on compact sets to f_a, there exists N_1 such that

$$|f_a^n(z) - P_{a,d}^n(z)| < \varepsilon$$

for all $n > N_1$ and all $z \in K$. The Carathéodory convergence of D_d to F_a shows that there exists $N \ge N_1$ such that $\overline{U_\varepsilon(f_a^n(K))} \subset D_d$ for all $d > N$. This implies $P_{a,d}^n(K) \subset D_d$ for all $d > N$. Hence we obtain that $K \subset O_d$ for all $d > N$ from the complete invariance of Fatou sets. Therefore O_d converges to the Baker domain in the sense of Carathéodory.

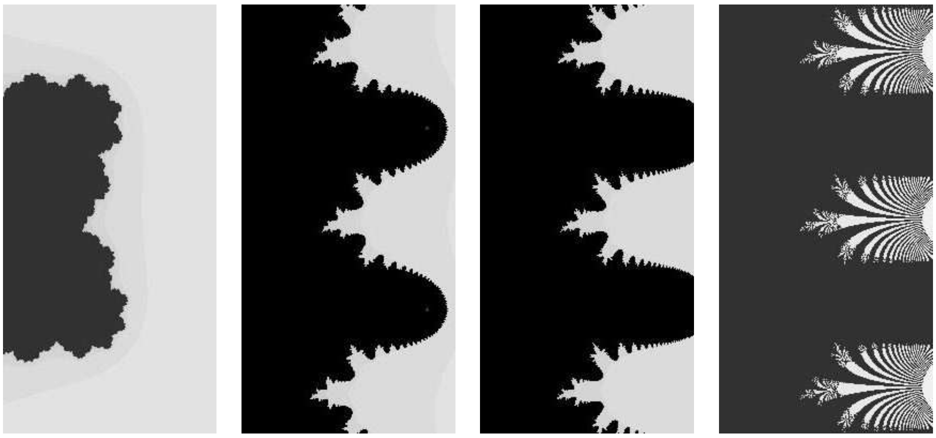

Figure 3. $a = -0.9 + 0.5i$. Left: the Fatou set of $P_{a,5}$ which has an attracting fixed point -5, middle-left: the Fatou set of $P_{a,50}$ which has an attracting fixed point -50, middle-right: the Fatou set of $P_{a,150}$ which has an attracting fixed point -150, right: the Fatou set of f_a which has a Baker domain. Range: $-3 \le \operatorname{Re} z \le 5$, $-8 \le \operatorname{Im} z \le 8$.

Assume $a \in M$ is contained in a w-component of M. We show that $P_{a,d}$ has attracting periodic points for all sufficiently large d. For $\lambda = e^a$, g_λ has an attracting periodic point, say η. Its period is denoted by p. Let ζ_0 be a point of the logarithmic lift of η the absolute value of whose imaginary part is the smallest. Since ζ_0 is a point contained in a wandering domain, there exists $k \in \mathbf{Z}^*$ such that

$$f_a^p(\zeta_0) = \zeta_0 + 2\pi k i,$$

which we denote by ζ_1. Since η is an attracting periodic point, by using an absorbing disk of η, we have r_0 and r_1 satisfying $0 < r_1 < r_0$ such that

$$f_a^p(D_0) \subset D_1 \subset D_0 + 2\pi ki$$

for $D_0 = \{z \mid |z - \zeta_0| < r_0\}$ and $D_1 = \{z \mid |z - \zeta_1| < r_1\}$. Let $4\varepsilon = r_0 - r_1$. Writing $\xi_d = \varphi_d^k(\zeta_0)$, we easily see that ξ_d tends to ζ_1 as $d \to \infty$. Hence there exists N_1 such that $|\xi_d - \zeta_1| < \varepsilon$ for all $d > N_1$. We define a set $E_d = \{z \mid |z - \xi_d| < r_1 + 2\varepsilon\}$. It follows that $D_1 \subset E_d \subset D_0 + 2\pi ki$ for all $d > N_1$. Since $P_{a,d}$ converges uniformly on compact sets to f_a, there exists $N_2 \geq N_1$ such that

$$|P_{a,d}^p(z) - f_a^p(z)| < \varepsilon$$

for all $d > N_2$ and all $z \in D_0$. This implies

$$P_{a,d}^p(D_0) \subset E_d.$$

Writing

$$G = \varphi_d^{-k}(E_d) = \{z \mid |z - \zeta_0| < r_1 + 2\varepsilon\},$$

we have $\overline{P_{a,d}^p(G)} \subset E_d = \varphi_d^k(G)$ since $\overline{G} \subset D_0$. Hence, by Proposition 9, we obtain

$$\begin{aligned}\overline{P_{a,d}^p(E_d)} &\subset P_{a,d}^p(\varphi_d^k(G)) = \varphi_d^k(P_{a,d}^p(G)) \\ &\subset \varphi_d^k(E_d) = \{z \mid |z - \varphi_d^{2k}(\zeta_0)| < r_1 + 2\varepsilon\}.\end{aligned}$$

This gives

$$\overline{P_{a,d}^{2p}(G)} \subset \varphi_d^{2k}(G).$$

Iterating this procedure d times, we have

$$\overline{P_{a,d}^{d\cdot p}(G)} \subset \varphi_d^{d\cdot k}(G) = G.$$

It follows that, for every $d > N_2$, there exists an attracting periodic point of $P_{a,d}$ in G, say η_d. More precisely, η_d is a periodic point of period $p_d = d/\ell$ of $P_{a,d}^p$, where ℓ is the greatest common divisor of d and k. Since we can choose an arbitrary small r_0, η_d tends to ζ_0 as $d \to \infty$. From the inclusion above, we also have

$$\overline{P_{a,d}^{p_d\cdot p}(\varphi_d^{t\cdot k}(G))} \subset \varphi_d^{t\cdot k}(G)$$

for every t $(0 \leq t \leq p_d - 1)$. Since every $\varphi_d^{t\cdot k}(G)$ has only one attracting periodic point of $P_{a,d}$, we have $P_{a,d}^{t\cdot p}(\eta_d) = \varphi_d^{t\cdot k}(\eta_d)$ for every t $(0 \leq t \leq p_d - 1)$. For $-p_d/2 < t \leq p_d/2$ if p_d is even and $-[p_d/2] \leq t \leq [p_d/2]$ if p_d is odd, we denote $\varphi_d^{t\cdot k}(\eta_d)$ by η_d^t. Fixing $t \in \mathbf{Z}$, we see that η_d^t tends to $\zeta_0 + 2\pi tki$ as $d \to \infty$ since $\varphi_d^{t\cdot k}(\zeta_0)$ tends to $\zeta_0 + 2\pi tki$ as $d \to \infty$. From the argument above, we choose $N_3 \geq N_2$ and $r_3 > 0$ such that $H = \{z \mid |z - \zeta_0| < r_3\} \subset F(f_a)$ and $H_d^0 = \{z \mid |z - \eta_d| < r_3\} \subset F(P_{a,d})$ for all $d > N_3$. We also write $H_d^t = \{z \mid |z - \eta_d^t| < r_3\}$. It is clear that H_d^t converges to $H + 2\pi kti$ in the sense of Carathéodory as $d \to \infty$ for each $t \in \mathbf{Z}$. The set $\bigcup_{t=-(p_d/2)+1}^{p_d/2} H_d^t$ for even p_d or $\bigcup_{t=-[p_d/2]}^{[p_d/2]} H_d^t$ for odd p_d is denoted by H_d. Let $D_d = \{z \mid |z + d| < R_d\}$, where $R_d = |\zeta_0 + d| - (r_1 + r_3 + 2\varepsilon)$. We may assume $r_1 + r_3 + 2\varepsilon < \pi/2$. By the choice of ζ_0, we see that $D_d \cap \{\zeta_0 + 2\pi kti \mid t \in \mathbf{Z}\} = \emptyset$ for all $d > N_3$. It follows that, for each d, the number of H_n^t satisfying $D_d \cap H_n^t \neq \emptyset$ is finite. Since D_d converges to $D = \{z \mid \operatorname{Re} z < \operatorname{Re} \zeta_0 - (r_1 + r_3 + 2\varepsilon)\}$, the limit of Carathéodory convergence of H_d does not intersect with D. Assume that there exists a connected open set U

in D^c which is contained in infinitely many H_d. For each $M > 0$, there exist a finite number of t such that $D^c \cap \{z \mid |\operatorname{Im} z| < M\} \cap (\bigcup_d H_d^t) \neq \emptyset$. Furthermore we see

$$\varphi_d^k(\zeta_0) = \operatorname{Re}\zeta_0 - \frac{2\pi k}{d}\operatorname{Im}\zeta_0 + O(d^{-2}) + i\left(2\pi k + \operatorname{Im}\zeta_0 + \frac{2\pi k}{d}\operatorname{Re}\zeta_0 + O(d^{-2})\right).$$

Hence there exists a unique t such that $U \subset H_d^t$ for infinitely many d. Hence U is contained in $H + 2\pi kti$. This shows that H_d converges to $\bigcup_{k\in\mathbf{Z}}(H + 2\pi ki)$ in the sense of Carathéodory.

For each attracting periodic point ζ of g_λ, we denote the logarithmic lift of ζ by L_ζ and, in addition, $\bigcup_\zeta L_\zeta$ by L. By an argument similar to the above, we have $N \in \mathbf{N}$ and $r > 0$ such that:

(i) for each $\eta \in L$, $\{z \mid |z - \eta| < r\} \subset F(f_a)$,
(ii) for $d > N$, $P_{a,d}$ has attracting cycles,
(iii) for every attracting periodic point ξ of $P_{a,d}$ of period p, $P_{a,d}^p(\{z \mid |z - \xi| < r\}) \subset \{z \mid |z - \xi| < r\}$.

We denote the set of all the attracting periodic points of $P_{a,d}$ by S_d. Then $\bigcup_{\xi\in S_d}\{z \mid |z-\xi| < r\}$ converges to $\bigcup_{\eta\in L}\{z \mid |z-\eta| < r\}$ as $d \to \infty$ in the sense of Carathéodory.

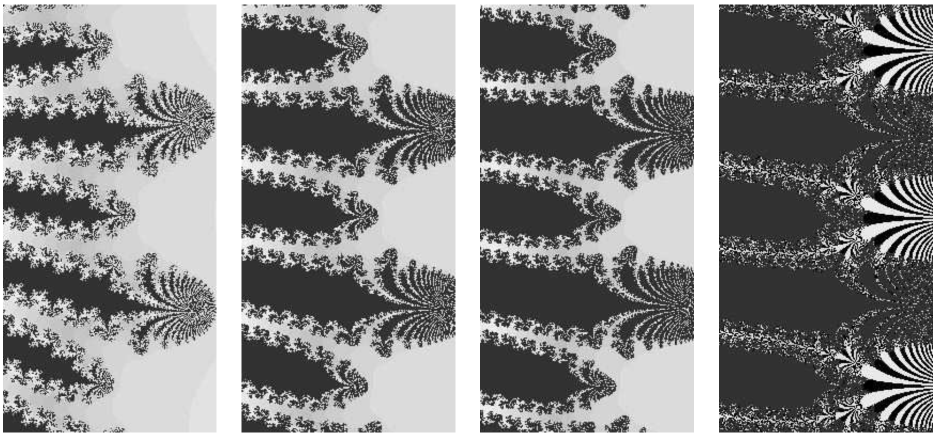

Figure 4. $a = 0.32 + 3.1i$. Left: the Fatou set of $P_{a,25}$ which has an attracting cycle of period 50, middle-left: the Fatou set of $P_{a,50}$ which has an attracting cycle of period 100, middle-right: the Fatou set of $P_{a,100}$ which has an attracting cycle of period 200, right: the Fatou set of f_a which has wandering domains. Range: $-4 \le \operatorname{Re} z \le 4$, $-8 \le \operatorname{Im} z \le 8$.

Let K be a compact set in $F(f_a)$ and denote $\exp(K)$ by K'. Then there exists an attracting periodic point ζ of g_λ which is contained in the derived set of $\{g_\lambda^n(K')\}_{n=0}^\infty$. Choose a positive number ε satisfying $r > 2\varepsilon$. Take an absorbing disk D of ζ of g_λ so that every component of the its logarithmic lift is contained in $\{z \mid |z - \eta| < r - 2\varepsilon\}$ for some $\eta \in L$. There exists $n \in \mathbf{N} \cup \{0\}$ such that $g_\lambda^n(K') \subset D$. This implies $f_a^n(K) \subset \{z \mid |z - \eta| < r - 2\varepsilon\}$ for some $\eta \in L$. Since $P_{a,d}$ converges uniformly on compact sets to f_a, there exists $N_1 \in \mathbf{N}$ such that

$$|P_{a,d}(z) - f_a(z)| < \varepsilon$$

for all $d > N_1$ and all $z \in K$. It follows that

$$P_{a,d}^n(K) \subset \overline{U_\varepsilon(f_a^n(K))} \subset \{z \mid |z - \eta| < r\}$$

for all $d > N_1$. From the Carathéodory convergence we proved above, there exists $N \geq N_1$ such that, for all $d > N$, $P^n_{a,d}(K) \subset \{z \mid |z - \zeta_d| < r\}$ for some attracting periodic point ζ_d of $P_{a,d}$. Since Fatou sets are completely invariant, we have $K \subset F(P_{a,d})$ for all $d > N$. Therefore $F(P_{a,d})$ converges to $F(f_a)$ in the sense of Carathéodory and thus $J(P_{a,d})$ converges to $J(f_a)$ in the Hausdorff distance. □

Remark. We take another family of polynomials $R_{a,d}(z) = z + (1 + z/d)^d + a$. It is clear that $R_{a,d}$ converges uniformly on compact sets to f_a as $d \to \infty$. If $|a| < 1$, that is, $F(f_a)$ only consists of attracting basins, then $J(R_{a,d})$ converges to $\widehat{J(f_a)}$ in the Hausdorff metric by the similar argument of Kisaka [14]. However, it was shown in [23] that if $a = -1$, that is, f_{-1} has a Baker domain, then $J(R_{-1,2d})$ does not converge to $\widehat{J(f_{-1})}$ in the Hausdorff metric.

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