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SOBOLEV AND TRUDINGER TYPE INEQUALITIES ON GRAND MUSIELAK–ORLICZ–MORREY SPACES

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Abstract. Our aim in this paper is to establish generalizations of Sobolev's inequality and Trudinger's inequality for general potentials of functions in grand Musielak–Orlicz–Morrey spaces.

1. Introduction

Grand Lebesgue spaces were introduced in [15] for the study of Jacobian. They play important roles also in the theory of partial differential equations (see [10], [16] and [29], etc.). The generalized grand Lebesgue spaces appeared in [12], where the existence and uniqueness of the non-homogeneous N-harmonic equations were studied. The boundedness of the maximal operator on the grand Lebesgue spaces was studied in [9]. For variable exponent Lebesgue spaces, see [6] and [7]. In [21] and [17], grand Morrey spaces and generalized grand Morrey spaces were introduced. For Morrey spaces, we refer to [24] and [27]. Further, grand Morrey spaces of variable exponent were considered in [11].

On the other hand, the classical Sobolev's inequality for Riesz potentials of L^p-functions (see, e.g. [2, Theorem 3.1.4 (b)]) has been extended to various function spaces. For Morrey spaces, Sobolev's inequality was studied in [1], [27], [5], [25], etc., for Morrey spaces of variable exponent in [3], [13], [14], [22], [23], etc., for grand Morrey spaces in [21] and [17], and also for grand Morrey spaces of variable exponent in [11]. Recently, Sobolev's inequality has been extended by the authors [19] to an inequality for general potentials of functions in Musielak–Orlicz–Morrey spaces.

The classical Trudinger's inequality for Riesz potentials of L^p-functions (see, e.g. [2, Theorem 3.1.4 (c)]) has been also extended to function spaces as above; see [22], [23] for Morrey spaces of variable exponent, [11] for grand Morrey spaces of variable exponent and [20] for Musielak–Orlicz–Morrey spaces.

In this paper, we define (generalized) grand Musielak–Orlicz–Morrey space on a bounded open set in $\mathbf{R}^N$ and give a Sobolev type inequality as well as a Trudinger type inequality for general potentials of functions in such spaces.

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Throughout this paper, let C denote various constants independent of the variables in question. The symbols $g \lesssim h$ and $g \sim h$ means that $g \le Ch$ and $C^{-1}h \le g \le Ch$ for some constant $C > 0$ respectively.

2. Preliminaries

Let G be a bounded open set in $\mathbf{R}^N$ and let d_G denote the diameter of G. We consider a function

$$\Phi(x,t) = t\phi(x,t) \colon G \times [0,\infty) \to [0,\infty)$$

satisfying the following conditions (Φ1)–(Φ4):

(Φ1) $\phi(\,\cdot\,,t)$ is measurable on G for each $t \ge 0$ and $\phi(x,\,\cdot\,)$ is continuous on $[0,\infty)$ for each $x \in G$;

(Φ2) there exists a constant $A_1 \ge 1$ such that

$$A_1^{-1} \le \phi(x,1) \le A_1 \quad \text{for all } x \in G;$$

(Φ3) there exists a constant $\varepsilon_0 > 0$ such that $t \mapsto t^{-\varepsilon_0}\phi(x,t)$ is uniformly almost increasing, namely there exists a constant $A_2 \ge 1$ such that

$$t^{-\varepsilon_0}\phi(x,t) \le A_2 s^{-\varepsilon_0}\phi(x,s)$$

for all $x \in G$ whenever $0 < t < s$;

(Φ4) there exists a constant $A_3 \ge 1$ such that

$$\phi(x,2t) \le A_3\phi(x,t) \quad \text{for all } x \in G \text{ and } t > 0.$$

Note that (Φ3) implies that

$$t^{-\varepsilon}\phi(x,t) \le A_2 s^{-\varepsilon}\phi(x,s)$$

for all $x \in G$ and $0 < \varepsilon \le \varepsilon_0$ whenever $0 < t < s$. Also note that (Φ2), (Φ3) and (Φ4) imply

$$0 < \inf_{x\in G} \phi(x,t) \le \sup_{x \in G} \phi(x,t) < \infty$$

for each $t > 0$ and there exists $\omega > 1$ such that

$$(A_1A_2)^{-1}t^{1+\varepsilon_0} \le \Phi(x,t) \le A_1A_2A_3t^{\omega} \tag{2.1}$$

for $t \ge 1$; in fact we can take $\omega \ge 1 + \log A_3/\log 2$.

We shall also consider the following condition:

(Φ5) for every $\gamma > 0$, there exists a constant $B_\gamma \ge 1$ such that

$$\phi(x,t) \le B_\gamma \phi(y,t)$$

whenever $|x-y| \le \gamma t^{-1/N}$ and $t \ge 1$.

Let $\bar{\phi}(x,t) = \sup_{0 \le s \le t} \phi(x,s)$ and

$$\overline{\Phi}(x,t) = \int_0^t \bar{\phi}(x,r)\,dr.$$

Then $\overline{\Phi}(x,\cdot)$ is convex and

$$\frac{1}{2A_3}\Phi(x,t) \le \overline{\Phi}(x,t) \le A_2\Phi(x,t)$$

for all $x \in G$ and $t \ge 0$.

Example 2.1. Let $p(\cdot)$ and $q_j(\cdot)$, $j = 1, \dots, k$, be measurable functions on G such that

$$1 < p^- := \inf_{x\in G} p(x) \le \sup_{x\in G} p(x) =: p^+ < \infty$$

and

$$-\infty < q_j^- := \inf_{x\in G} q_j(x) \le \sup_{x\in G} q_j(x) =: q_j^+ < \infty, \quad j = 1, \dots k.$$

Set $L(t) := \log(e+t)$, $L^{(1)}(t) = L(t)$ and $L^{(j)}(t) = L(L^{(j-1)}(t))$, $j = 2, \dots$. Then,

$$\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t) = t^{p(x)} \prod_{j=1}^{k} \big(L^{(j)}(t)\big)^{q_j(x)}$$

satisfies $(\Phi1)$, $(\Phi2)$, $(\Phi3)$ with $0 < \varepsilon_0 < p^- - 1$ and $(\Phi4)$. (2.1) holds for any $\omega > p^+$. $\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)$ satisfies $(\Phi5)$ if $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{C_p}{L(1/|x-y|)} \quad (x \ne y)$$

and $q_j(\cdot)$ is $(j+1)$-log-Hölder continuous, namely

$$|q_j(x) - q_j(y)| \le \frac{C_{q_j}}{L^{(j+1)}(1/|x-y|)} \quad (x \ne y)$$

for $j = 1, \dots, k$ (cf. [19, Example 2.1]).

We also consider a function $\kappa(x,r)\colon G \times (0, d_G) \to (0,\infty)$ satisfying the following conditions:

(κ1) $\kappa(x,\cdot)$ is continuous on $(0, d_G)$ for each $x \in G$ and satisfies the uniform doubling condition: there is a constant $Q_1 \ge 1$ such that

$$Q_1^{-1}\kappa(x,r) \le \kappa(x,r') \le Q_1 \kappa(x,r)$$

for all $x \in G$ whenever $0 < r \le r' \le 2r < d_G$;

(κ2) $r \mapsto r^{-\delta}\kappa(x,r)$ is uniformly almost increasing for some $\delta > 0$, namely there is a constant $Q_2 > 0$ such that

$$r^{-\delta}\kappa(x,r) \le Q_2 s^{-\delta}\kappa(x,s)$$

for all $x \in G$ whenever $0 < r < s < d_G$;

(κ3) there is a constant $Q_3 \ge 1$ such that

$$Q_3^{-1} \min(1, r^N) \le \kappa(x,r) \le Q_3$$

for all $x \in G$ and $0 < r < d_G$.

Example 2.2. Let $\nu(\cdot)$ and $\beta(\cdot)$ be functions on G such that $\nu^- := \inf_{x\in G} \nu(x) > 0$, $\nu^+ := \sup_{x\in G} \nu(x) \le N$ and $-c(N - \nu(x)) \le \beta(x) \le c$ for all $x \in G$ and some constant $c > 0$. Then $\kappa(x,r) = r^{\nu(x)}(\log(e+1/r))^{\beta(x)}$ satisfies (κ1), (κ2) and (κ3); we can take any $0 < \delta < \nu^-$ for (κ2).

Given $\Phi(x,t)$ and $\kappa(x,r)$, we define the Musielak–Orlicz–Morrey space $L^{\Phi,\kappa}(G)$ by

$$L^{\Phi,\kappa}(G) = \left\{ f \in L^1_{loc}(G);\ \sup_{x\in G,\, 0<r<d_G} \frac{\kappa(x,r)}{|B(x,r)|} \int_{B(x,r)\cap G} \Phi\big(y, |f(y)|\big)\, dy < \infty \right\}.$$

It is a Banach space with respect to the norm

$$\|f\|_{\Phi,\kappa;G} = \inf\left\{\lambda > 0;\ \sup_{x\in G,\,0<r<d_G} \frac{\kappa(x,r)}{|B(x,r)|}\int_{B(x,r)\cap G} \overline{\Phi}\big(y,|f(y)|/\lambda\big)\,dy \le 1\right\}$$

(cf. [26]).

In case $\kappa(x,r) = r^N$, $L^{\Phi,\kappa}(G)$ is the Musielak–Orlicz space

$$L^{\Phi}(G) = \left\{f \in L^1_{\mathrm{loc}}(G);\ \int_G \Phi(y,|f(y)|)\,dy < \infty\right\}$$

with the norm

$$\|f\|_{\Phi;G} = \inf\left\{\lambda > 0;\ \int_G \overline{\Phi}\big(y,|f(y)|/\lambda\big)\,dy \le 1\right\}.$$

Remark 2.3. The Musielak–Orlicz spaces $L^{\Phi}(G)$ include

- Orlicz spaces defined by Young functions satisfying the doubling condition;
- variable exponent Lebesgue spaces.

The Musielak–Orlicz–Morrey spaces $L^{\Phi,\kappa}(G)$ include Morrey spaces as well as variable exponent Morrey spaces.

3. Grand Musielak–Orlicz–Morrey space

For $\varepsilon \ge 0$, set $\Phi_\varepsilon(x,t) := t^{-\varepsilon}\Phi(x,t) = t^{1-\varepsilon}\phi(x,t)$. Then, $\Phi_\varepsilon(x,t)$ satisfies (Φ1), (Φ2) with the same A_1 and (Φ4) with the same A_3. If $\Phi(x,t)$ satisfies (Φ5), then so does $\Phi_\varepsilon(x,t)$ with the same $\{B_\gamma\}_{\gamma>0}$.

If $0 \le \varepsilon < \varepsilon_0$, then $\Phi_\varepsilon(x,t)$ satisfies (Φ3) with ε_0 replaced by $\varepsilon_0 - \varepsilon$ and the same A_2. It follows that

$$\frac{1}{2A_3}\Phi_\varepsilon(x,t) \le \overline{\Phi_\varepsilon}(x,t) \le A_2\Phi_\varepsilon(x,t) \tag{3.1}$$

for all $x \in G$, $t \ge 0$ and $0 \le \varepsilon \le \varepsilon_0$. By ($\Phi$3), we see that for $0 \le \varepsilon \le \varepsilon_0$

$$\Phi_\varepsilon(x,at)\begin{cases} \le A_2 a\Phi_\varepsilon(x,t) & \text{if } 0 \le a \le 1,\\ \ge A_2^{-1} a\Phi_\varepsilon(x,t) & \text{if } a \ge 1.\end{cases} \tag{3.2}$$

Let

$$\tilde\sigma = \sup\{\sigma \ge 0 \colon r^{N-\sigma}\kappa(x,r)^{-1} \text{ is bounded on } G \times (0,\min(1,d_G))\}.$$

By (κ2), $0 \le \tilde\sigma \le N$. If $\tilde\sigma = 0$, then let $\sigma_0 = 0$; otherwise fix any $\sigma_0 \in (0,\tilde\sigma)$. We also take δ_0 such that $0 < \delta_0 < \delta$ for δ in (κ2).

For $-\delta_0 \le \sigma \le \sigma_0$, set

$$\kappa_\sigma(x,r) = r^\sigma\kappa(x,r)$$

for $x \in G$ and $0 < r < d_G$. Then $\kappa_\sigma(x,r)$ satisfies (κ1), (κ2) and (κ3) with constants independent of σ.

Lemma 3.1. *For $0 \le \varepsilon \le \varepsilon_0$, let*

$$\Phi_\varepsilon^{-1}(x,s) = \sup\{t > 0 : \Phi_\varepsilon(x,t) < s\} \quad (x \in G,\ s > 0).$$

Then there exists $r_0 \in (0,\min(1,d_G))$ such that $\kappa_\sigma(x,r) \le 1$ and

$$\Phi_\varepsilon^{-1}\big(x,\kappa_\sigma(x,r)^{-1}\big) \ge 1$$

for all $x \in G$, $0 < r \le r_0$, $-\delta_0 \le \sigma \le \sigma_0$ and $0 < \varepsilon \le \varepsilon_0$.

Proof. By ($\kappa 2$) and ($\kappa 3$),

$$\kappa_\sigma(x,r) \le Q_2 Q_3 \min(1,d_G)^{-\delta} r^{\delta+\sigma} \le Q_2 Q_3 \min(1,d_G)^{-\delta} r^{\delta-\delta_0}$$

for $x \in G$, $0 < r < \min(1,d_G)$ and $-\delta_0 \le \sigma \le \sigma_0$. Hence, there is $r' \in (0, \min(1,d_G))$ such that $\kappa_\sigma(x,r) \le 1$ for $x \in G$, $0 < r \le r'$ and $-\delta_0 \le \sigma \le \sigma_0$. By (2.1), we see that

$$\Phi_\varepsilon^{-1}\big(x, \kappa_\sigma(x,r)^{-1}\big) \ge C^{-1} \kappa_\sigma(x,r)^{-1/\omega} \ge C'^{-1} r^{-(\delta-\delta_0)/\omega}$$

whenever $x \in G$, $0 < r \le r'$, $-\delta_0 \le \sigma \le \sigma_0$ and $0 < \varepsilon \le \varepsilon_0$ with constants C, $C' > 0$ independent of x, r, σ, ε. Hence the assertion of the lemma holds if we take $r_0 \in (0, r']$ satisfying $r_0^{-(\delta-\delta_0)/\omega} \ge C'$. □

Proposition 3.2. *Assume that* $\Phi(x,t)$ *satisfies* ($\Phi 5$). *If* $0 \le \varepsilon_1 \le \varepsilon_2 \le \varepsilon_0$, $-\delta_0 \le \sigma_j \le \sigma_0$, $j = 1, 2$ *and*

$$\sigma_1 + \frac{\delta-\delta_0}{\omega}\varepsilon_1 \le \sigma_2 + \frac{\delta-\delta_0}{\omega}\varepsilon_2,$$

then $L^{\Phi_{\varepsilon_1},\kappa_{\sigma_1}}(G) \subset L^{\Phi_{\varepsilon_2},\kappa_{\sigma_2}}(G)$ *and*

$$\|f\|_{\Phi_{\varepsilon_2},\kappa_{\sigma_2};G} \le C \|f\|_{\Phi_{\varepsilon_1},\kappa_{\sigma_1};G}$$

for all $f \in L^{\Phi_{\varepsilon_1},\kappa_{\sigma_1}}(G)$ *with* $C > 0$ *independent of* ε_1, ε_2, σ_1, σ_2. *In particular,*

$$L^{\Phi,\kappa}(G) \subset L^{\Phi_\varepsilon,\kappa_\sigma}(G)$$

if $0 \le \varepsilon \le \varepsilon_0$, $-\delta_0 \le \sigma \le \sigma_0$ *and* $\sigma + ((\delta-\delta_0)/\omega)\varepsilon \ge 0$.

Proof. Let $\|f\|_{\Phi_{\varepsilon_1},\kappa_{\sigma_1};G} \le 1$. Then

$$\frac{\kappa_{\sigma_1}(x,r)}{|B(x,r)|} \int_{B(x,r)} \Phi_{\varepsilon_1}\big(y, |f(y)|\big)\, dy \le 1$$

for $x \in G$ and $0 < r < d_G$.

For $x \in G$ and $0 < r < d_G$, let

$$k(x,r) = \Phi_{\varepsilon_1}^{-1}\big(x, \kappa_{\sigma_1}(x,r)^{-1}\big)$$

and

$$I(x,r) = \int_{B(x,r)} \Phi_{\varepsilon_2}\big(y, |f(y)|\big)\, dy.$$

We write $I(x,r) = I_1(x,r) + I_2(x,r)$, where

$$I_1(x,r) = \int_{B(x,r)\cap\{y: |f(y)| \le k(x,r)\}} \Phi_{\varepsilon_2}\big(y, |f(y)|\big)\, dy$$

and

$$I_2(x,r) = \int_{B(x,r)\cap\{y: |f(y)| > k(x,r)\}} \Phi_{\varepsilon_2}\big(y, |f(y)|\big)\, dy.$$

If $|f(y)| \le k(x,r)$, then

$$\Phi_{\varepsilon_2}(y, |f(y)|) \le A_2 \Phi_{\varepsilon_2}(y, k(x,r)) = A_2 k(x,r)^{\varepsilon_1-\varepsilon_2} \Phi_{\varepsilon_1}(y, k(x,r)).$$

Let $r_0 \in (0, \min(1,d_G))$ be the number given in Lemma 3.1. Then, (3.2) implies

$$k(x,r) \le C\kappa_{\sigma_1}(x,r)^{-1} \le C r^{-N}$$

for $0 < r \le r_0$ with constants independent of x, σ_1, ε_1. Hence, by (Φ5), there is a constant $B > 0$ independent of x, σ_1, ε_1, such that

$$\Phi_{\varepsilon_1}(y, k(x,r)) \le B\Phi_{\varepsilon_1}(x, k(x,r))$$

whenever $|x-y| < r \le r_0$. Therefore,

$$I_1(x,r) \le C|B(x,r)|k(x,r)^{\varepsilon_1-\varepsilon_2}\Phi_{\varepsilon_1}(x, k(x,r)) = C|B(x,r)|k(x,r)^{\varepsilon_1-\varepsilon_2}\kappa_{\sigma_1}(x,r)^{-1}$$

for $0 < r \le r_0$.

On the other hand, if $|f(y)| > k(x,r)$, then

$$\Phi_{\varepsilon_2}(y, |f(y)|) = |f(y)|^{\varepsilon_1-\varepsilon_2}\Phi_{\varepsilon_1}(y, |f(y)|) \le k(x,r)^{\varepsilon_1-\varepsilon_2}\Phi_{\varepsilon_1}(y, |f(y)|),$$

so that

$$I_2(x,r) \le k(x,r)^{\varepsilon_1-\varepsilon_2}\int_{B(x,r)} \Phi_{\varepsilon_1}(y, |f(y)|)\,dy \le |B(x,r)|k(x,r)^{\varepsilon_1-\varepsilon_2}\kappa_{\sigma_1}(x,r)^{-1}$$

for $0 < r \le r_0$.

Therefore,

$$I(x,r) \le C|B(x,r)|k(x,r)^{\varepsilon_1-\varepsilon_2}\kappa_{\sigma_1}(x,r)^{-1},$$

which implies

$$\frac{\kappa_{\sigma_2}(x,r)}{|B(x,r)|}\int_{B(x,r)} \Phi_{\varepsilon_2}\big(y, |f(y)|\big)\,dy \le Cr^{\sigma_2-\sigma_1}k(x,r)^{\varepsilon_1-\varepsilon_2}$$

for $0 < r \le r_0$. Since

$$k(x,r)^{-1} \le Cr^{(\delta-\delta_0)/\omega}$$

and $\sigma_2 - \sigma_1 + ((\delta-\delta_0)/\omega)(\varepsilon_2-\varepsilon_1) \ge 0$ by assumption,

$$\frac{\kappa_{\sigma_2}(x,r)}{|B(x,r)|}\int_{B(x,r)} \Phi_{\varepsilon_2}\big(y, |f(y)|\big)\,dy \le Cr^{\sigma_2-\sigma_1+((\delta-\delta_0)/\omega)(\varepsilon_2-\varepsilon_1)} \le C$$

for $0 < r \le r_0$ with positive constants C's independent of x, σ_j, ε_j $(j = 1, 2)$.

In case $r_0 < r < d_G$, we see

$$\begin{aligned} I(x,r) &\le A_2\int_{B(x,r)} \Phi_{\varepsilon_2}(y, 1)\,dy + \int_{B(x,r)} \Phi_{\varepsilon_1}(y, |f(y)|)\,dy \\ &\le A_1A_2|B(x,r)| + |B(x,r)|\kappa_{\sigma_1}(x,r)^{-1}, \end{aligned}$$

so that

$$\frac{\kappa_{\sigma_2}(x,r)}{|B(x,r)|}\int_{B(x,r)} \Phi_{\varepsilon_2}\big(y, |f(y)|\big)\,dy \le A_1A_2\kappa_{\sigma_2}(x,r) + r^{\sigma_2-\sigma_1} \le C$$

with C independent of r, x, σ_1 σ_2.

Therefore, $\|f\|_{\Phi_{\varepsilon_2},\kappa_{\sigma_2};G} \le C$ with $C > 0$ independent of ε_1, ε_2, σ_1, σ_2. □

Let $\eta(\varepsilon)$ be an increasing positive function on $(0,\infty)$ such that $\eta(0+) = 0$. Let $\xi(\varepsilon)$ be a function on $(0, \varepsilon_1]$ with some $\varepsilon_1 \in (0, \varepsilon_0/2]$ such that $-\delta_0 \le \xi(\varepsilon) \le \sigma_0$ for $0 < \varepsilon \le \varepsilon_1$, $\xi(0+) = 0$ and $\varepsilon \mapsto \xi(\varepsilon) + ((\delta-\delta_0)/\omega)\varepsilon$ is non-decreasing; in particular, $\xi(\varepsilon) + ((\delta-\delta_0)/\omega)\varepsilon \ge 0$ for $0 < \varepsilon \le \varepsilon_1$.

Given $\Phi(x,t)$, $\kappa(x,r)$, $\eta(\varepsilon)$ and $\xi(\varepsilon)$, the associated (generalized) grand Musielak–Orlicz–Morrey space is defined by (cf. [17] for generalized grand Morrey space)

$$\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G) = \left\{ f \in \bigcap_{0<\varepsilon\le\varepsilon_1} L^{\Phi_\varepsilon,\kappa_{\xi(\varepsilon)}}(G);\ \|f\|_{\Phi,\kappa;\eta,\xi;G} < \infty \right\},$$

where

$$\|f\|_{\Phi,\kappa;\eta,\xi;G} = \sup_{0<\varepsilon\le\varepsilon_1} \eta(\varepsilon)\|f\|_{\Phi_\varepsilon,\kappa_{\xi(\varepsilon)};G}.$$

$\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ is a Banach space with the norm $\|f\|_{\Phi,\kappa;\eta,\xi;G}$. Note that, in view of Proposition 3.2, this space is determined independent of the choice of ε_1.

In case $\xi(\varepsilon)\equiv 0$, the symbol ξ may be omitted. If $\kappa(x,r) = r^N$ and $\xi(\varepsilon)\equiv 0$, then the symbol κ will be also omitted; namely

$$\widetilde{L}^{\Phi}_{\eta}(G) = \left\{ f \in \bigcap_{0<\varepsilon\le\varepsilon_0} L^{\Phi_\varepsilon}(G);\ \|f\|_{\Phi;\eta;G} := \sup_{0<\varepsilon\le\varepsilon_0} \eta(\varepsilon)\|f\|_{\Phi_\varepsilon;G} < \infty \right\}.$$

This space may be called a grand Musielak–Orlicz space.

Remark 3.3. The grand Musielak–Orlicz space $\widetilde{L}^{\Phi}_{\eta}(G)$ include the following spaces:

- generalized grand Lebesgue spaces introduced in [4];
- grand Orlicz spaces introduced in [18] where $\Phi(x,t) = \Phi(t)$ satisfying

$$\sup_{0<\varepsilon\le\varepsilon_0} \eta(\varepsilon) \int_1^\infty t^{-N-\varepsilon}\Phi(t)\,\frac{dt}{t} < \infty$$

  (see also [8]).

The (generalized) grand Musielak–Orlicz–Morrey space $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ include also the following spaces:

- grand Morrey spaces introduced in [21] where $\xi(\varepsilon)\equiv 0$;
- grand grand Morrey spaces introduced in [28] and generalized grand Morrey spaces introduced in [17] where $\xi(\varepsilon)$ is an increasing positive function on $(0,\infty)$.

4. Boundedness of the maximal operator

Hereafter, we shall always assume that $\Phi(x,t)$ satisfies (Φ5). For a nonnegative $f \in L^1_{\mathrm{loc}}(G)$, $x\in G$, $0<r<d_G$ and $\varepsilon>0$, set

$$I(f;x,r) := \frac{1}{|B(x,r)|}\int_{B(x,r)\cap G} f(y)\,dy$$

and

$$J_\varepsilon(f;x,r) := \frac{1}{|B(x,r)|}\int_{B(x,r)\cap G} \Phi_\varepsilon\big(y,f(y)\big)\,dy.$$

We show a Jensen type inequality for functions in $L^{\Phi_\varepsilon,\kappa_\sigma}(G)$.

Lemma 4.1. *There exists a constant $C>0$ (independent of ε and σ) such that*

$$\Phi_\varepsilon(x, I(f;x,r)) \le C J_\varepsilon(f;x,r)$$

for all $x\in G$, $0<r<d_G$, $0<\varepsilon\le\varepsilon_0$ and for all nonnegative $f\in L^1_{\mathrm{loc}}(G)$ such that $f(y)\ge 1$ or $f(y)=0$ for each $y\in G$ and $\|f\|_{\Phi_\varepsilon,\kappa_\sigma;G}\le 1$ with $-\delta_0\le\sigma\le\sigma_0$.

Proof. Let f be as in the statement of the lemma and let $I = I(f;x,r)$ and $J_\varepsilon = J_\varepsilon(f;x,r)$ for $x\in G$, $0<r<d_G$ and $0<\varepsilon\le\varepsilon_0$. Note that $\|f\|_{\Phi_\varepsilon,\kappa_\sigma;G}\le 1$ implies $J_\varepsilon \le 2A_3\kappa_\sigma(x,r)^{-1}$ by (3.1).

By (Φ2) and (3.2), $\Phi_\varepsilon\big(y, f(y)\big) \ge (A_1A_2)^{-1} f(y)$, since $f(y) \ge 1$ or $f(y) = 0$. Hence $I \le A_1A_2J_\varepsilon$. Thus, if $J_\varepsilon \le 1$, then

$$\Phi_\varepsilon(x, I) \le (A_1A_2J_\varepsilon)A_2\phi(x, A_1A_2) \le CJ_\varepsilon.$$

Next, suppose $J_\varepsilon > 1$. Since $\Phi_\varepsilon(x,t) \to \infty$ as $t \to \infty$, there exists $K_\varepsilon \ge 1$ such that

$$\Phi_\varepsilon(x, K_\varepsilon) = \Phi_\varepsilon(x, 1)J_\varepsilon.$$

Then $K_\varepsilon \le A_2J_\varepsilon$ by (3.2). With this K_ε, we have

$$\int_{B(x,r)\cap G} f(y)\,dy \le K_\varepsilon|B(x,r)| + A_2 \int_{B(x,r)\cap G} f(y) \frac{f(y)^{-\varepsilon}\phi\big(y, f(y)\big)}{K_\varepsilon^{-\varepsilon}\phi(y, K_\varepsilon)}\,dy.$$

Since $\kappa_\sigma(x,r)J_\varepsilon \le 2A_3$,

$$1 \le K_\varepsilon \le A_2J_\varepsilon \le 2A_2A_3\kappa_\sigma(x,r)^{-1} \le Cr^{-N}$$

with a constant $C > 0$ independent of ε and σ. Hence, by (Φ5) there is $\beta \ge 1$, independent of f, x, r, ε and σ such that

$$\phi(x, K_\varepsilon) \le \beta\phi(y, K_\varepsilon)$$

for all $y \in B(x,r)$. Thus, we have

$$\begin{aligned}\int_{B(x,r)\cap G} f(y)\,dy &\le K_\varepsilon|B(x,r)| + \frac{A_2\beta}{K_\varepsilon^{-\varepsilon}\phi(x,K_\varepsilon)} \int_{B(x,r)\cap G} \Phi_\varepsilon\big(y, f(y)\big)\,dy \\ &= K_\varepsilon|B(x,r)| + A_2\beta|B(x,r)|\frac{J_\varepsilon}{K_\varepsilon^{-\varepsilon}\phi(x,K_\varepsilon)}.\end{aligned}$$

Since

$$K_\varepsilon^{-\varepsilon}\phi(x, K_\varepsilon) = K_\varepsilon^{-1}\Phi_\varepsilon(x, K_\varepsilon) = K_\varepsilon^{-1}J_\varepsilon\Phi_\varepsilon(x,1) \ge A_1^{-1}K_\varepsilon^{-1}J_\varepsilon,$$

it follows that

$$I \le (1 + A_1A_2\beta)K_\varepsilon,$$

so that by (Φ2), (Φ3) and (Φ4)

$$\Phi_\varepsilon(x, I) \le C\Phi_\varepsilon(x, K_\varepsilon) \le CJ_\varepsilon$$

with constants $C > 0$ independent of f, x, r, ε and σ as required. □

For a locally integrable function f on G, the Hardy–Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)\cap G} |f(y)|\,dy.$$

The following lemma can be proved in a way similar to the proof of [25, Theorem 1]:

Lemma 4.2. *Let $p_0 > 1$ and $-\delta_0 \le \sigma \le \sigma_0$. Then there exists a constant $C > 0$ independent of σ for which the following holds: If f is a measurable function such that*

$$\int_{B(x,r)\cap G} |f(y)|^{p_0}\,dy \le |B(x,r)|\kappa_\sigma(x,r)^{-1}$$

for all $x \in G$ and $0 < r < d_G$, then

$$\int_{B(x,r)\cap G} [Mf(y)]^{p_0}\,dy \le C|B(x,r)|\kappa_\sigma(x,r)^{-1}$$

for all $x \in G$ and $0 < r < d_G$.

Lemma 4.3. *There is a constant $C > 0$ (independent of ε and σ) such that*

$$\|Mf\|_{\Phi_\varepsilon,\kappa_\sigma;G} \le C\|f\|_{\Phi_\varepsilon,\kappa_\sigma;G}$$

for all $f \in L^{\Phi_\varepsilon,\kappa_\sigma}(G)$ whenever $0 < \varepsilon \le \varepsilon_0/2$ and $-\delta_0 \le \sigma \le \sigma_0$.

Proof. Set $p_0 = 1 + \varepsilon_0/2$ and consider the function

$$\tilde{\Phi}(x,t) = \Phi(x,t)^{1/p_0}.$$

Then $\tilde{\Phi}(x,t)$ also satisfies all the conditions (Φj), $j = 1, 2, \ldots, 5$ with ε_0 replaced by $\varepsilon_0' = \varepsilon_0/(2+\varepsilon_0)$. In fact, it trivially satisfies (Φj) for $j = 1, 2, 4, 5$. Since

$$t^{-\varepsilon_0'}t^{-1}\tilde{\Phi}(x,t) = [t^{-\varepsilon_0}\phi(x,t)]^{1/p_0},$$

condition $(\Phi 3)$ implies that $\tilde{\Phi}(x,t)$ satisfies $(\Phi 3)$ with ε_0 replaced by ε_0'.

Let $0 < \varepsilon \le \varepsilon_0/2$, $-\delta_0 \le \sigma \le \sigma_0$ and $f \ge 0$ and $\|f\|_{\Phi_\varepsilon,\kappa_\sigma;G} \le 1$. Let $f_1 = f\chi_{\{z: f(z)\ge 1\}}$ and $f_2 = f - f_1$, where χ_E is the characteristic function of E.

Since $\Phi_\varepsilon(x,t) \ge 1/(A_1A_2)$ for $t \ge 1$, we see that

$$\tilde{\Phi}_{\varepsilon/p_0}(x,t) = \Phi_\varepsilon(x,t)^{1/p_0} \le (A_1A_2)^{1-1/p_0}\Phi_\varepsilon(x,t)$$

if $t \ge 1$, so that

$$\int_{B(x,r)\cap G} \tilde{\Phi}_{\varepsilon/p_0}(y, f_1(y))\, dy \le 2(A_1A_2)^{1-1/p_0}A_3|B(x,r)|\kappa_\sigma(x,r)^{-1}$$

for every $x \in G$ and $0 < r < d_G$. Hence $\|f_1\|_{\tilde{\Phi}_{\varepsilon/p_0},\kappa_\sigma;G} \le c_0$ with $c_0 > 0$ independent of ε and σ.

Let $F_\varepsilon(x) = \Phi_\varepsilon(x, f(x))$. Then $\tilde{\Phi}_{\varepsilon/p_0}(x, f(x)) = F_\varepsilon(x)^{1/p_0}$. Applying Lemma 4.1 to $\tilde{\Phi}_{\varepsilon/p_0}$ and f_1/c_0, we have

$$\Phi_\varepsilon(x, Mf_1(x)) = \left[\tilde{\Phi}_{\varepsilon/p_0}\big(x, Mf_1(x)\big)\right]^{p_0} \le C[M(F_\varepsilon^{1/p_0})(x)]^{p_0}.$$

On the other hand, since $Mf_2 \le 1$, we have by $(\Phi 2)$ and $(\Phi 3)$

$$\Phi_\varepsilon(x, Mf_2(x)) \le A_1A_2.$$

Thus, we obtain

$$\Phi_\varepsilon\big(x, Mf(x)\big) \le C\left\{\left[M(F_\varepsilon^{1/p_0})(x)\right]^{p_0} + 1\right\}$$

for $x \in G$ with a constant $C > 0$ independent of f and ε. Hence

$$\int_{B(x,r)\cap G} \Phi_\varepsilon\big(y, Mf(y)\big)\, dy \le C\left\{\int_{B(x,r)\cap G} \left[M(F_\varepsilon^{1/p_0})(y)\right]^{p_0} dy + |B(x,r)|\right\}$$

for $x \in G$ and $0 < r < d_G$. Since $\|f\|_{\Phi_\varepsilon,\kappa_\sigma;G} \le 1$ and $\Phi_\varepsilon(y, f(y)) = F_\varepsilon(y) = (F_\varepsilon^{1/p_0}(y))^{p_0}$, Lemma 4.2 implies

$$\int_{B(x,r)\cap G} \left[M(F_\varepsilon^{1/p_0})(y)\right]^{p_0} dy \le C|B(x,r)|\kappa_\sigma(x,r)^{-1}$$

with a constant $C > 0$ independent of x, r, ε, σ. Hence,

$$\int_{B(x,r)\cap G} \Phi_\varepsilon\big(y, Mf(y)\big)\, dy \le C|B(x,r)|\kappa_\sigma(x,r)^{-1},$$

which shows

$$\|Mf\|_{\Phi_\varepsilon,\kappa_\sigma;G} \le C\|f\|_{\Phi_\varepsilon,\kappa_\sigma;G}$$

with a constant $C > 0$ independent of ε and σ. □

From this lemma we obtain the boundedness of the maximal operator on $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$.

Theorem 4.4. *The maximal operator M is bounded from $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ into itself; namely there exists a constant $C > 0$ such that*

$$\|Mf\|_{\Phi,\kappa;\eta,\xi;G} \le C\|f\|_{\Phi,\kappa;\eta,\xi;G}$$

for all $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$.

Corollary 4.5. *If $\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)$ satisfies the conditions in Example 2.1, then the maximal operator M is bounded from $\widetilde{L}^{\Phi_{p(\cdot),\{q_j(\cdot)\}}}_{\eta}(G)$ into itself.*

5. Sobolev type inequality

Lemma 5.1. [19, Lemma 5.1] *Let $F(x,t)$ be a positive function on $G \times (0,\infty)$ satisfying the following conditions:*

(F1) *$F(x,\cdot)$ is continuous on $(0,\infty)$ for each $x \in G$;*

(F2) *there exists a constant $K_1 \ge 1$ such that*

$$K_1^{-1} \le F(x,1) \le K_1 \quad \text{for all } x \in G;$$

(F3) *$t \mapsto t^{-\varepsilon'}F(x,t)$ is uniformly almost increasing for $\varepsilon' > 0$; namely there exists a constant $K_2 \ge 1$ such that*

$$t^{-\varepsilon'}F(x,t) \le K_2 s^{-\varepsilon'}F(x,s) \quad \text{for all } x \in G \quad \text{whenever } 0 < t < s.$$

Set

$$F^{-1}(x,s) = \sup\{t > 0\,;\, F(x,t) < s\}$$

for $x \in G$ and $s > 0$. Then:

(1) *$F^{-1}(x,\cdot)$ is non-decreasing;*

(2) *$F^{-1}(x,\lambda s) \le (K_2\lambda)^{1/\varepsilon'}F^{-1}(x,s)$ for all $x \in G$, $s > 0$ and $\lambda \ge 1$;*

(3) *$F(x,F^{-1}(x,t)) = t$ for all $x \in G$ and $t > 0$;*

(4) *$K_2^{-1/\varepsilon'}t \le F^{-1}(x,F(x,t)) \le K_2^{2/\varepsilon'}t$ for all $x \in G$ and $t > 0$;*

(5) *$\min\left\{1, \left(\dfrac{s}{K_1K_2}\right)^{1/\varepsilon'}\right\} \le F^{-1}(x,s) \le \max\{1,(K_1K_2s)^{1/\varepsilon'}\}$ for all $x \in G$ and $s > 0$.*

Remark 5.2. $F(x,t) = \Phi_\varepsilon(x,t)$ $(0 < \varepsilon \le \varepsilon_0)$ satisfies (F1), (F2) and (F3) with $\varepsilon' = 1$, $K_1 = A_1$ and $K_2 = A_2$ and $F(x,t) = \kappa_\sigma(x,t)$ $(-\delta_0 \le \sigma \le \sigma_0)$ satisfies (F1), (F2) and (F3) with $\varepsilon' = \delta - \delta_0$, $K_1 = Q_1$ and $K_2 = Q_2$.

Lemma 5.3. *There exists a constant $C > 0$ such that*

$$\frac{\eta(\varepsilon)}{|B(x,r)|}\int_{B(x,r)\cap G} f(y)\,dy \le C\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,r)^{-1})$$

for all $x \in G$, $0 < r < d_G$, $0 < \varepsilon \le \varepsilon_1$ and nonnegative functions f on G such that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

Proof. Let f be a nonnegative function on G such that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$. Then we have by (3.1)

$$\frac{\kappa_{\xi(\varepsilon)}(x,r)}{|B(x,r)|}\int_{B(x,r)\cap G}\Phi_\varepsilon(y,\eta(\varepsilon)f(y))\,dy \le 2A_3$$

for all $x\in G$, $0<r<d_G$ and $0<\varepsilon\le\varepsilon_1$. Fix ε and let $f_1 = f\chi_{\{x:\eta(\varepsilon)f(x)\ge 1\}}$ and $f_2 = f - f_1$. By Lemma 4.1,

$$\Phi_\varepsilon\left(x,\frac{1}{|B(x,r)|}\int_{B(x,r)\cap G}\eta(\varepsilon)f_1(y)\,dy\right)\le C\kappa_{\xi(\varepsilon)}(x,r)^{-1}$$

for all $x\in G$, $0<r<d_G$ and $0<\varepsilon\le\varepsilon_0/2$ with a constant $C>0$ independent of x, r, ε. Since

$$\Phi_\varepsilon\left(x,\frac{1}{|B(x,r)|}\int_{B(x,r)\cap G}\eta(\varepsilon)f_2(y)\,dy\right)\le A_2\Phi_\varepsilon(x,1)\le A_1A_2,$$

we have

$$\Phi_\varepsilon\left(x,\frac{1}{|B(x,r)|}\int_{B(x,r)\cap G}\eta(\varepsilon)f(y)\,dy\right)\le C_1\kappa_{\xi(\varepsilon)}(x,r)^{-1}$$

with a constant $C_1\ge 1$ independent of x, r, ε. Hence, we find by Lemma 5.1 with $F=\Phi_\varepsilon$ and $\varepsilon'=1$

$$\begin{aligned}\frac{1}{|B(x,r)|}\int_{B(x,r)\cap G}\eta(\varepsilon)f(y)\,dy &\le A_2\Phi_\varepsilon^{-1}\left(x,C_1\kappa_{\xi(\varepsilon)}(x,r)^{-1}\right)\\ &\le C_1A_2^2\Phi_\varepsilon^{-1}\left(x,\kappa_{\xi(\varepsilon)}(x,r)^{-1}\right),\end{aligned}$$

as required. □

As a potential kernel, we consider a function

$$J(x,r)\colon G\times(0,d_G)\to[0,\infty)$$

satisfying the following conditions:

(J1) $J(\cdot,r)$ is measurable on G for each $r\in(0,d_G)$;
(J2) $J(x,\cdot)$ is non-increasing on $(0,d_G)$ for each $x\in G$;
(J3) $\int_0^{d_G}J(x,r)r^{N-1}dr\le J_0<\infty$ for every $x\in G$.

Example 5.4. Let $\alpha(\cdot)$ be a measurable function on G such that

$$0<\alpha^- := \inf_{x\in G}\alpha(x)\le\sup_{x\in G}\alpha(x) =: \alpha^+ < N.$$

Then, $J(x,r)=r^{\alpha(x)-N}$ satisfies (J1), (J2) and (J3).

For a nonnegative measurable function f on G, its J-potential Jf is defined by

$$Jf(x)=\int_G J(x,|x-y|)f(y)\,dy\quad(x\in G).$$

Set

$$\overline{J}(x,r)=\frac{N}{r^N}\int_0^r J(x,\rho)\rho^{N-1}d\rho$$

for $x\in G$ and $0<r<d_G$. Then $J(x,r)\le\overline{J}(x,r)$. Further, $\overline{J}(x,\cdot)$ is non-increasing and continuous on $(0,d_G)$ for each $x\in G$. Also, set

$$Y_J(x,r)=r^N\overline{J}(x,r)$$

for $x \in G$ and $0 < r < d_G$.

We consider the following condition:

($\Phi\kappa J$) there exist constants $\delta' > 0$ and $A_4 \geq 1$ such that

$$s^{\delta'} Y_J(x,s)\Phi_\varepsilon^{-1}(x,\kappa_\sigma(x,s)^{-1}) \leq A_4 t^{\delta'} Y_J(x,t)\Phi_\varepsilon^{-1}(x,\kappa_\sigma(x,t)^{-1})$$

for all $x \in G$ whenever $0 < t < s < d_G$, $0 < \varepsilon \leq \varepsilon_0/2$, $-\delta_0 \leq \sigma \leq \sigma_0$ and $\sigma + ((\delta - \delta_0)/\omega)\varepsilon \geq 0$.

Lemma 5.5. *Assume* ($\Phi\kappa J$). *Then there exists a constant* $C > 0$ *such that*

$$\int_r^{d_G} \rho^N \Phi_\varepsilon^{-1}(x,\kappa_\sigma(x,\rho)^{-1})\, d(-\overline{J}(x,\cdot))(\rho) \leq C Y_J(x,r)\Phi_\varepsilon^{-1}(x,\kappa_\sigma(x,r)^{-1})$$

for all $x \in G$, $0 < r \leq d_G/2$, $0 < \varepsilon \leq \varepsilon_0/2$ *and* $-\min(\delta_0, ((\delta - \delta_0)/\omega)\varepsilon) \leq \sigma \leq \sigma_0$.

Proof. We follow the proof of [19, Lemma 6.2], noting that the constants are independent of ε and σ. □

Lemma 5.6. *Assume* ($\Phi\kappa J$). *Then there exists a constant* $C > 0$ *such that*

$$\eta(\varepsilon)\int_{G\setminus B(x,r)} J(x,|x-y|)f(y)\,dy \leq C Y_J(x,r)\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,r)^{-1})$$

for all $x \in G$, $0 < r \leq d_G/2$, $0 < \varepsilon \leq \varepsilon_1$ *and* $f \geq 0$ *satisfying* $\|f\|_{\Phi,\kappa;\eta,\xi;G} \leq 1$.

Proof. By the integration by parts, we have

$$\begin{aligned}
&\int_{G\setminus B(x,r)} J(x,|x-y|)f(y)\,dy \\
&\leq J(x,d_G - 0)\int_G f(y)\,dy + \int_r^{d_G}\left(\int_{B(x,\rho)\cap G} f(y)\,dy\right) d(-J(x,\cdot))(\rho),
\end{aligned}$$

where $J(x,d_G - 0) = \lim_{\rho\to d_G-0} J(x,\rho)$. Hence, by Lemma 5.3, we have

$$\begin{aligned}
\eta(\varepsilon)\int_{G\setminus B(x,r)} J(x,|x-y|)f(y)\,dy \leq C\Big\{& Y_J(x,d_G)\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,d_G)^{-1}) \\
&+ \int_r^{d_G} |B(x,\rho)|\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,\rho)^{-1})\, d(-J(x,\cdot))(\rho)\Big\}.
\end{aligned}$$

Hence by ($\Phi\kappa J$) and the previous lemma we obtain the required result. □

Lemma 5.7. *Assume* ($\Phi\kappa J$). *Then there exists a constant* $C > 0$ *such that*

$$\eta(\varepsilon)Jf(x) \leq C\left\{\eta(\varepsilon)Mf(x)Y_J\Big(x,\kappa_{\xi(\varepsilon)}^{-1}\big(x,\Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))^{-1}\big)\Big) + 1\right\}$$

for all $x \in G$, $0 < \varepsilon \leq \varepsilon_1$ *and* $f \geq 0$ *satisfying* $\|f\|_{\Phi,\kappa;\eta,\xi;G} \leq 1$.

Proof. Let f be a nonnegative function on G such that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \leq 1$. For $0 < r \leq d_G/2$, we write

$$\begin{aligned}
Jf(x) &= \int_{B(x,r)\cap G} J(x,|x-y|)f(y)\,dy + \int_{G\setminus B(x,r)} J(x,|x-y|)f(y)\,dy \\
&= J_1(x) + J_2(x).
\end{aligned}$$

First note that

$$J_1(x) \leq C Y_J(x,r)Mf(x)$$

(see, e.g., [30, p. 63, (16)]). By Lemma 5.6, we have

$$\eta(\varepsilon)J_2(x) \le CY_J(x,r)\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,r)^{-1}).$$

Hence

$$\eta(\varepsilon)Jf(x) \le CY_J(x,r)\left\{\eta(\varepsilon)Mf(x) + \Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,r)^{-1})\right\} \tag{5.1}$$

for $x \in G$, $0 < r \le d_G/2$ and $0 < \varepsilon \le \varepsilon_1$.

We consider two cases.

Case 1: $d_G/2 < \kappa_{\xi(\varepsilon)}^{-1}\big(x, \Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))^{-1}\big)$. In this case, let $r = d_G/2$. Since

$$\Phi_\varepsilon(x,\eta(\varepsilon)Mf(x)) \le Q_2\kappa_{\xi(\varepsilon)}(x,d_G/2)^{-1} \le Q_2Q_3\max\big(1,(d_G/2)^{-N}\big),$$

it follows that $\eta(\varepsilon)Mf(x) \le C_1$ with a constant $C_1 > 0$ independent of x and ε. Also,

$$\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,r)^{-1}) = \Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,d_G/2)^{-1}) \le C_2$$

with a constant $C_2 > 0$ independent of x and ε. Hence, by (5.1) and (J3),

$$\eta(\varepsilon)Jf(x) \le C$$

with a constant $C > 0$ independent of x and ε.

Case 2: $d_G/2 \ge \kappa_{\xi(\varepsilon)}^{-1}\big(x, \Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))^{-1}\big)$. In this case, take

$$r = \kappa_{\xi(\varepsilon)}^{-1}\big(x, \Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))^{-1}\big).$$

Then $\kappa_{\xi(\varepsilon)}(x,r)^{-1} = \Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))$, so that by Lemma 5.1(4)

$$\Phi_\varepsilon^{-1}(x,\kappa_{\xi(\varepsilon)}(x,r)^{-1}) \le C\eta(\varepsilon)Mf(x)$$

with a constant $C > 0$ independent of x and ε. Hence, by (5.1)

$$\begin{aligned}\eta(\varepsilon)Jf(x) &\le CY_J(x,r)\eta(\varepsilon)Mf(x)\\ &= C\eta(\varepsilon)Mf(x)Y_J\Big(x,\kappa_{\xi(\varepsilon)}^{-1}\big(x,\Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))^{-1}\big)\Big)\end{aligned}$$

with a constant $C > 0$ independent of x and ε. □

The following theorem gives a Sobolev type inequality for potentials Jf of $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$. Example 5.9 below shows that this theorem includes known Sobolev type inequalities as special cases.

Theorem 5.8. *Assume* $(\Phi\kappa J)$. *Suppose a function*

$$\Psi(x,t)\colon G\times[0,\infty)\to[0,\infty)$$

satisfies $(\Phi1)$–$(\Phi4)$ *with* ε_0 *replaced by some* ε_0' *in* $(\Phi3)$ *and*

$(\Psi\Phi)$ *there exist a constant* $A' \ge 1$ *and a strictly increasing continuous function* $\zeta(\varepsilon)$ *on* $[0,\varepsilon_1]$ *such that* $\zeta(0)=0$, $\varepsilon\mapsto\xi(\varepsilon)+((\delta-\delta_0)/\omega^*)\zeta(\varepsilon)$ *is non-decreasing with* $\omega^*>1$ *such that* $\Psi(x,t)\le Ct^{\omega^*}$ *for* $t\ge1$, *and*

$$\Psi_{\zeta(\varepsilon)}\Big(x, tY_J\Big(x,\kappa_{\xi(\varepsilon)}^{-1}\big(x,\Phi_\varepsilon(x,t)^{-1}\big)\Big)\Big) \le A'\Phi_\varepsilon(x,t)$$

for all $x\in G$, $t\ge1$ *and* $0<\varepsilon\le\varepsilon_1$.

Then there exists a constant $C>0$ *such that*

$$\|Jf\|_{\Psi,\kappa;\eta\circ\zeta^{-1},\xi\circ\zeta^{-1};G} \le C\|f\|_{\Phi,\kappa;\eta,\xi;G}$$

for all $f\in\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$.

Proof. Let f be a nonnegative function on G such that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$. Choose $\varepsilon_1' \in (0,\varepsilon_1]$ such that $\zeta(\varepsilon_1') \le \varepsilon_0'$. Let $x \in G$, $0 < r < d_G$ and $0 < \varepsilon \le \varepsilon_1'$. By Lemma 5.7 and ($\Psi\Phi$) we have

$$\begin{aligned}&\Psi_{\zeta(\varepsilon)}\big(x, \eta(\varepsilon)Jf(x)\big)\\ &\le C\left\{\Psi_{\zeta(\varepsilon)}\left(x, \eta(\varepsilon)Mf(x)Y_J\Big(x, \kappa^{-1}_{\xi(\varepsilon)}\big(x, \Phi_\varepsilon(x,\eta(\varepsilon)Mf(x))^{-1}\big)\Big)\right)+1\right\}\\ &\le C\left\{\Phi_\varepsilon\big(x,\eta(\varepsilon)Mf(x)\big)+1\right\}.\end{aligned}$$

Note that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$ implies $\|Mf\|_{\Phi,\kappa;\eta,\xi;G} \le C$ by Theorem 4.4. Hence there is a constant $C_1' > 0$ such that

$$\frac{\kappa_{\xi(\varepsilon)}(x,r)}{|B(x,r)|}\int_{B(x,r)\cap G}\Phi_\varepsilon\big(y,\eta(\varepsilon)Mf(y)\big)\,dy \le C_1'$$

for all $x \in G$, $0 < r < d_G$ and $0 < \varepsilon \le \varepsilon_1'$. Therefore, there is another constant $C_2' > 0$ such that

$$\frac{\kappa_{\xi(\varepsilon)}(x,r)}{|B(x,r)|}\int_{B(x,r)\cap G}\Psi_{\zeta(\varepsilon)}\big(y,\eta(\varepsilon)Jf(y)\big)\,dy \le C_2'$$

for all $x \in G$, $0 < r < d_G$ and $0 < \varepsilon \le \varepsilon_1'$, so that

$$\frac{\kappa_{(\xi\circ\zeta^{-1})(\varepsilon')}(x,r)}{|B(x,r)|}\int_{B(x,r)\cap G}\Psi_{\varepsilon'}\big(y,(\eta\circ\zeta^{-1})(\varepsilon')Jf(y)\big)\,dy \le C_2'$$

for all $x \in G$, $0 < r < d_G$ and $0 < \varepsilon' \le \zeta(\varepsilon_1')$, which implies the required result. □

Example 5.9. Let $\Phi(x,t) = \Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)$ be as in Example 2.1, $\kappa(x,r) = r^{\nu(x)}(\log(e+1/r))^{\beta(x)}$ be as in Example 2.2 and $J(x,r) = r^{\alpha(x)-N}$ be as in Example 5.4.

Note that $\sigma_0 = 0$ if $\nu^+ := \sup_{x\in G}\nu(x) = N$ and $0 < \sigma_0 < N-\nu^+$ if $\nu^+ < N$. We may take $0 < \delta_0 < \delta < \nu^-$ and $\omega > p^+$. Then,

$$\sigma + \frac{\delta-\delta_0}{\omega}\varepsilon < \sigma + \frac{\nu^-}{p^+}\varepsilon \le \sigma + \frac{\nu(x)}{p(x)}\varepsilon.$$

Hence, if $\sigma + ((\delta-\delta_0)/\omega)\varepsilon \ge 0$, then

$$\frac{\nu(x)+\sigma}{p(x)-\varepsilon} \ge \frac{\nu(x)}{p(x)}. \tag{5.2}$$

Since

$$Y_J(x,r)\Phi_\varepsilon^{-1}(x,\kappa_\sigma(x,r)^{-1}) \sim r^{\alpha(x)-(\nu(x)+\sigma)/(p(x)-\varepsilon)}\left[Q(x,1/r)(\log(e+1/r))^{\beta(x)}\right]^{-1/(p(x)-\varepsilon)},$$

where $Q(x,t) = \prod_{j=1}^k \big(L^{(j)}(t)\big)^{q_j(x)}$, we see that condition ($\Phi\kappa J$) holds if

$$\inf_{x\in G}\left(\frac{\nu(x)}{p(x)} - \alpha(x)\right) > 0.$$

Set

$$\Psi(x,t) = \left[\Phi_{p(\cdot),\{q_j(\cdot)\}}(x,t)\right]^{p^*(x)/p(x)}(\log(e+t))^{p^*(x)\alpha(x)\beta(x)/\nu(x)},$$

where $1/p^*(x) = 1/p(x) - \alpha(x)/\nu(x)$. We see

$$tY_J\big(x,\kappa_\sigma^{-1}(x,\Phi_\varepsilon(x,t)^{-1})\big) \sim t^{p(x)/p^*_\sigma(x)+\varepsilon\alpha(x)/(\nu(x)+\sigma)}\Big[Q(x,t)(\log(e+t))^{\beta(x)}\Big]^{-\alpha(x)/(\nu(x)+\sigma)},$$

where $1/p^*_\sigma(x) = 1/p(x) - \alpha(x)/(\nu(x)+\sigma)$. Hence

$$\begin{aligned}
&\Psi\Big(x,\, tY_J\big(x, \kappa_\sigma^{-1}(x, \Phi_\varepsilon(x,t)^{-1})\big)\Big) \\
&\sim t^{p(x)p^*(x)/p^*_\sigma(x)+\varepsilon p^*(x)\alpha(x)/(\nu(x)+\sigma)} \left[Q(x,t)(\log(e+t))^{\beta(x)}\right]^{-p^*(x)\alpha(x)/(\nu(x)+\sigma)} \\
&\quad \cdot Q(x,t)^{p^*(x)/p(x)}(\log(e+t))^{p^*(x)\alpha(x)\beta(x)/\nu(x)} \\
&= \Phi_\varepsilon(x,t) t^{\sigma(p^*(x)-p(x))/(\nu(x)+\sigma)+\varepsilon[p^*(x)\alpha(x)/(\nu(x)+\sigma)+1]} \\
&\quad \cdot Q(x,t)^{\sigma(p^*(x)-p(x))/[p(x)(\nu(x)+\sigma)]}(\log(e+t))^{\sigma p^*(x)\alpha(x)\beta(x)/[\nu(x)(\nu(x)+\sigma)]}.
\end{aligned}$$

Here, note that $\xi(\varepsilon) + (\nu^-/p^+)\varepsilon \ge 0$ implies $\nu(x)+\xi(\varepsilon) > \nu(x)/2$ if $0 < \varepsilon \le 1/2$. Let $0 < \varepsilon \le \min(1/2, \varepsilon_1)$. Let $\theta = (\delta - \delta_0)/\omega$. Since

$$\frac{\xi(\varepsilon)}{\nu(x)+\xi(\varepsilon)} \le \frac{\xi(\varepsilon)+\theta\varepsilon}{\nu(x)} \quad \text{and} \quad \frac{p^*(x)\alpha(x)}{\nu(x)+\xi(\varepsilon)} + 1 \le 2\frac{p^*(x)}{p(x)},$$

$$\begin{aligned}
&\Psi\Big(x,\, tY_J\big(x, \kappa_{\xi(\varepsilon)}^{-1}(x, \Phi_\varepsilon(x,t)^{-1})\big)\Big) \\
&\lesssim \Phi_\varepsilon(x,t) t^{(\xi(\varepsilon)+\theta\varepsilon)(p^*(x)-p(x))/\nu(x)+2\varepsilon p^*(x)/p(x)}[\log(e+t)]^{m_1(\xi(\varepsilon)+\theta\varepsilon)}
\end{aligned}$$

for $t \ge 1$ with a constant $m_1 \ge 0$. In view of (5.2), we also see that

$$tY_J\big(x, \kappa_{\xi(\varepsilon)}^{-1}(x, \Phi_\varepsilon(x,t)^{-1})\big) \gtrsim t^{p(x)/p^*(x)}[\log(e+t)]^{-m_2}$$

with a constant $m_2 \ge 0$, which implies

$$\begin{aligned}
&\Psi_{\zeta(\varepsilon)}\Big(x,\, tY_J\big(x, \kappa_{\xi(\varepsilon)}^{-1}(x, \Phi_\varepsilon(x,t)^{-1})\big)\Big) \\
&\lesssim \Phi_\varepsilon(x,t)\left\{t^{p(x)/p^*(x)}[\log(e+t)]^{-m_2}\right\}^{-\zeta(\varepsilon)} \\
&\quad \cdot t^{2p^*(x)/p(x)\varepsilon}\left\{t^{(p^*(x)-p(x))/\nu(x)}[\log(e+t)]^{m_1}\right\}^{(\xi(\varepsilon)+\theta\varepsilon)}
\end{aligned}$$

for $t \ge 1$.

Now, let $\zeta(\varepsilon) = a\varepsilon + b(\xi(\varepsilon)+\theta\varepsilon)$ $(a, b > 0)$. If $a > 2\sup_{x\in G}(p^*(x)/p(x))^2$, then

$$\sup_{x\in G,\ t\ge 1}\left\{t^{p(x)/p^*(x)}[\log(e+t)]^{-m_2}\right\}^{-a} t^{2p^*(x)/p(x)} < \infty$$

and if $b > \sup_{x\in G} p^*(x)(p^*(x)-p(x))/(p(x)\nu(x))$, then

$$\sup_{x\in G,\ t\ge 1}\left\{t^{p(x)/p^*(x)}[\log(e+t)]^{-m_2}\right\}^{-b}\left\{t^{(p^*(x)-p(x))/\nu(x)}[\log(e+t)]^{m_1}\right\} < \infty,$$

so that $\Psi(x,t)$ satisfies condition $(\Psi\Phi)$ with $\zeta(\varepsilon) = a\varepsilon + b(\xi(\varepsilon)+\theta\varepsilon)$ $(0 < \varepsilon \le \min(1/2, \varepsilon_1))$.

6. Trudinger type inequality

In this section, we consider Trudinger type inequality on $\widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$.

Lemma 6.1. *Let $t_1,\ t_2 > 0$. If*

$$\Phi(x, t_1) \le K\Phi(x, t_2)$$

for some $x \in G$ with $K \ge A_2^{-1}$, then $t_1 \le A_2Kt_2$.

Proof. Assume $t_1 > A_2 K t_2$. Note that $t_1 > t_2$. Using ($\Phi 3$), we have

$$\Phi(x, t_1) = t_1 \phi(x, t_1) > K t_2 \phi(x, t_2) = K\Phi(x, t_2),$$

which contradicts the assumption. □

In this section, we assume:

(Ξ) $\xi(\varepsilon) \le a\varepsilon$ for $0 < \varepsilon \le \varepsilon_1$ with some $a \ge 0$.

Recall that $\xi(\varepsilon) \ge -((\delta - \delta_0)/\omega)\varepsilon$ by assumption. Let

$$\varepsilon(r) = (\log(e + 1/r))^{-1}$$

for $r > 0$ and let $r_1 \in (0, \min(1, d_G))$ be such that $\varepsilon(r) \le \varepsilon_1$ for $0 < r \le r_1$.

Lemma 6.2. *There exists a constant $C \ge 1$ such that*

$$C^{-1}\Phi^{-1}(x, \kappa(x, r)^{-1}) \le \Phi^{-1}_{\varepsilon(r)}(x, \kappa_{\xi(\varepsilon(r))}(x, r)^{-1}) \le C\Phi^{-1}(x, \kappa(x, r)^{-1})$$

for all $x \in G$ and $0 < r \le r_1$.

Proof. Fix $x \in G$ and set

$$t_0(r) = \Phi^{-1}(x, \kappa(x, r)^{-1}) \quad \text{and} \quad t(r) = \Phi^{-1}_{\varepsilon(r)}(x, \kappa_{\xi(\varepsilon(r))}(x, r)^{-1})$$

for $0 < r \le r_1$. Then

$$\begin{aligned} \Phi(x, t_0(r)) &= \kappa(x, r)^{-1} = r^{\xi(\varepsilon(r))}\kappa_{\xi(\varepsilon(r))}(x, r)^{-1} \\ &= r^{\xi(\varepsilon(r))}\Phi_{\varepsilon(r)}(x, t(r)) = r^{\xi(\varepsilon(r))} t(r)^{-\varepsilon(r)}\Phi(x, t(r)). \end{aligned} \tag{6.1}$$

Thus, in view of Lemma 6.1, it is enough to show that there exists a constant $K \ge 1$ independent of x such that

$$K^{-1} \le r^{\xi(\varepsilon(r))} t(r)^{-\varepsilon(r)} \le K \tag{6.2}$$

for all $0 < r \le r_1$.

Note that

$$e^{-a} \le r^{a\varepsilon(r)} \le r^{\xi(\varepsilon(r))} \le r^{-((\delta-\delta_0)/\omega)\varepsilon(r)} \le e^{(\delta-\delta_0)/\omega} \tag{6.3}$$

for $0 < r \le r_1$ and that

$$Q_3^{-1} \le \kappa(x, r)^{-1} \le Q_3 \left(1 + \frac{1}{r}\right)^N$$

by ($\kappa 3$).

If $t(r) \le 1$, then by (6.1) and (6.3)

$$\begin{aligned} Q_3^{-1} &\le \kappa(x, r)^{-1} = r^{\xi(\varepsilon(r))} t(r)^{-\varepsilon(r)}\Phi(x, t(r)) \\ &\le e^{(\delta-\delta_0)/\omega} t(r)^{1-\varepsilon(r)}\phi(x, t(r)) \le e^{(\delta-\delta_0)/\omega} A_1 A_2 t(r)^{1-\varepsilon(d_G)}, \end{aligned}$$

so that $t(r) \ge C_1^{-1}$ with a constant $C_1 \ge 1$ independent of x. Thus

$$C_1^{-\varepsilon(d_G)} \le t(r)^{\varepsilon(r)} \le 1$$

if $t(r) \le 1$.

If $t(r) \ge 1$, then by (6.1) and (6.3) again

$$Q_3 \left(1 + \frac{1}{r}\right)^N \ge \kappa(x, r)^{-1} \ge e^{-a} t(r)^{1-\varepsilon(r)}\phi(x, t(r)) \ge e^{-a}(A_1 A_2)^{-1} t(r)^{1-\varepsilon(d_G)},$$

so that $t(r) \le C_2[(1+1/r)^N]^{1/(1-\varepsilon(d_G))}$ with $C_2 \ge 1$ independent of x. Since $(1+1/r)^{\varepsilon(r)}$ is bounded for $r > 0$, it follows that

$$1 \le t(r)^{\varepsilon(r)} \le C_2^{\varepsilon(d_G)} \left[\left(1 + \frac{1}{r}\right)^N \right]^{\varepsilon(r)/(1-\varepsilon(d_G))} \le C_3$$

if $t(r) \ge 1$, with a constant $C_3 \ge 1$ independent of x.

Therefore, (6.2) holds with $K = \max\{e^{(\delta-\delta_0)/\omega} C_1^{\varepsilon(d_G)}, e^a C_3\}$. □

Lemma 6.3. *There exists a constant $C > 0$ such that*

$$\frac{1}{|B(x,r)|} \int_{B(x,r)\cap G} f(y)\,dy \le C\Phi^{-1}(x, \kappa(x,r)^{-1}) \eta\left((\log(e+1/r))^{-1}\right)^{-1} \tag{6.4}$$

for all $x \in G$, $0 < r < d_G$ and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

Proof. Let f be a nonnegative measurable function on G such that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

If $0 < r \le r_1$, then by Lemma 5.3

$$\frac{1}{|B(x,r)|} \int_{B(x,r)\cap G} f(y)\,dy \le C\Phi^{-1}_{\varepsilon(r)}(x, \kappa_{\xi(\varepsilon(r))}(x,r)^{-1}) \eta(\varepsilon(r))^{-1}$$

for all $x \in G$. Hence, using the above lemma we obtain (6.4).

In case $r_1 < r < d_G$, note that

$$\Phi^{-1}_{\varepsilon(r_1)}\left(x, \kappa_{\xi(\varepsilon(r_1))}(x, r_1)^{-1}\right) \le C\Phi^{-1}(x, \kappa(x,r)^{-1})$$

by (κ3) and Lemma 5.1(5). Hence, by Lemma 5.3 with $\varepsilon = \varepsilon(r_1)$, we obtain (6.4) in this case, too. □

In this section, we also assume that

- (J3′) $J(x,r) \le C_J r^{-\varsigma}$ for $x \in G$ and $0 < r \le d_G$ with constants $0 \le \varsigma < N$ and $C_J > 0$;
- (J4) there is $r_0 \in (0, d_G)$ such that

$$\inf_{x\in G} J(x, r_0) > 0 \quad \text{and} \quad \inf_{x\in G} \frac{\overline{J}(x, r_0)}{\overline{J}(x, d_G)} > 1.$$

Here note that (J3′) implies (J3).

Example 6.4. Let $\alpha(\cdot)$ and $J(x,r)$ be as in Example 5.4. Then, $J(x,r)$ satisfies (J3′) and (J4) (with $\varsigma = N - \alpha^-$). In particular, it satisfies (J4) with any $r_0 \in (0, d_G)$.

We consider the function

$$\Gamma(x,s) = \begin{cases} \displaystyle\int_{1/s}^{d_G} \rho^N \Phi^{-1}\left(x, \kappa(x,\rho)^{-1}\right) \eta\left((\log(e+1/\rho))^{-1}\right)^{-1} d(-\overline{J}(x,\cdot))(\rho) & \text{if } s \ge \dfrac{1}{r_0}, \\ \Gamma(x, 1/r_0) r_0 s \quad \text{if } 0 \le s \le 1/r_0 \end{cases}$$

for every $x \in G$, where r_0 is the number given in (J4). $\Gamma(x,\cdot)$ is strictly increasing and continuous for each $x \in G$.

Lemma 6.5. *There exist positive constants C' and C'' such that*

- (a) $\Gamma(x,s) \le C' s^{\varsigma} \eta\left((\log(e+s))^{-1}\right)^{-1}$ *for all $x \in G$ and $s \ge 1/r_0$ with ς in condition* (J3′);

(b) $\Gamma(x, 1/r_0) \geq C'' > 0$ *for all* $x \in G$.

Proof. First note from (κ3) and Lemma 5.1(5) that

$$C^{-1} \leq \Phi^{-1}(x, \kappa(x,r)^{-1}) \leq Cr^{-N}. \tag{6.5}$$

By (6.5) and (J3′),

$$\begin{aligned}\Gamma(x,s) &\leq C\eta\left((\log(e+s))^{-1}\right)^{-1}\int_{1/s}^{d_G} d(-\overline{J}(x,\cdot))(\rho)\\ &\leq C\eta\left((\log(e+s))^{-1}\right)^{-1}\overline{J}(x,1/s) \leq C's^{\varsigma}\eta\left((\log(e+s))^{-1}\right)^{-1}\end{aligned}$$

for all $x \in G$ and $s \geq 1/r_0$; and

$$\begin{aligned}\Gamma(x,1/r_0) &\geq C^{-1}\int_{r_0}^{d_G}\rho^N d(-\overline{J}(x,\cdot))(\rho) \geq C^{-1}r_0^N\int_{r_0}^{d_G} d(-\overline{J}(x,\cdot))(\rho)\\ &= C^{-1}r_0^N\left(\overline{J}(x,r_0)-\overline{J}(x,d_G)\right) \geq C'' > 0,\end{aligned}$$

where we used (J4) to obtain the inequalities in the last line. □

Lemma 6.6. *There exists a constant* $C > 0$ *such that*

$$\int_{G\setminus B(x,\delta)} J(x,|x-y|)f(y)\,dy \leq C\Gamma\left(x,\frac{1}{\delta}\right)$$

for all $x \in G$, $0 < \delta \leq r_0$ *and nonnegative* $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ *with* $\|f\|_{\Phi,\kappa;\eta,\xi;G} \leq 1$.

Proof. By integration by parts, Lemma 6.3, (6.5), (J3′) and Lemma 6.5(b), we have

$$\begin{aligned}\int_{G\setminus B(x,\delta)} J(x,|x-y|)f(y)\,dy &\leq \int_{G\setminus B(x,\delta)} \overline{J}(x,|x-y|)f(y)\,dy\\ &\leq C\Big\{d_G^N\overline{J}(x,d_G)\Phi^{-1}\big(x,\kappa(x,d_G)^{-1}\big)\eta\left((\log(e+1/d_G))^{-1}\right)^{-1}\\ &\quad+\int_{\delta}^{d_G}\rho^N\Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}d(-\overline{J}(x,\cdot))(\rho)\Big\}\\ &\leq C\big\{\Gamma(x,1/r_0)+\Gamma(x,1/\delta)\big\} \leq C\Gamma(x,1/\delta).\end{aligned}$$

□

Lemma 6.7. *Let* $0 < \lambda < N$ *and define*

$$I_\lambda f(x) = \int_G |x-y|^{\lambda-N}f(y)\,dy$$

for a nonnegative measurable function f *on* G *and*

$$\omega_\lambda(z,r) = \frac{1}{1+\displaystyle\int_r^{d_G}\rho^\lambda\Phi^{-1}(z,\kappa(z,\rho)^{-1})\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}\frac{d\rho}{\rho}}$$

for $z \in G$. *Then there exists a constant* $C_{I,\lambda} > 0$ *such that*

$$\frac{\omega_\lambda(z,r)}{|B(z,r)|}\int_{B(z,r)\cap G} I_\lambda f(x)\,dx \leq C_{I,\lambda}$$

for all $z \in G$, $0 < r < d_G$ *and nonnegative* $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ *with* $\|f\|_{\Phi,\kappa;\eta,\xi;G} \leq 1$.

Proof. Let $z \in G$. Let $f(x) = 0$ for $x \in \mathbf{R}^N \setminus G$ and write

$$I_\lambda f(x) = \int_{B(z,2r)} |x-y|^{\lambda-N} f(y)\,dy + \int_{G\setminus B(z,2r)} |x-y|^{\lambda-N} f(y)\,dy$$
$$= I_1(x) + I_2(x)$$

for $x \in G$. By Fubini's theorem,

$$\int_{B(z,r)\cap G} I_1(x)\,dx = \int_{B(z,2r)} \left(\int_{B(z,r)\cap G} |x-y|^{\lambda-N}\,dx \right) f(y)\,dy$$
$$\le \int_{B(z,2r)} \left(\int_{B(y,3r)} |x-y|^{\lambda-N}\,dx \right) f(y)\,dy$$
$$\le C \int_{B(z,2r)} \left(\int_0^{3r} t^\lambda \frac{dt}{t} \right) f(y)\,dy \le \frac{C}{\lambda} r^\lambda \int_{B(z,2r)} f(y)\,dy.$$

Now, by Lemma 6.3, ($\kappa 2$) and Lemma 5.1(2), we have

$$r^\lambda \int_{B(z,2r)} f(y)\,dy \le C r^\lambda |B(z,2r)| \Phi^{-1}(z, \kappa(z,2r)^{-1}) \eta \left((\log(e + 1/(2r)))^{-1} \right)^{-1}$$
$$\le C|B(z,r)| \int_r^{2r} \rho^\lambda \Phi^{-1}(z, \kappa(z,\rho)^{-1}) \eta \left((\log(e+1/\rho))^{-1} \right)^{-1} \frac{d\rho}{\rho}$$

if $0 < r < d_G/2$ and, by Lemma 6.3 and (6.5), we have

$$r^\lambda \int_{B(z,2r)} f(y)\,dy = r^\lambda \int_{B(z,d_G)} f(y)\,dy$$
$$\le C {d_G}^\lambda |B(z,d_G)| \Phi^{-1}(z, \kappa(z,d_G)^{-1}) \eta \left((\log(e+1/d_G))^{-1} \right)^{-1} \le C|B(z,r)|$$

if $d_G/2 \le r < d_G$. Therefore

$$\int_{B(z,r)\cap G} I_1(x)\,dx \le \frac{C}{\lambda} \frac{|B(z,r)|}{\omega_\lambda(z,r)}$$

for all $0 < r < d_G$.

For I_2, first note that $I_2(x) = 0$ if $x \in G$ and $r \ge d_G/2$. Let $0 < r < d_G/2$. Since

$$I_2(x) \le C \int_{G\setminus B(z,2r)} |z-y|^{\lambda-N} f(y)\,dy \quad \text{for} \quad x \in B(z,r) \cap G,$$

by integration by parts and Lemma 6.3, we have

$$I_2(x) \le C \Bigg\{ {d_G}^\lambda \Phi^{-1}(z, \kappa(z,d_G)^{-1}) \eta \left((\log(e+1/d_G))^{-1} \right)^{-1}$$
$$+ \int_{2r}^{d_G} \rho^\lambda \Phi^{-1}(z, \kappa(z,\rho)^{-1}) \eta \left((\log(e+1/\rho))^{-1} \right)^{-1} \frac{d\rho}{\rho} \Bigg\}$$
$$\le \frac{C}{\omega_\lambda(z,r)}$$

for all $x \in B(z,r) \cap G$. Hence

$$\int_{B(z,r)\cap G} I_2(x)\,dx \le C \frac{|B(z,r)|}{\omega_\lambda(z,r)}.$$

Thus this lemma is proved. □

From now on, we deal with the case $\Gamma(x,r)$ satisfies the uniform log-type condition:

($\Gamma_{\log}$) there exists a constant $c_\Gamma > 0$ such that

$$\Gamma(x, s^2) \le c_\Gamma \Gamma(x, s)$$

for all $x \in G$ and $s \ge 1$.

By ($\Gamma_{\log}$), together with Lemma 6.5, we see that $\Gamma(x,s)$ satisfies the uniform doubling condition in s:

Lemma 6.8. [20, Lemma 4.2] *For every $a > 1$, there exists $b > 0$ such that $\Gamma(x, as) \le b\Gamma(x,s)$ for all $x \in G$ and $s > 0$.*

Now we consider the following condition (J5):

(J5) there exists $0 < \lambda < N - \varsigma$ such that $r \mapsto r^{N-\lambda} J(x,r)$ is uniformly almost increasing on $(0, d_G)$ for ς in condition (J3′).

Example 6.9. Let J be as in Example 5.4. It satisfies (J5) with $0 < \lambda < \alpha^-$.

Theorem 6.10. *Assume that Γ satisfies ($\Gamma_{\log}$) and J satisfies (J5). For each $x \in G$, let $\gamma(x) = \sup_{s>0} \Gamma(x,s)$. Suppose $\Lambda(x,t)\colon G \times [0,\infty) \to [0,\infty]$ satisfies the following conditions:*

(Λ1) *$\Lambda(\cdot, t)$ is measurable on G for each $t \in [0,\infty)$; $\Lambda(x,\cdot)$ is continuous on $[0,\infty)$ for each $x \in G$;*

(Λ2) *there is a constant $A_1' \ge 1$ such that $\Lambda(x,t) \le \Lambda(x, A_1' s)$ for all $x \in G$ whenever $0 < t < s$;*

(Λ3) *$\Lambda(x, \Gamma(x,s)/A_2') \le A_3' s$ for all $x \in G$ and $s > 0$ with constants A_2', $A_3' \ge 1$ independent of x.*

Then, for λ given in (J5)*, there exists a constant $C^* > 0$ such that $Jf(x)/C^* \le \gamma(x)$ for a.e. $x \in G$ and*

$$\frac{\omega_\lambda(z,r)}{|B(z,r)|} \int_{B(z,r)\cap G} \Lambda\left(x, \frac{Jf(x)}{C^*}\right) dx \le 1$$

for all $z \in G$, $0 < r < d_G$ and nonnegative $f \in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$.

By ($\Gamma_{\log}$) and (Λ3), the assertion of this theorem can be considered as exponential integrability of Jf; cf. Corollary 6.12 below.

Proof. Let f be a nonnegative measurable function on G such that $\|f\|_{\Phi,\kappa;\eta,\xi;G} \le 1$. Fix $x \in G$. For $0 < \delta \le r_0$, Lemma 6.6, (J5) and (J3′) imply

$$\begin{aligned}
Jf(x) &\le \int_{B(x,\delta)} J(x,|x-y|) f(y)\,dy + C\Gamma\left(x, \frac{1}{\delta}\right) \\
&= \int_{B(x,\delta)} |x-y|^{N-\lambda} J(x,|x-y|)|x-y|^{\lambda-N} f(y)\,dy + C\Gamma\left(x, \frac{1}{\delta}\right) \\
&\le C\left\{\delta^{N-\lambda} J(x,\delta) I_\lambda f(x) + \Gamma\left(x, \frac{1}{\delta}\right)\right\} \\
&\le C\left\{\delta^{N-\varsigma-\lambda} I_\lambda f(x) + \Gamma\left(x, \frac{1}{\delta}\right)\right\}
\end{aligned}$$

with constants $C > 0$ independent of x.

If $I_\lambda f(x) \le 1/r_0$, then we take $\delta = r_0$. Then, by Lemma 6.5(b)

$$Jf(x) \le C\Gamma\left(x, \frac{1}{r_0}\right).$$

By Lemma 6.8, there exists $C_1^* > 0$ independent of x such that

$$Jf(x) \le C_1^*\Gamma\left(x, \frac{1}{2A_3'}\right) \quad \text{if } I_\lambda f(x) \le 1/r_0. \tag{6.6}$$

Next, suppose $1/r_0 < I_\lambda f(x) < \infty$. Let $m = \sup_{s \ge 1/r_0, x \in G} \Gamma(x,s)/s$. By $(\Gamma_{\log})$, $m < \infty$. Define δ by

$$\delta^{N-\varsigma-\lambda} = \frac{r_0^{N-\varsigma-\lambda}}{m}\Gamma(x, I_\lambda f(x))(I_\lambda f(x))^{-1}.$$

Since $\Gamma(x, I_\lambda f(x))(I_\lambda f(x))^{-1} \le m$, $0 < \delta \le r_0$. Then by Lemma 6.5(b)

$$\begin{aligned}\frac{1}{\delta} &\le C\Gamma(x, I_\lambda f(x))^{-1/(N-\varsigma-\lambda)}(I_\lambda f(x))^{1/(N-\varsigma-\lambda)} \\ &\le C\Gamma(x, 1/r_0)^{-1/(N-\varsigma-\lambda)}(I_\lambda f(x))^{1/(N-\varsigma-\lambda)} \le C(I_\lambda f(x))^{1/(N-\varsigma-\lambda)}.\end{aligned}$$

Hence, using $(\Gamma_{\log})$ and Lemma 6.8, we obtain

$$\Gamma\left(x, \frac{1}{\delta}\right) \le \Gamma\big(x, C(I_\lambda f(x))^{1/(N-\varsigma-\lambda)}\big) \le C\Gamma(x, I_\lambda f(x)).$$

By Lemma 6.8 again, we see that there exists a constant $C_2^* > 0$ independent of x such that

$$Jf(x) \le C_2^*\Gamma\left(x, \frac{1}{2C_{I,\lambda}A_3'}I_\lambda f(x)\right) \quad \text{if } 1/r_0 < I_\lambda f(x) < \infty, \tag{6.7}$$

where $C_{I,\lambda}$ is the constant given in Lemma 6.7.

Now, let $C^* = A_1'A_2'\max(C_1^*, C_2^*)$. Then, by (6.6) and (6.7),

$$\frac{Jf(x)}{C^*} \le \frac{1}{A_1'A_2'}\max\left\{\Gamma\left(x, \frac{1}{2A_3'}\right), \Gamma\left(x, \frac{1}{2C_{I,\lambda}A_3'}I_\lambda f(x)\right)\right\}$$

whenever $I_\lambda f(x) < \infty$. Since $I_\lambda f(x) < \infty$ for a.e. $x \in G$ by Lemma 6.7, $Jf(x)/C^* \le \gamma(x)$ a.e. $x \in G$, and by $(\Lambda 2)$ and $(\Lambda 3)$, we have

$$\begin{aligned}\Lambda\left(x, \frac{Jf(x)}{C^*}\right) &\le \max\left\{\Lambda\left(x, \Gamma\left(x, \frac{1}{2A_3'}\right)/A_2'\right), \Lambda\left(x, \Gamma\left(x, \frac{1}{2C_{I,\lambda}A_3'}I_\lambda f(x)\right)/A_2'\right)\right\} \\ &\le \frac{1}{2} + \frac{1}{2C_{I,\lambda}}I_\lambda f(x)\end{aligned}$$

for a.e. $x \in G$. Thus, noting that $\omega_\lambda(z,r) \le 1$ and using Lemma 6.7, we have

$$\begin{aligned}&\frac{\omega_\lambda(z,r)}{|B(z,r)|}\int_{B(z,r)\cap G}\Lambda\left(x, \frac{Jf(x)}{C^*}\right)\,dx \\ &\le \frac{1}{2}\omega_\lambda(z,r) + \frac{1}{2C_{I,\lambda}}\frac{\omega_\lambda(z,r)}{|B(z,r)|}\int_{B(z,r)\cap G} I_\lambda f(x)\,dx \le \frac{1}{2} + \frac{1}{2} = 1\end{aligned}$$

for all $z \in G$ and $0 < r < d_G$. □

Remark 6.11. If $\Gamma(x,s)$ is bounded, that is,

$$\sup_{x\in G}\int_0^{d_G} \rho^N \Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}\, d(-\overline{J}(x,\cdot))(\rho)<\infty,$$

then by Lemma 6.6 we see that $J|f|$ is bounded for every $f\in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$. In particular, if $\omega_{N-\varsigma}(x,r)^{-1}$ is bounded, that is,

$$\sup_{x\in G}\int_0^{d_G} \rho^{N-\varsigma} \Phi^{-1}\big(x,\kappa(x,\rho)^{-1}\big)\eta\left((\log(e+1/\rho))^{-1}\right)^{-1}\, \frac{d\rho}{\rho}<\infty,$$

then $\Gamma(x,s)$ is bounded by (J3′), and hence $J|f|$ is bounded for every $f\in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$.

If we further assume a continuity of the potential kernel J like condition (J5) in our paper [20], then we can show a continuity of Jf for $f\in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$, as in [20, Theorem 5.3].

Applying Theorem 6.10 to special Φ, κ and J, we obtain the following corollary:

Corollary 6.12. *Let $\kappa(x,r)$ and $\alpha(x)$ be as in Examples 2.2 and 5.4 and let $p(x)$ and $q(x)$ be as in Examples 2.1. Set $\eta(t)=t^\theta$ for $\theta>0$, $\Phi(x,t)=t^{p(x)}(\log(e+t))^{q(x)}$ and*

$$I_{\alpha(\cdot)}f(x)=\int_G |x-y|^{\alpha(x)-N}f(y)\,dy$$

for a nonnegative locally integrable function f on G. Assume that

$$\alpha(x)-\nu(x)/p(x)=0 \quad \text{for all } x\in G.$$

(1) *Suppose that*

$$\inf_{x\in G}(-q(x)/p(x)-\beta(x)/p(x)+\theta+1)>0.$$

Then for $0<\lambda<\alpha^-$ there exist constants $C^*>0$ and $C^{**}>0$ such that

$$\frac{r^{\nu(z)/p(z)-\lambda}}{|B(z,r)|}\int_{B(z,r)\cap G}\exp\left(\left(\frac{I_{\alpha(\cdot)}f(x)}{C^*}\right)^{p(x)/(p(x)+\theta p(x)-\beta(x)-q(x))}\right)dx\le C^{**}$$

for all $z\in G$, $0<r<d_G$ and nonnegative $f\in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;G}\le 1$.

(2) *If*

$$\sup_{x\in G}(-q(x)/p(x)-\beta(x)/p(x)+\theta+1)\le 0,$$

then for $0<\lambda<\alpha^-$ there exist constants $C^*>0$ and $C^{**}>0$ such that

$$\frac{r^{\nu(z)/p(z)-\lambda}}{|B(z,r)|}\int_{B(z,r)\cap G}\exp\left(\exp\left(\frac{I_{\alpha(\cdot)}f(x)}{C^*}\right)\right)dx\le C^{**}$$

for all $z\in G$, $0<r<d_G$ and nonnegative $f\in \widetilde{L}^{\Phi,\kappa}_{\eta,\xi}(G)$ with $\|f\|_{\Phi,\kappa;\eta,\xi;G}\le 1$.

Proof. In the present situation, we see that

$$\Gamma(x,s)\sim\begin{cases}(\log(e+s))^{-q(x)/p(x)-\beta(x)/p(x)+\theta+1} & \text{in case (1),}\\ \log\left(\log(e+s)\right) & \text{in case (2)}\end{cases}$$

for all $x\in G$ and $s\ge 1/r_0=2/d_G$. Hence, we may take

$$\Lambda(x,t)=\begin{cases}\exp\left(t^{p(x)/(p(x)+\theta p(x)-q(x)-\beta(x))}\right) & \text{in case (1),}\\ \exp(\exp t) & \text{in case (2).}\end{cases}$$

On the other hand,

$$\omega_{\lambda'}(z,r) \sim r^{\nu(z)/p(z)-\lambda'}\left(\log(e+1/r)\right)^{-q(x)/p(x)-\beta(x)/p(x)+\theta}$$

for all $z \in G, 0 < s < d_G$ and $0 < \lambda' < \alpha^-$, so that

$$r^{\nu(z)/p(z)-\lambda} \le C\omega_{\lambda'}(z,r)$$

if $0 < \lambda < \lambda' < \alpha^-$. Thus, given $0 < \lambda < \alpha^-$, Theorem 6.10 implies the required results. □

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