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HAUSDORFF AND HARMONIC MEASURES ON NON-HOMOGENEOUS CANTOR SETS

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Abstract. We consider (not self-similar) Cantor sets defined by a sequence of piecewise linear functions. We prove that the dimension of the harmonic measure on such a set is strictly smaller than its Hausdorff dimension. Some Hausdorff measure estimates for these sets are also provided.

1. Introduction and statement of results

In this paper, we deal with the Hausdorff dimension and the harmonic measure of a certain type of Cantor sets X in the plane. Recall the definition of the Hausdorff dimension of a (probability) Borel measure μ:

$$\dim_{\mathcal{H}}(\mu) = \inf_{\{Z:\mu(Z)=1\}} \dim_{\mathcal{H}}(Z)$$

where infimum is taken over all Borel subsets Z with $\mu(Z) = 1$.

Let ω be the harmonic measure on $\hat{\mathbf{C}} \setminus X$ evaluated at ∞. By celebrated results of Makarov [Ma] and Jones and Wolff [JV] the Hausdorff dimension of ω is not larger than one. On the other hand, it is clear that the Hausdorff dimension of ω is at most $\dim_{\mathcal{H}}(X)$. Obviously, if $\dim_{\mathcal{H}}(X) > 1$ then $\dim_{\mathcal{H}}(\omega) < \dim_{\mathcal{H}}(X)$.

The questions we answer in our paper are motivated by previous results obtained for classes of self-similar and self-conformal (Cantor) sets (see remark 6 for the definition of a self-conformal Cantor set). Note that the Hausdorff dimension of a self-conformal Cantor set is always positive, and given by the so-called Bowen's formula, see e.g. [Ru] for an elementary proof and many generalisations.

It has been observed, for several self-similar and self-conformal (Cantor) sets, that $\dim_{\mathcal{H}}(\omega) < \dim_{\mathcal{H}}(X)$, see, e.g. [Ba1, Ca, MV, Vol1, Vol2, Zd1, Zd2, UZ]. Nevertheless, the intriguing question about the inequality of dimensions for an arbitrary self-conformal Cantor repeller, remains open.

Let us also recall that in $\mathbf{R}^d$, $d \geq 3$, a general result of Bourgain [Bou] states that for all domains Ω, the dimension of harmonic measure is bounded above by $d - \epsilon(d)$, where $\epsilon(d)$ is a positive constant depending only on d, whose exact value remains unknown.

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All the proofs of the strict inequality $\dim_{\mathcal{H}}(\omega) < \dim_{\mathcal{H}}(X)$ for conformal repellers rely on ergodic theory tools: one constructs an invariant measure equivalent to the harmonic measure and its ergodic properties play a crucial role in the arguments (see also [LV]).

On the other hand, the inequality $\dim_{\mathcal{H}}(\omega) < \dim_{\mathcal{H}}(X)$ is not true for more general Cantor sets, even after assuming a strict regularity of the construction [Ba1].

In this paper we prove the inequality $\dim_{\mathcal{H}}(\omega) < \dim_{\mathcal{H}}(X)$ for a class of non-homogeneous Cantor sets. In this case there is no invariant ergodic measure equivalent to harmonic measure and hence previously mentioned tools are inapplicable. This has also been the case of [Ba1], where an analogous result was proved for a class of non-homogeneous 4-corner "translation invariant" Cantor sets. That proof made use of special symmetries of the set. In the present paper, using an entirely different approach, we prove a general result. In fact, the result of [Ba1] is a special case of our Theorem A.

Before stating our results, we recall the notion of a modulus of a topological annulus in $\mathbf{C}$.

Definition 1. Let $A \subset \mathbf{C}$ be a doubly connected domain (a topological annulus) in the plane. Every such domain A is conformally equivalent to some geometric annulus of the form $\{z : r < |z| < R\}$. The value

$$\operatorname{mod}(A) = \frac{1}{2\pi} \log\left(\frac{R}{r}\right)$$

is called a modulus of the topological annulus A.

We consider the following class of Cantor sets in the plane (even though proofs can be easily generalized to higher dimensions).

Let Q be a Jordan domain in $\mathbf{C}$. Let $M > 0$, $0 < \underline{a} < \overline{a} < 1$ be fixed. We fix a positive integer $N > 1$.

Definition 2. Let $\mathcal{Q} = (Q_1, \dots Q_N)$ be a family of disjoint Jordan domains such that each Q_i is a preimage of Q under some (expanding) similitude $(a_i)^{-1}z + b_i$. We call a family $\mathcal{Q} = (Q_1, \dots Q_N)$ *admissible* if the following holds:

(1) $\underline{a} \le |a_i| \le \overline{a}$,
(2) $\operatorname{cl} Q_i \subset Q$,
(3) there exists a topological annulus $A \subset Q$ with $\operatorname{mod}(A) > M$ and separating ∂Q from $\bigcup_j Q_j$ (i.e. ∂Q and $\bigcup_j Q_j$ are in different components of $\mathbf{C} \setminus A$.

Definition 3. Note that, in this way, we have introduced a piecewise linear map f defined on the union of admissible discs: $f\colon \bigcup_{Q_i \in \mathcal{Q}} Q_i \to Q$ by the formula

$$f(z) = (a_i^{-1} z + b_i) \quad \text{for } z \in Q_i,$$

where $a_i^{-1} Q_i + b_i = Q$. If $\mathcal{Q}$ satisfies the conditions in Definition 2 then we call the map f admissible.

Definition 4. A set $X_0 \subset \mathbf{C}$ is called admissible if

$$X_0 = \bigcap_{n=1}^{\infty} \left(f_n \circ f_{n-1} \circ \dots \circ f_1 \circ f_0\right)^{-1}(Q).$$

for some sequence of admissible families $(\mathcal{Q}_k)_{k=0}^{\infty}$, $\mathcal{Q}_k = (Q_{k,1}, \ldots Q_{k,N})$, and maps f_k defined as

$$f_k(z) = a_{k,i}^{-1} z + b_{k,i} \quad \text{for } z \in Q_{k,i},$$

where $a_{k,i}^{-1} Q_{k,i} + b_{k,i} = Q$. So, the map f_k is defined on the union of the domains $\{Q_{k,i}\}_{i=1}^{N}$, and $f_k(Q_{k,i}) = Q$, for all $i = 1, \ldots, N$.

Remark 5. Note that $(f_n \circ f_{n-1} \circ \cdots \circ f_0)^{-1}(Q)$ is a descending family of sets. Moreover, since $f^{-1}(\mathrm{cl}Q) \subset Q$ for every admissible map, we have

$$X_0 = \bigcap_{n=1}^{\infty} (f_n \circ f_{n-1} \circ \cdots \circ f_0)^{-1} (\mathrm{cl}\, Q),$$

thus X_0 is a compact set. The expanding property (item (1) in the definition of an admissible family) implies easily that X is perfect and totally disconnected. Thus, X_0 is homeomorphic to a Cantor set.

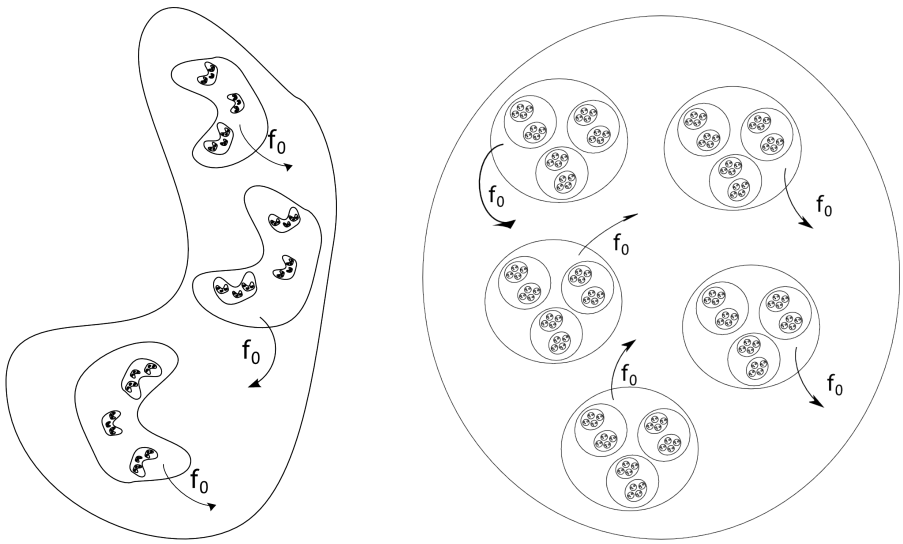

Figure 1. An illustration of an admissible set X_0 and of a generalized admissible set as defined in Section 11.

Remark 6. If the sequence f_n is given just by one admissible map f, $f_n = f$, for all $n \geq 0$ then the resulting Cantor set is called a *self-similar Cantor set*.

A more general definition of an admissible conformal map can be described similarly as in Definition 2: we modify Definition 2, allowing the map $f\colon Q_i \to Q$ to be a conformal isomorphism onto Q. Now, if the sequence f_n is given just by one admissible conformal map f, $f_n = f$, for all $n \geq 0$ then the resulting Cantor set is called a *self-conformal Cantor set*.

Remark 7. Notice that for admissible Cantor sets there are uniform bounds Λ and $\overline{\Lambda}$, depending only on $N, M, \underline{a}, \overline{a}$ such that

$$0 < \underline{\Lambda} < \dim_{\mathcal{H}}(X) < \overline{\Lambda} < 2.$$

A short proof of this fact will be given later, in Corollary 18.

In the present paper we prove the following

Theorem A. *Let X be an admissible Cantor set. Let ω be the harmonic measure on X. Then*

$$\dim_{\mathcal{H}}(\omega) < \dim_{\mathcal{H}}(X).$$

This is the main result of this paper. The idea is to create an alternative between two situations, the one implying the result (section 8) and the other being impossible (as we prove in sections 9 and 10). In the first situation we make use of a tool due to Bourgain [Bou]. In the second situation we refer to some ideas due to Volberg [Vol2].

Note also that we can find a uniform strictly positive lower bound of $\dim_{\mathcal{H}} X - \dim_{\mathcal{H}} \omega$ that only depends on $\underline{a}$, M and N as will be pointed out in section 11.

Moreover, we have the following result of independent interest:

Theorem B. *Let $(f_k)(z) = \sum_{i=1}^{N}(a_{k,i}^{-1}z + b_{k,i})\mathbb{1}_{Q_{k,i}}$ be a sequence of admissible maps and let $X = X_0$ be the associated admissible Cantor set. There exist a sequence of admissible functions $(\tilde{f}_k)$, $(\tilde{f}_k)(z) = \sum_{i=1}^{N}(\tilde{a}_{k,i}^{-1}z + \tilde{b}_{k,i})\mathbb{1}_{\tilde{Q}_{k,i}}$ such that*

(1) $\lim_{k\to\infty} \max_i(|\tilde{a}_{k,i} - a_{k,i}|) = 0$, $\tilde{b}_{k,i} = b_{k,i}$,
(2) *the associated Cantor set $\tilde{X}$ is admissible and* $\dim_{\mathcal{H}}(\tilde{X}) = \dim_{\mathcal{H}}(X)$,
(3) $0 < H_{\dim_{\mathcal{H}}(\tilde{X})}(\tilde{X}) < \infty$,
(4) *if ω and $\tilde{\omega}$ are the harmonic measures of X and $\tilde{X}$ respectively, then* $\dim_{\mathcal{H}} \omega = \dim_{\mathcal{H}} \tilde{\omega}$.

The proof of items (1), (2) and (3) of this theorem are carried out in Section 5. Item (4) follows from results of [Ba2] and [BaHa].

The paper is organized in 11 sections. Section 2 contains some well known facts and introduces notation. Some basic remarks on Hausdorff dimension of the Cantor sets considered here and on conformal measures can be found in Sections 3 and 4. Adapted tools from potential theory are presented in Section 6 and in Section 7 we apply all previous results to study limits of sequences of Cantor sets.

The proof of the main theorem is carried out in Sections 8, 9, 10. Section 8 provides a sufficient condition to have $\dim_{\mathcal{H}} X > \dim_{\mathcal{H}} \omega$. In Sections 9 and 10 we study the alternative case, when condition of Section 8 fails.

Finally, in Section 11, we show that the assumptions of the main theorem are somehow optimal: we construct a Cantor set X slightly different from the ones studied here, for which $\dim_{\mathcal{H}} X = \dim_{\mathcal{H}} \omega$.

2. Definitions and basic facts

In this Section we present the notation and some introductory remarks.

Remark 8. Using the Harnack inequality and the condition (3) in definition 2 we conclude that there exists a universal constant C (depending only on M) with the following property: Let $\mathcal{Q} = (Q_1, \dots Q_N)$ be an arbitrary admissible family of domains. Then there exists a smooth Jordan curve $\gamma_{\mathcal{Q}} \subset Q \setminus \bigcup_j Q_j$ (depending on the family $\mathcal{Q}$ of domains), and separating ∂Q from $\bigcup_j Q_j$ such that, for every positive harmonic function $\phi\colon Q \setminus \bigcup Q_j \to \mathbf{R}$,

$$\frac{\sup_{\gamma_{\mathcal{Q}}} \phi}{\inf_{\gamma_{\mathcal{Q}}} \phi} < C. \tag{1}$$

Notation. Note that f_0 maps X_0 onto the Cantor set $X_1 := \bigcap_{n=1}^{\infty}(f_n \circ f_{n-1} \circ \cdots \circ f_1)^{-1}(Q)$, and, generally, denoting

$$X_k = \bigcap_{n=k}^{\infty} (f_n \circ f_{n-1} \circ \cdots \circ f_{k+1} \circ f_k)^{-1} (Q)$$

we have

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots X_k \xrightarrow{f_k} X_{k+1} \ldots . \tag{2}$$

We shall use the notation f^k for the composition $f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1 \circ f_0$.

Let $x \in X_{k+1}$. Then, for every $i = 1, \ldots N$ there exists a unique point $y_{k,i} \in Q_{k,i}$ such that $f_k(y_{k,i}) = x$.

Definition 9. Let $\mathcal{L}_{k,s} \colon C(X_k) \to C(X_{k+1})$ be the operator defined as

$$\mathcal{L}_{k,s}(\phi)(x) = \sum_{i=1}^{N} \phi(y_{k,i}) |a_{k,i}|^s$$

(where we use the common notation $C(X)$ to denote the space of continuous functions defined on a compact metric space X).

Definition 10. We shall use the natural coding C_0 of the set X_0 by the symbolic space Σ, consisting of infinite sequences of digits $j \in \{1, \ldots, N\}$. As usual, the k-th digit in the code $C_0(x)$ equals j if $f_{k-1} \circ f_{k-2} \circ \cdots \circ f_1 \circ f_0(x) \in Q_{k,j}$. Similarly, the coding of the set X_k is defined, so that $C_{k+1}(f_k(x)) = \sigma(C_k(x))$ where $x \in X_0$ and σ is the left shift.

Notation. In what follows, we often identify the symbolic cylinder I and the corresponding subset of the Cantor set $C_0^{-1}(I)$. The family of all cylinders $I \subset \Sigma$, of length n will be denoted by $\mathcal{E}_n$. Each cylinder I of length n defines a branch of the map $(f_{n-1} \circ \cdots \circ f_1 \circ f_0)^{-1}$. The image of Q under this branch will be denoted by Q_I. Note that

$$Q_I \cap X_0 = C_0^{-1}(I)$$

and the sets Q_I are just the connected components of the set $(f_{n-1} \circ \cdots \circ f_0)^{-1}(Q)$.

We will denote by the same letter C a constant which may vary in the proofs.

3. Hausdorff dimension

The following simple proposition gives an explicit bound for the Hausdorff dimension of the set X.

Proposition 11. *Let $|a_{k,1}|, \ldots |a_{k,N}|$ be the sequence of "scales" used in the construction of X_0. Then $\dim_{\mathcal{H}}(X_0) \leq \rho$ where ρ is the number characterized in the following way:*

$$\rho = \inf\{s \colon \liminf_{n\to\infty} \prod_{k=1}^{n} (|a_{k,1}|^s + |a_{k,2}|^s + \ldots |a_{k,N}|^s) = 0\}. \tag{3}$$

Proof. First, note that $\liminf_{n\to\infty}\prod_{k=1}^{n}(|a_{k,1}|^s+|a_{k,2}|^s+\dots|a_{k,N}|^s)=0$ for all $s>\rho$. Pick some $s>\rho$. There exists a subsequence $n_j\to\infty$ with

$$\prod_{k=1}^{n_j}(|a_{k,1}|^s+|a_{k,2}|^s+\dots|a_{k,N}|^s)\to 0$$

Let $\mathcal{D}_n$ be the family of the domains $\{Q_I\colon I\in\mathcal{E}_n\}$ which appear at the n'th step of the construction of the Cantor set X. Then, after expanding and reordering, the product above can be rewritten as

$$\frac{1}{(\operatorname{diam}Q)^s}\sum_{Q_I\in\mathcal{D}_{n_j}}(\operatorname{diam}Q_I)^s. \tag{4}$$

So we have $\sum_{Q_I\in\mathcal{D}_{n_j}}(\operatorname{diam}Q_I)^s\to 0$. This shows that $\dim_{\mathcal{H}}(X)\le\rho$. □

Remark 12. In the next Section we shall prove that, actually, $\dim_{\mathcal{H}}(X)=\rho$, see Proposition 16 and Corollary 17.

The observation in Proposition 13 below will be used is Section 6.

Proposition 13. *There exist $K\in\mathbf{N}$, $C>0$ such that the following holds. Let X be an admissible Cantor set, I is a cylinder in the symbolic space Σ and J is another cylinder of length K (so IJ is a subcylinder of I, with K symbols added). Let $z\in Q_{IJ}$. Then*

$$\operatorname{dist}(z,\partial Q_I)>C\operatorname{diam}Q_I.$$

Proof. It is well known that every topological annulus A with sufficiently large modulus N contains a round annulus R separating the boundary components of A, with a modulus $\tilde N>N-C$ where C is some absolute constant (see, e.g. [McM, Thm. 2.1.]) Fix N so large that $\tilde N>1$. Fix K such that $KM>N$. Consider the annulus $A=Q_I\setminus Q_{IJ}$. It follows from the definition of an admissible Cantor set that $\operatorname{mod}(A)>KM>N$. Since this annulus separates Q_{IJ} from ∂Q_I, we conclude that, for $z\in Q_{IJ}$, $\operatorname{dist}(z,\partial Q_I)>e^{\tilde N}\operatorname{diam}Q_{IJ}>\operatorname{diam}Q_{IJ}>\underline{a}^K\operatorname{diam}Q_I$. □

4. Conformal measures

Let, as above, X_0 be an admissible set, and set $X_k=f^k(X_0)$.

Definition 14. Fix $h>0$. The sequence of probability measures $\nu_0,\nu_1,\dots$ is called a collection of h-conformal measures if $\operatorname{supp}\nu_k=X_k$ and the following holds: there exists a sequence $\lambda_{k,h}$ of positive "scaling factors" such that

$$\mathcal{L}^*_{k,h}(\nu_{k+1})=\lambda_{k,h}\nu_k. \tag{5}$$

Note that the condition (5) is equivalent to the following: if B is a Borel measurable set, $B\subset Q_{k,i}$ then

$$\nu_{k+1}(f_k(B))=\lambda_{k,h}\cdot(|a_{k,i}|^{-h})\cdot\nu_k(B)=\lambda_{k,h}\int_B|f_k'|^h\,d\nu_k. \tag{6}$$

The collection of h-conformal measures exists for every $h\ge 0$. The measure ν_0 is uniquely determined by assigning to every cylinder I, of length m, the value of the measure $\nu_0(I)$, or, more precisely, of the set $C_0(I) \subset X_0$:

$$\nu_0(I) = \frac{\left(|(f_{m-1} \circ \cdots \circ f_1 \circ f_0)'|^{-h}\right)_{|I}}{\lambda_{0,h}\lambda_{1,h} \dots \lambda_{m-1,h}}. \tag{7}$$

The measures ν_k, $k > 0$, are defined in a similar way:

$$\nu_k(I) = \frac{\left(|(f_{m-1+k} \circ \cdots \circ f_{1+k} \circ f_k)'|^{-h}\right)_{|I}}{\lambda_{k,h}\lambda_{1+k,h} \dots \lambda_{m-1+k,h}}. \tag{8}$$

The normalizing factors are given explicitly:

$$\lambda_{n,h} = (|a_{n,1}|^h + \dots |a_{n,N}|^h), \tag{9}$$

$n = 0, 1, 2, \dots$.

Let us note the following straightforward

Proposition 15. *For every h, the sequence of h-conformal measures ν_k is invariant, i.e.*

$$(f_k)_*(\nu_k) = \nu_{k+1}.$$

Proof. This follows directly from the conformality condition (5). It is enough to check it for $k = 0$. Let $A \subset X_1$ be an arbitrary Borel set. Then $f_0^{-1}(A) = A_1 \cup A_2 \cup \dots \cup A_N$, where $A_j \subset Q_{0,j}$. Using (6) we write

$$\nu_0(A_j) = |a_{0,j}|^h \cdot \frac{1}{\lambda_{0,h}} \nu_1(A)$$

and

$$\nu_0(f_0^{-1}(A)) = \sum_{j=1}^{N} \nu_0(A_j) = \frac{1}{\lambda_{0,h}} \left(\sum_{j=0}^{N} |a_{0,j}|^h \right) \nu_1(A) = \nu_1(A). \qquad \square$$

We note the following.

Proposition 16. *Let ρ be the real number defined by (3). If ν_k is the sequence of ρ- conformal measures then, for every $k \geq 0$*

$$\dim_{\mathcal{H}}(\nu_k) = \rho \tag{10}$$

Proof. It is obvious that the dimensions of all the measures ν_k are the same. Indeed, let A be a Borel measurable set such that $\nu_k(A) = 1$. Then $\nu_{k+1}(f_k(A)) = 1$ and, since f_k, restricted to every set $Q_{k,i}$ is linear, $\dim_{\mathcal{H}}(f_k(A)) = \dim_{\mathcal{H}}(A)$. Conversely, let B be a Borel measurable set such that $\nu_{k+1}(B) = 1$. Then $\nu_k(f_k^{-1}(B)) = 1$ and $\dim_{\mathcal{H}}(f_k^{-1}(B)) = \dim_{\mathcal{H}}(B)$.

So, we check (10) for ν_0. Fix an arbitrary $s < \rho$. It follows from condition (3) in Definition 2 that there exists $r_0 < \operatorname{diam} Q$ such that, if $z \in X_k$ then the ball $B(z, r_0)$ is contained in some domain $Q_{k,i}$ (so the map f_k is injective and continuous in $B(z, r_0)$).

Now, take an arbitrary ball $B = B(z, r)$ with $z \in X_0$ and $r < r_0$ and let n be the least iterate such that the diameter of $f_{n-1} \circ \cdots \circ f_1 \circ f_0(B)$ becomes larger than r_0. Then we have, using (6),

$$\nu_0(B) = \frac{\int_{f^n(B)} |(f^{-n})'|^\rho \, d\nu_n}{\lambda_{0,\rho} \dots \lambda_{n-1,\rho}}.$$

The nominator of the last fraction is just, up to a bounded factor, $(\operatorname{diam}(B))^\rho \asymp r^\rho \leq (\operatorname{diam}(B))^s$.

After neglecting this bounded factor we can write the above ratio as

$$(\mathrm{diam}(B))^s \cdot \frac{\mathrm{diam}(B)^{\rho-s}}{\lambda_{0,\rho}\lambda_{1,\rho}\dots\lambda_{n-1,\rho}}. \tag{11}$$

Since all the maps f_k are expanding, with expansion factor bounded from below by $\frac{1}{\overline{a}} > 1$, n is related to $\mathrm{diam}\, B = 2r$, namely $r \le \exp(-n\delta)$ for some positive δ, and we can estimate the second factor in (11) from above by

$$C\exp(-n(\rho-s)\delta)\frac{1}{\lambda_{0,\rho}\lambda_{1,\rho}\dots\lambda_{n-1,\rho}}, \tag{12}$$

where $C > 0$ is a constant. Now, choose $s' \in (s,\rho)$ sufficiently close to ρ so that, for all k, $\lambda_{k,s'} \le \lambda_{k,\rho}\exp(\delta(\rho-s))$. Then

$$\exp(-n(\rho-s)\delta)\frac{1}{\lambda_{0,\rho}\lambda_{1,\rho}\dots\lambda_{n-1,\rho}} \le \frac{1}{\lambda_{0,s'}\lambda_{1,s'}\dots\lambda_{n-1,s'}}.$$

Since ρ was a "transition parameter", $\lambda_{0,s'}\lambda_{1,s'}\dots\lambda_{n-1,s'} \to \infty$ for every $s' < \rho$. This proves that for all $z \in X_0$

$$\lim_{r\to 0}\frac{\nu_0(B(z,r))}{r^s} = 0,$$

which implies that $\dim_{\mathcal{H}}(\nu_0) \ge s$ and, consequently, $\dim_{\mathcal{H}}(\nu_0) \ge \rho$. Together with the estimate $\dim_{\mathcal{H}}(X_0) \le \rho$, this gives $\dim_{\mathcal{H}}(\nu_0) = \rho$. □

Corollary 17. $\rho = \dim_{\mathcal{H}}(X_0)$.

Proof. Proposition 11 gives the estimate $\dim_{\mathcal{H}}(X_0) \le \rho$, while Proposition 10 implies that $\dim_{\mathcal{H}}(X_0) \ge \dim_{\mathcal{H}}(\nu_0) = \rho$. □

We end this section with the following corollary, providing the bounds formulated in Remark 7.

Corollary 18. *Let X be an admissible and $N, M, \underline{a}, \overline{a}$ the associated bounds. There are exist constants $0 < \underline{\Lambda} < \overline{\Lambda} < 2$, depending only on $N, M, \underline{a}, \overline{a}$ and on the domain Q such that,*

$$\underline{\Lambda} < \dim_{\mathcal{H}}(X) < \overline{\Lambda}.$$

Proof. Fix $\underline{\Lambda} > 0$ satisfying $N(\underline{a})^{\underline{\Lambda}} = 1$ and let $s < \underline{\Lambda}$. Then, for every $k \ge 0$, $|a_{k,1}|^s + \dots |a_{k,n}|^s > 1$. Therefore, $\rho \ge s$, and, since s was an arbitrary number in $(0, \underline{\Lambda})$, we conclude that $\rho \ge \underline{\Lambda}$.

To get the upper bound of the dimension , we use Proposition 13. Applying Proposition 13 to $Q_I = Q$ we see that there exist constants $C > 0$, $K \in \mathbf{N}$ such that for every admissible Cantor set X, and every cylinder J of length K, $\mathrm{dist(Q_J, \partial Q) > C}$. This implies that there exists another constant $c < 1$ such that

$$\frac{\mathrm{area}(\bigcup_{|J|=K} Q_J)}{\mathrm{area}(Q)} < c$$

or, equivalently,

$$\prod_{k=0}^{K-1}(|a_{k,1}|^2 + \dots + |a_{k,N}|^2) < c.$$

Applying the same bound for consecutive admissible sets $X_0, X_K, X_{2K}\ldots$ and multiplying, we get, with possibly larger constant $\delta < 1$:

$$\prod_{k=0}^{n-1}(|a_{k,1}|^2 + \cdots + |a_{k,N}|^2) < \delta^n$$

for every $n \in \mathbf{N}$.

Therefore,

$$\prod_{k=0}^{n-1}(|a_{k,1}|^s + \cdots + |a_{k,N}|^s) < \prod_{k=0}^{n-1}(|a_{k,1}|^2 + \cdots + |a_{k,N}|^2) \cdot (\underline{a}^{s-2})^n < \delta^n \cdot (|\underline{a}|^{s-2})^n.$$

Define $\overline{\Lambda} < 2$ as the value satisfying the equation $\underline{a}^{\overline{\Lambda}-2} \cdot \delta = 1$. For every $s > \Lambda$, we conclude, using the formula for Hausdorff dimension of X (see Corollary 17) that $\dim_H(X) \leq s$. Thus, $\dim_H(X) \leq \overline{\Lambda}$. □

5. Hausdorff and harmonic measures

In this section we prove Theorem B. We start with

Theorem 19. *Let (f_n) be a sequence of admissible maps and let X the associated Cantor set. There exist a sequence of admissible functions $(\tilde{f}_n) = \sum_{i=1}^N (\tilde{a}_{k,i}^{-1} z + \tilde{b}_{k,i})1\!\!1_{\tilde{Q}_{k,i}}$ such that*

(1) $\lim_{k\to\infty} \max_{i=1,\ldots N}(|\tilde{a}_{k,i} - a_{k,i}|) = 0$,
(2) $\tilde{b}_{k,i} = b_{k,i}$ *for* $k \in \mathbf{N}$, $i = 1,\ldots N$,
(3) *the associated Cantor set* $\tilde{X}$ *satisfies* $\dim_{\mathcal{H}}(\tilde{X}) = \dim_{\mathcal{H}}(X)$,
(4) $0 < H_{\dim_{\mathcal{H}}(\tilde{X})}(\tilde{X}) < \infty$.

We can also deduce

Corollary 20. *Let $\tilde{X}$ be the admissible Cantor set, constructed in Theorem 19. If ω and $\tilde{\omega}$ are the harmonic measures of X and $\tilde{X}$ respectively, then $\dim_{\mathcal{H}} \omega = \dim_{\mathcal{H}} \tilde{\omega}$.*

In [Ba2] the author proved that if all squares of a given generation k are of equal size a_k (i.e. $a_{k,i} = a_k$, for any $i, j = 1, \ldots, N$ and for all k), then the dimension of harmonic measure is a continuous function with respect to the ℓ^∞ norm of the sequence (a_k). More recently, in [BaHa] the authors extended this result to Cantor sets defined by a sequence of conformal maps. In particular, applied to our case, this implies that if two Cantor sets X, X' are defined by sequences $(a_{k,i}, b_{k,i}), (a'_{k,i}, b'_{k,i})$ respectively, such that $\lim_k \max_i\{|a_{k,i} - a'_{k,i}| + |b_{k,i} - b'_{k,i}|\} = 0$, then the associated harmonic measures have the same dimension.

Thus, Theorem 19 and Corollary 20 imply Theorem B. The rest of this section is devoted to the proof of Theorem 19.

The following proposition is a refinement of Proposition 11.

Proposition 21. *Let $a_{k,1}, \ldots a_{k,N}$ be the sequence of "scales" used in the construction of X. For all $h > 0$ then there is a constant $C >$ such that*

$$\frac{1}{C} \liminf_{n\to\infty} \prod_{k=1}^{n} \lambda_{k,h} \leq H_h(X) \leq \liminf_{n\to\infty} \prod_{k=1}^{n} \lambda_{k,h}$$

Proof. Subsets of the Cantor set X and the cylinders on the symbolic space Σ are identified through the coding. The upper bound of $H_h(X)$ is immediate since $\prod_{k=1}^n \lambda_{k,h}$ corresponds to the natural covering of X by its cylinders of the nth generation.

To prove the lower bound take any ball U intersecting X and define I^U to be the cylinder of the highest generation s containing $U \cap X$. More precisely, take

$$s(U) = \max\{n\, ;\ \exists I_n^U \in \mathcal{E}_n \colon U \cap X \subset I_n^U\},$$

and let $I^U = I^U_{s(U)}$.

Clearly, $\operatorname{diam}(U \cap X) \le \operatorname{diam}(I^U)$. On the other hand, U intersects two distinct subcylinders of I_s^U. By the modulus separation condition (3) in Definition 2, we deduce that there is a constant $C = C(M,Q)$ such that $\operatorname{diam}(U) \ge \underline{a}C \operatorname{diam}(I^U)$.

This implies that we can replace all balls U of a given covering $\mathcal{R}$ of X by cylinders I_U of similar size and still control the variation of the sum $\sum_{U\in\mathcal{R}} \operatorname{diam}(U)^h \ge (\underline{a}C)^h \sum_{U\in\mathcal{R}} \operatorname{diam}(I_U)^h$.

Since we can only consider coverings with cylinders, one can easily check that we get optimal coverings using cylinders of the same generation. Indeed, for $n \in \mathbf{N}$ we say that a covering $\mathcal{R}$ with cylinders is n-optimal for H_h if

$$\sum_{I\in\mathcal{R}} \operatorname{diam}(I)^h = \min\left\{\sum_{\mathcal{R}'} \operatorname{diam}(I)^h\, ;\ \mathcal{R}' \text{ covering with cylinders of generation } \le n\right\}.$$

Take an n- optimal covering $\mathcal{R}$, of minimal cardinality. Choose I a cylinder in $\mathcal{R}$ of the minimal generation and let I' be any cylinder of the same generation not contained in $\mathcal{R}$. There is hence a subcovering $\mathcal{R} \cap I' = \{I'J_1, \dots, I'J_\ell\}$ of I' with subcylinders of I'.

By the definition of $\mathcal{R}$ we have $\operatorname{diam}(I')^h > \sum_{i=1}^\ell \operatorname{diam}(I'J_i)^h$ or, equivalently,

$$\sum_{i=1}^{\ell} \frac{\operatorname{diam}(I'J_i)^h}{\operatorname{diam}(I')^h} < 1.$$

But this latter sum is equal to $\sum_{i=1}^\ell \frac{\operatorname{diam}(IJ_i)^h}{\operatorname{diam}(I)^h}$ and hence $\operatorname{diam}(I)^h > \sum_{i=1}^\ell \operatorname{diam}(IJ_i)^h$ which contradicts $I \in \mathcal{R}$. It follows that all cylinders of the same generation as I, say, n, are in $\mathcal{R}$.

We conclude that there exists a constant $C > 0$ such that for every cover $\mathcal{R}$ of the set X, by balls of diameters smaller than ε, there exists a cover $\mathcal{R}'$ by cylinder sets of the same generation n, and of diameter smaller than $C\varepsilon$, such that

$$\sum_{U\in\mathcal{R}} \operatorname{diam}(U)^h \ge \frac{1}{C} \sum_{I\in\mathcal{R}'} \operatorname{diam}(I)^h$$

Note that, as in (4), we can write

$$\sum_{I\in\mathcal{R}'} \operatorname{diam}(I)^h = \operatorname{diam}(X)^h \cdot \prod_{k=1}^n \lambda_{k,h}$$

and the proof is complete. □

Let us now turn to the proof of Theorem 19.

Proof. We construct the sequence $\tilde{f}_n$ satisfying (1) and (3). Recall that ρ denotes the dimension of X.

Let us distinguish two cases:

Case 1: $H_\rho(X) = 0$. Since $H_{\rho-\varepsilon}(X) = +\infty$ for all $\varepsilon > 0$, Proposition 21 implies that

$$\lim_{n\to\infty} \prod_{k=1}^{n} \lambda_{k,\rho-\varepsilon} = +\infty. \tag{13}$$

The construction is carried out by induction.

Step 1. Define, for $n \in \mathbf{N}$, $\varepsilon_{1,n}$ to be a real number such that

$$\prod_{k=1}^{n} \lambda_{k,\rho-\varepsilon_{1,n}} = 1.$$

Note that $\varepsilon_{1,n}$ does not have to be positive. However, since $H_\rho(X) = 0$, we have, using Proposition 21 that $\liminf_n \Pi_{k=1}^n \lambda_{k,\rho} = 0$. Thus, $\varepsilon_{1,n}$ is positive for infinitely many n's.

By (13) $\lim\limits_{n\to\infty} \varepsilon_{1,n} = 0^+$. We can therefore choose n_1 such that

$$\varepsilon_{1,n_1} = \max\{\varepsilon_{1,n}\,;\ n \in \mathbf{N}\} > 0.$$

For $k = 1, \ldots, n_1$ and $i = 1, \ldots, N$, put

$$\tilde{a}_{k,i} = a_{k,i}|a_{k,i}|^{-\frac{\varepsilon_{1,n_1}}{\rho}}.$$

This implies $\prod_{k=1}^{n_1} (|\tilde{a}_{k,1}|^\rho + |\tilde{a}_{k,2}|^\rho + \cdots + |\tilde{a}_{k,N}|^\rho) = 1$ and, by the choice of ε_{1,n_1}, $\prod_{k=1}^{n} (|\tilde{a}_{k,1}|^\rho + \tilde{a}_{k,2}|^\rho + \cdots + |\tilde{a}_{k,N}|^\rho) \geq 1$, for $n \leq n_1$. Remark also that, $|\tilde{a}_{k,i}| \geq |a_{k,i}|$.

Step 2. Define for $n > n_1$, $\varepsilon_{2,n}$ to be a real number such that

$$\prod_{k=n_1+1}^{n} \lambda_{k,\rho-\varepsilon_{2,n}} = 1.$$

Clearly, $\lim_{n\to\infty} \varepsilon_{2,n} = 0$. As before we can now choose n_2 such that $\varepsilon_{2,n_2} = \max\{\varepsilon_{2,n}\ ;\ n > n_1\} > 0$. Now, we have, for $n \geq n_1$

$$1 = \prod_{k=1}^{n} \lambda_{k,\rho-\varepsilon_{1,n}} = \prod_{k=1}^{n_1} \lambda_{k,\rho-\varepsilon_{1,n}} \prod_{k=n_1+1}^{n} \lambda_{k,\rho-\varepsilon_{1,n}}.$$

Since, for $n > n_1$, $\varepsilon_{1,n} \leq \varepsilon_{1,n_1}$ we get

$$\prod_{k=1}^{n_1} \lambda_{k,\rho-\varepsilon_{1,n}} \leq \prod_{k=1}^{n_1} \lambda_{k,\rho-\varepsilon_{1,n_1}} = 1.$$

This implies that

$$\prod_{k=n_1+1}^{n} \lambda_{k,\rho-\varepsilon_{1,n}} \geq 1$$

and therefore $\varepsilon_{2,n} \leq \varepsilon_{1,n}$, for all $n > n_1$. In particular, $\varepsilon_{2,n_2} \leq \varepsilon_{1,n_1}$.

For $k = n_1+1, \ldots, n_2$ and $i = 1, \ldots, N$ put

$$\tilde{a}_{k,i} = a_{k,i}|a_{k,i}|^{-\frac{\varepsilon_{2,n_2}}{\rho}}.$$

The same reasoning as above now gives $\prod_{k=1}^{n_2}(|\tilde{a}_{k,1}|^\rho + |\tilde{a}_{k,2}|^\rho + \cdots + |\tilde{a}_{k,N}|^\rho) = 1$ and by the choice of ε_{2,n_2}, $\prod_{k=1}^{n}(|\tilde{a}_{k,1}|^\rho + |\tilde{a}_{k,2}|^\rho + \cdots + |\tilde{a}_{k,N}|^\rho) \geq 1$, for $n \leq n_1$. Again, $|\tilde{a}_{k,i}| \geq |a_{k,i}|$.

Step 3. Proceed by induction.

Since $\varepsilon_{1,n} \geq \varepsilon_{k,n}$ for all k,n we have that $\lim_{k\to\infty} \varepsilon_{k,n_k} = 0$. This implies that $|\tilde{a}_{k,i} - a_{k,i}| \to 0$ as $k \to \infty$. We define the maps $(\tilde{f}_n)$ and the domains $\tilde{Q}_{k,i}$ using the modified constants $\tilde{a}_{k,i}$:

$$\tilde{f}_n(z) = \sum_{i=1}^{N} (\tilde{a}_{k,i}^{-1} z + \tilde{b}_{k,i}) \mathbb{1}_{\tilde{Q}_{k,i}}$$

where $\tilde{Q}_{k,i} = \tilde{a}_{k,i} Q + b_{k,i}$. Let $\tilde{X}$ be the corresponding Cantor set. We have

$$\liminf_{n\to\infty} \prod_{k=1}^{n} (|\tilde{a}_{k,1}|^\rho + |\tilde{a}_{k,2}|^\rho + \cdots + |\tilde{a}_{k,N}|^\rho) = 1,$$

which proves $H_\rho(\tilde{X}) = 1$.

Case 2: $H_\rho(X) = +\infty$. This case can be treated in the same way as Case 1. Nevertheless, there is a simple way to deal with it. Clearly, since ρ is the dimension of the set, for all $\delta < 1$ we get that $\liminf_{n\to\infty} \delta^n \prod_{k=1}^{n} \lambda_{k,\rho} = 0$ and therefore we can find a sequence $(\delta_j)_j < 1$, $\lim_{j\to\infty} \delta_j = 1$ and a strictly increasing sequence of positive integers n_j such that

$$\liminf_{K\to\infty} \prod_{j=1}^{K} \prod_{\ell=n_j}^{n_{j+1}} \delta_j^{n_{j+1}-n_j} \lambda_{\ell,\rho} = 0.$$

We can now modify the sequence $(a_{k,i})$, by putting for all $j \in \mathbf{N}$ and $k = n_j + 1, \ldots, n_{j+1}$

$$a'_{k,i} = \delta_j^{\frac{1}{\rho}} a_{k,i},$$

the sequence $(b_{k,i})$ is left unchanged. This yields a Cantor set X' (of the same Hausdorff dimension) satisfying $\lim_k \max_i\{|a_{k,i} - a'_{k,i}|\} = 0$ and $\liminf_{n\to\infty} \prod_{k=1}^{n} \lambda'_{k,\rho} = 0 = H_\rho(X')$, which puts the situation back to case one. $\square$

6. Green's functions and capacity

We denote by $\mathrm{Cap}(X)$ the logarithmic capacity of X. We start with the following.

Proposition 22. *There exists a constant $\kappa > 0$, depending only on $M, \overline{a}, \underline{a}, Q, N$, such that, if X is an admissible Cantor set then $\mathrm{Cap}(X) > \kappa$.*

Proof. One can assume that $\operatorname{diam} Q = 1$. Fix h positive and so small that $P = N\underline{a}^h > 1$. We shall use the measure ν_h to estimate the capacity from below. Then, using (7) we get, for every cylinder I,

$$\nu_h(I) \leq (\operatorname{diam} Q_I)^h \frac{1}{P^n} < (\operatorname{diam} Q_I)^h$$

The logarithmic potential of the measure ν_h can be estimated pointwise. Let $z \in X$; denote by $I_n(z)$ the cylinder containing x (under the identification of X with the symbolic space Σ). Then, using Proposition 13, we get

$$U_{\nu_h}(z) = \int \log \frac{1}{|z-w|}\, d\nu_h(w) \le \sum_n \nu_h(I_n(z)) \cdot \inf_{w\in I_n(z)\setminus I_{n+1}(z)} \log \frac{1}{|z-w|}$$
$$\le \sum_n \nu_h(I_n(z)) \log \frac{1}{C \operatorname{diam} Q_{I_{n+1}(z)}} \le \sum_n \operatorname{diam} Q_{I_n(z)} \log \frac{1}{C \operatorname{diam} Q_{I_{n+1}(z)}}.$$

Since $\operatorname{diam} Q_{I_n(z)} < \overline{a}^n$ and $\operatorname{diam} Q_{I_n(z)} > \underline{a}^n$, this easily gives a common bound on $U_{\nu_h}(z)$. Consequently, we get a common bound for the energy function:

$$I(\nu_h) = \int U_{\nu_h}(z) d\nu_h(z) \le I_0 < \infty$$

and $\operatorname{Cap}(X) \ge \exp(-I_0)$. □

Let $X = X_0$ be an admissible Cantor set, and let $(X_k)_{k=0}^{\infty}$ be the associated sequence of consecutive Cantor sets, according to (2). Denote by ω_k the harmonic measure on the Cantor set X_k, evaluated at ∞. Denote by G_k the Green's function in $\mathbf{C} \setminus X_k$. Note that all the sets X_k are regular in the sense of Dirichlet, thus each function G_k has a continuous extension to the whole plane $\mathbf{C}$ and $G_{k|X_k} = 0$. We clearly have $\omega_k = \Delta G_k$.

Given an admissible Cantor set X, we denote by $\mathcal{G}_X$ the family of all functions $F\colon Q \to \mathbf{R}$ such that F is continuous in Q, $F_{|Q\setminus X}$ is harmonic and strictly positive, while $F_{|X} = 0$. Obviously, such a function is subharmonic in Q and we require, additionally, that for $F \in \mathcal{G}_X$, the measure $\mu_F = \Delta(F)$ to be normalized, i.e. $\mu_F(X) = 1$.

We introduce the following operators in a way similar to those proposed in [Zd1].

Definition 23. Let $\mathcal{P}_k\colon \mathcal{G}_{X_k} \to \mathcal{G}_{X_{k+1}}$ be defined as

$$\mathcal{P}_k(F)(x) = \sum_{y\in f_k^{-1}(x)} F(y)$$

Recall the notation: if μ is a measure in X_k then $(f_k)_*\mu$ is the image of the measure μ under f_k; in other words $(f_k)_*\mu = \mu \circ f_k^{-1}$.

Proposition 24. *If $F \in \mathcal{G}_{X_k}$ then*

$$(f_k)_*(\mu_F) = \Delta \mathcal{P}_k(F).$$

Proof. Let $\phi \in C_0^\infty(Q)$ be a test function. Then

$$\Delta\mathcal{P}_k(F)(\phi) = \int_Q \Delta\phi \cdot \mathcal{P}_k(F) = \sum_{i=1}^N \int_{Q_{k,i}} \Delta\phi \circ f_k \cdot F \cdot |f_k'|^2$$
$$= \sum_{i=1}^N \int_{Q_{k,i}} \Delta(\phi \circ f_k) \cdot F = \int_Q \phi \circ f_k \, d\mu_F = (f_k)_*(\phi),$$

which proves the statement. □

Remark 8 and the Maximum Principle give the following observation (see also [MV], [Zd3]).

Proposition 25. *There exists a universal constant $D > 0$ such that if X is an admissible Cantor set and $F_1, F_2 \in \mathcal{G}_X$ then the measures μ_{F_1}, μ_{F_2} are mutually absolutely continuous, with density bounded by D.*

Proof. Let $(\mathcal{Q}_k)_{k=0}^{\infty}$, be the sequence of admissible families, defining the set X (see Definition 4). Let $F \in \mathcal{G}_X$, let G be the standard Green's function for X. Let $\gamma = \gamma_{\mathcal{Q}_0}$ be the Jordan curve described in Remark 8. Since γ separates ∂Q from the union $\bigcup_{j=1}^{N} Q_{0,j}$, and since the set X is contained in the union $\bigcup_{j=1}^{N} Q_{0,j}$, we conclude that X is contained in the Jordan domain bounded by the curve γ. In the sequel, we denote this curve γ by $\gamma(X)$. Since μ_F is a probability measure, the ratio $\frac{G(x)}{F(x)}$ cannot be larger than 1 everywhere in $\gamma(X)$. Indeed, if $\frac{G(x)}{F(x)} \geq L > 1$ in γ, then the Maximum Principle implies that the inequality $G(x) \geq LF(x)$ holds everywhere in Q. This would imply $\mu(X) \geq L\omega(X) = 1$, a contradiction. By the same reason, the above ratio cannot be smaller than 1 everywhere in $\gamma(X)$. Together with Remark 8 this implies that there exists a constant $C > 0$, independent of both the set X and $F \in \mathcal{G}_X$ such that, for an arbitrary function $F \in \mathcal{G}_X$, $\frac{1}{C} \leq F_{|\gamma(X)} \leq C$. Using the Maximum Principle again, we conclude that $\frac{1}{C^2} \leq \frac{d\mu_{F_1}}{d\mu_{F_2}} \leq C^2$. □

Proposition 26. (Uniform decay of Green's functions) *There exist constants $0 < \eta < 1$, $C > 0$ (depending on $Q, M, \underline{a}, \overline{a}, N$) such that, for every admissible Cantor set X, for an arbitrary function $F \in \mathcal{G}_X$, and an arbitrary cylinder I of length n,*

$$\sup_{z \in Q_I} F(z) \leq C\eta^n. \tag{14}$$

Proof. First, notice that there is a common bound on $F_{|\gamma(X)}$, over all admissible sets X, and all functions $F \in \mathcal{G}_X$ (see the proof of Proposition 25). This implies that there exists a constant $C > 0$ such that $F_{Q_I} \leq C$ for every cylinder I of length 1.

Now, let I be an arbitrary cylinder of length n and IJ its subcylinder of length $n+1$. Let $z \in \partial Q_{IJ}$. Put $X_I = Q_I \cap X$. Then

$$F(z) = \int_{\partial Q_I} F(w)\omega(z, dw, Q_I \setminus X_I).$$

Thus,

$$\sup_{z \in \partial Q_{IJ}} F(z) \leq \sup_{w \in \partial Q_I} F(w) \cdot \omega(z, \partial Q_I, Q_I \setminus X_I) \tag{15}$$

It remains to check that

$$\omega(z, \partial Q_I, Q_I \setminus X_I) < \eta \tag{16}$$

for some $0 < \eta < 1$. This follows from the standard estimate (from below) of the harmonic measure by the capacity (see [GM], Theorem III. 9.1).

Indeed, since the required estimate is invariant under conformal maps, and the pair (Q_I, X_I) is mapped under f^n onto the pair (Q, X_n), it is enough to prove that there exists $\eta \in (0, 1)$ such that, for an arbitrary admissible Cantor set X,

$$\omega(z, X, Q \setminus X) > 1 - \eta$$

where $z \in Q_J$ and $|J| = 1$. Since we have the estimate of the capacity $\mathrm{Cap}(X)$ from below by κ, and since the set X is separated from ∂Q by some annulus with modulus larger than M, the estimate (16) follows. Thus, (15) implies, by induction, that, if I is a cylinder of length n then

$$\sup_{z\in\partial Q_I} F(z) < C\eta^n.$$

The required estimate on $\sup_{z\in Q_I} F(z)$ follows now from the Maximum Principle. $\square$

7. Sequences and convergence of admissible Cantor sets

Recall that Q is a fixed Jordan domain. Recall that a non-homogeneous Cantor set is given by a sequence of maps $f_k(z) = \sum_{i=1}^N (a_{k,i}^{-1} z + b_{k,i}) 1\!\!1_{Q_{k,i}}$, where $a_{k,i}^{-1} Q_{k,i} + b_{k,i} = Q$ and $k = 0, 1, 2 \dots$. Obviously, f_k is N-to-one and the branches $(f_k)_i^{-1} \colon Q \to Q_{k,i}$ are given by $(f_k)_i^{-1}(w) = a_{k,i}(w - b_{k,i})$.

Assume that we are given an infinite sequence of admissible Cantor sets $X^{(0)}, X^{(1)}, \dots, X^{(n)}, \dots$. Let us note the following:

Proposition 27. *Let $X^{(0)}, X^{(1)}, \dots X^{(n)}, \dots$ be a sequence of admissible Cantor sets of the same Hausdorff dimension ρ. For each n denote by $({}^n f_k)_{k=0}^{\infty}$, the sequence of maps defining the set $X^{(n)}$. Let $h > 0$ be given (not necessarily equal to the Hausdorff dimension of the sets $X^{(n)}$). For every n, let $\{\nu_k^{(n)}\}_{k=0}^{\infty}$ be the sequence of h-conformal measures associated to the set $X^{(n)}$. Then one can extract a subsequence n_s so that, for all $k \in \mathbf{N}$, and all $i = 1, \dots N$ the following holds:*

1. *The limit $\lim_{s\to\infty} ({}^{n_s} f_k)_i^{-1} = ({}^{\infty} f_k)_i^{-1}$ exists (which, equivalently, means simply that for all k the coefficients of the piecewise linear map ${}^{n_s} f_k$ converge to the coefficient of the piecewise linear map ${}^{\infty} f_k$). The Cantor set $X^{(\infty)}$, built with the maps ${}^{\infty} f_k$ is admissible.*
2. *For all $k \geq 0$, the following (weak-*) limits exist:*

$$\nu_k^{(n_s)} \to \nu_k^{(\infty)}$$

and $\nu_k^{(\infty)}$ is the system of h- conformal measures for $X^{(\infty)}$. The corresponding normalizing factors are

$$\lambda_{k,h}^{\infty} = \lim_{s\to\infty} \lambda_{k,h}^{n_s}.$$

Proof. The proof of convergence of the maps uses only the diagonal argument. Note that we do not require (and do not prove) this convergence to be uniform with respect to k.

To prove the convergence of the conformal measures, it is enough to recall the explicit formulas (7) and (8). Let us fix an arbitrary cylinder I, of length m. Then

$$\nu_0^{(n_s)}(I) = \frac{\left(|({}^{n_s} f_{m-1} \circ \dots \circ {}^{n_s} f_1 \circ {}^{n_s} f_0)'|^{-h}\right)_{|I}}{\lambda_{0,h}^{n_s} \lambda_{1,h}^{n_s} \dots \lambda_{m-1,h}^{n_s}}$$

and it is clear that the convergence of the coefficients of the maps ${}^{n_s} f_k$ for $k = 0, \dots m-1$ gives the convergence of $\nu_0^{(n_s)}(I)$ to $\nu_0^{(\infty)}(I)$. This easily implies that $\nu_0^{(n_s)}$ converge weakly to $\nu_0^{(\infty)}$, treated as measures in Σ and also as measures in $\mathbf{C}$. The same reasoning applies for the measures $\nu_k^{(n_s)}$. Here, as usual, we identify, through an appropriate coding, the measures on the Cantor sets $X_k^{(n_s)}$ and the measures on the symbolic space Σ. $\square$

Now, let $X^{(n)}$ be a sequence of admissible Cantor sets, converging to $X^{(\infty)}$ in the sense of item (1) in Proposition 27.

Proposition 28. *Let $X^{(0)}, X^{(1)}, \dots X^{(n)}, \dots$ be a sequence of admissible Cantor sets, converging to $X^{(\infty)}$ in the sense of item (1) in Proposition 27. Assume that a sequence of subharmonic functions $F^{(n)}\colon Q \to \mathbf{R}$ is given:*

$$F^{(n)} \in \mathcal{G}_{X^{(n)}}.$$

Then one extract a subsequence such that $F^{(n_s)}$ converges uniformly on compact subsets of Q to

$$F^{(\infty)} \in \mathcal{G}_{X^{(\infty)}}.$$

Moreover, the sequence of measures $\mu_{n_s} = \Delta(F^{(n_s)})$ converges weakly to $\mu^{(\infty)} = \Delta(F^{(\infty)})$.

Proof. The proof, again, uses the diagonal argument. Write $Q \setminus X^{(\infty)}$ as a countable union $\bigcup C_m$ of compact connected subsets of $Q \setminus X^{(\infty)}$, where $C_{m+1} \supset C_m$:

$$C_m = \overline{Q}'_m \setminus \bigcup_{|J|=m} Q_J$$

where Q_J correspond to the coding for the limit set $X^{(\infty)}$ and Q'_m is an increasing sequence of topological discs, with $X^{(\infty)} \subset Q'_m \subset \overline{Q}'_m \subset Q'_{m+1}$ and $\bigcup Q'_m = Q$.

Fix m. As $X^{(n)} \to X^{(\infty)}$, the functions $F^{(n)}$ form a uniformly bounded sequence of harmonic functions in a neighbourhood of C_m, starting from some $n = n(m)$. Thus, one can extract a subsequence converging uniformly in C_m to some function $F^{(\infty)}$ defined in C_m and harmonic in $\operatorname{int}(C_m)$. In the inductive construction, we choose yet another subsequence, converging uniformly in C_{m+1}. The limit must coincide in $\operatorname{int}(C_m)$ with the previously found limit $F^{(\infty)}$.

The required subsequence n_s is now chosen according to the Cantor diagonal argument. It is obvious from the construction that $F^{(\infty)}$ is positive and harmonic in $Q \setminus X^{(\infty)}$. It remains to check that setting $F^{(\infty)}(x) = 0$ for $x \in X^{(\infty)}$ gives a continuous (thus: also subharmonic) extension of $F^{(\infty)}$ to the whole domain Q.

Let I be an arbitrary cylinder, denote by l the length of I. Let I' be the cylinder of length $l-1$ containing I, and let Q_I (resp. $Q_{I'}$) be the domain corresponding to I (I'), defined by the coding for $X^{(\infty)}$. Similarly, denote by $Q_I^{(n)}$ (resp. $Q_{I'}^{(n)}$) the domain corresponding to I (resp. I'), defined by the coding for $X^{(n)}$.

Then, for large n_s, $Q_I \subset Q_{I'}^{(n_s)}$. Let $z \in Q_I$. Using the estimate (14) we get that

$$F^{(n_s)}(z) \le C\eta^{l-1}$$

and, therefore,

$$F^{(\infty)}(z) \le C\eta^{l-1}.$$

Thus $F^{(\infty)}(z)$ tends to 0 as $z \to X^{(\infty)}$.

The above reasoning shows also that the convergence $F^{(n_s)} \to F^{(\infty)}$ is uniform in each set $\overline{Q}'_m$. Once the convergence $F^{(n_s)} \rightrightarrows F^{(\infty)}$ has been established, the convergence of the measures μ_{n_s} is standard: if $\phi \in C_0^\infty(Q)$, then

$$\Delta\tilde{G}^{(n_s)}(\phi) = \int \Delta\phi \tilde{G}^{(n_s)} \to \int \Delta\phi \tilde{G}^{(\infty)} = \Delta\tilde{G}^{(\infty)}(\phi). \qquad \square$$

8. Sufficient condition for the inequality $\dim(X) > \dim(\omega)$

In this section we show how to adapt the argument proposed by Bourgain in [Bou] to prove the inequality $\dim(X) > \dim(\omega)$. In this way, we obtain some explicit sufficient condition which guarantees the inequality $\dim(X) > \dim(\omega)$ (see Proposition 29 below).

Recall that $\omega = \omega_0$ is the standard harmonic measure in X_0, evaluated at the point at ∞. Similarly, the harmonic measure on the set X_k is denoted by ω_k. We shall use the natural codings $C_0, C_1, \dots$ introduced in Definition 10.

In what follows, we often identify the symbolic cylinder I and the corresponding subset of the Cantor set $Q_I \cap X_0 = C_0^{-1}(I)$.

Proposition 29. *Let $X = X_0$ be the admissible Cantor set. Let, as above, $\omega = \omega_0$ be the harmonic measure on X_0, $\rho = dim_H(X)$ and let $\nu = \nu_0$ be the ρ-conformal measure on X_0. Assume the following: There exists $K > 0$ and $\gamma < 1$ such that for every cylinder $I = (I)_n \subset X$ of length n there exists a subcylinder $IJ = (IJ)_{n+K(I)}$, $K(I) \le K$ such that*

$$\max\left(\frac{\omega(IJ)}{\omega(I)} : \frac{\nu(IJ)}{\nu(I)}, \frac{\nu(IJ)}{\nu(I)} : \frac{\omega(IJ)}{\omega(I)}\right) > \frac{1}{\gamma}. \tag{17}$$

Then $\dim_{\mathcal{H}}(\omega) < \dim_{\mathcal{H}}(X) - \delta$ where δ is a constant depending only on $\underline{a}$, K, N, γ.

Proof. Given $I = I_n \in \mathcal{E}_n$, denote by $\mathcal{E}_{n+K(I)}(I)$ the family of all cylinders of generation $n + K(I)$, which are contained in I. First, we check that it follows from (17) that there exists $0 < \beta < 1$ such that, for every $I = I_n \in \mathcal{E}_n$,

$$\sum_{IJ \in \mathcal{E}_{n+K(I)}(I)} (\omega(IJ))^{\frac{1}{2}} (\nu(IJ))^{\frac{1}{2}} \le \beta \omega(I)^{\frac{1}{2}} \nu(I)^{\frac{1}{2}}. \tag{18}$$

The constant β depends on K, $\underline{a}$, $\overline{a}$ and γ. This can be seen as follows: Notice that, given two sequences of positive numbers $c_1, \dots c_\kappa$ and $d_1, \dots d_\kappa$ such that $\sum c_i = \sum d_i = 1$ we have, by Schwarz inequality, $\sum c_i^{\frac{1}{2}} d_i^{\frac{1}{2}} \le 1$ and the equality holds iff the sequences are equal.

Let κ be a positive integer and $B_0 = \{(p_1, \dots, p_\kappa, q_1, \dots q_\kappa) \in [0,1]^{2\kappa}\,;\ \sum_i p_i = \sum_i q_i = 1\}$ and, for $0 < \gamma < 1$ take the compact subset B_γ of B_0:

$$B_\gamma = \Big\{(p_1, \dots, p_\kappa, q_1, \dots q_\kappa) \in [0,1]^{2\kappa}\,;\ \sum_i p_i = \sum_i q_i = 1 \\ \text{and } \exists j \in \{1, \dots, \kappa\}\ p_j \le \gamma q_j\Big\}.$$

Since the function $(p_1, \dots, p_\kappa, q_1, \dots, q_\kappa) \mapsto \sum_i \sqrt{p_i q_i}$ is continuous, we get that there exists $\beta = \beta(\gamma, \kappa) < 1$ such that

$$\sup_{B_\gamma} \sum_i \sqrt{p_i q_i} \le \beta < 1.$$

Finally, to get (18), one can now apply the previous to $p_i = \omega(IJ)/\omega(I)$ and $q_i = \nu(IJ)/\nu(I)$.

Now, (18) implies easily that for $n > K$,

$$\sum_{I \in \mathcal{E}_n} \omega(I)^{\frac{1}{2}} \nu(I)^{\frac{1}{2}} \le \tilde{\beta}^n \tag{19}$$

with some $\beta < \tilde{\beta} < 1$. Next, fix some $s > \rho$ such that

$$\tilde{\beta}\underline{a}^{\rho-s} < 1 \tag{20}$$

Since $s > \rho = \dim_{\mathcal{H}}(X)$, we have

$$\liminf_{n\to\infty} \lambda_{1,s}\lambda_{2,s}\dots\lambda_{n,s} = 0.$$

Thus, there exists a sequence $n_i \to \infty$ such that $\lim_{i\to\infty} \lambda_{1,s}\lambda_{2,s}\dots\lambda_{n_i,s} = 0$. Fix such a sequence.

Obviously, one can assume that $\operatorname{diam} X = 1$. Now, formula (7) gives

$$\nu(I_{n_i}) = \frac{(\operatorname{diam} I_{n_i})^\rho}{\lambda_{1,\rho}\lambda_{2,\rho}\dots\lambda_{n_i,\rho}}.$$

Since $\lambda_{k,\rho} \le \underline{a}^{\rho-s}\lambda_{k,s}$, we can write, for every cylinder $I \in \mathcal{E}_{n_i}$,

$$\nu(I_{n_i}) \ge (\operatorname{diam} I_{n_i})^\rho(\underline{a})^{(s-\rho)n_i}\frac{1}{\lambda_{1,s}\lambda_{2,s}\dots\lambda_{n_i,s}} \ge (\operatorname{diam} I_{n_i})^\rho(\underline{a})^{(s-\rho)n_i},$$

for n_i large, since the value of the omitted fraction tends to ∞.

Inserting this inequality to (19) and using (20) we get, for small positive ε,

$$\begin{aligned}\sum_{J\in\mathcal{E}_{n_i}} (\omega(J))^{\frac12}(\operatorname{diam}(J))^{\frac{\rho-\varepsilon}{2}} &\le \sum_{J\in\mathcal{E}_{n_i}} (\omega(J))^{\frac12}(\nu(J))^{\frac12}\underline{a}^{\frac{\rho-s}{2}n_i}\operatorname{diam}(J)^{-\frac{\varepsilon}{2}}\\ &\le \sum_{J\in\mathcal{E}_{n_i}} (\omega(J))^{\frac12}(\nu(J))^{\frac12}(\underline{a})^{(\frac{\rho-s-\varepsilon}{2})n_i}\\ &\le \tilde{\beta}^{n_i}(\underline{a})^{(\frac{\rho-s-\varepsilon}{2})n_i} = \left(\tilde{\beta}\underline{a}^{\rho-s}\underline{a}^{\frac{s-\rho-\varepsilon}{2}}\right)^{n_i} < \hat{\beta}^{n_i}\end{aligned} \tag{21}$$

with some $\hat{\beta} < 1$, if ε is small (since s has been chosen so that $\tilde{\beta}\underline{a}^{\rho-s} < 1$).

We shall show that (21) implies that $\dim_{\mathcal{H}}\omega < \rho$. Denote by $\mathcal{F}_{n_i}$ the family of all cylinders $I \in \mathcal{E}_{n_i}$ for which $\omega(I) < \operatorname{diam}(I)^{\rho-\varepsilon}$, and by $\mathcal{H}_{n_i}$ the family of the remaining cylinders in $\mathcal{E}_{n_i}$. Then

$$\sum_{I\in\mathcal{H}_{n_i}} (\operatorname{diam} I)^{\rho-\varepsilon} \le \sum_{I\in\mathcal{H}_{n_i}} \omega(I) \le 1$$

and

$$\sum_{I\in\mathcal{F}_{n_i}} \omega(I) = \sum_{I\in\mathcal{F}_{n_i}} \omega(I)^{\frac12}\omega(I)^{\frac12} \le \sum_{I\in\mathcal{F}_{n_i}} \omega(I)^{\frac12}\operatorname{diam}(I)^{\frac{\rho-\varepsilon}{2}} \le \hat{\beta}^{n_i}.$$

Thus, by Borel–Cantelli lemma,

$$\omega\left(\bigcup_{i_0}\bigcap_{i=i_0}^{\infty}(\bigcup_{I\in\mathcal{H}_{n_i}} I)\right) = 1$$

On the other hand, we see, directly from the definition of Hausdorff measure, that $(\rho-\varepsilon)$- dimensional Hausdorff measure of the above set is σ–finite.

Therefore, $\dim_{\mathcal{H}}(\omega) \le \rho - \varepsilon$. □

9. The alternative case

We will investigate the case when condition (17) of Proposition 29 fails. We keep the notation of the previous sections. In particular, $X = X_0$ is an admissible Cantor set of dimension ρ. Let ν_k be the collection of ρ-conformal measures associated to X. Note that (although this fact in not used in our proof), we can assume, using Theorem B, that the starting measures ν_k are just the normalized ρ dimensional Hausdorff measures.

Proposition 30. *Let $X = X_0$ be an admissible Cantor set, let $\rho = \dim_{\mathcal{H}}(X)$, and let ν_k be the collection of ρ-conformal measures associated to X. Suppose that for all $1 > \gamma > 0$ and $K \in N$ there exist a cylinder I such that for all subcylinders IJ, where J is a word of length $\le K$ we have*

$$\gamma < \left| \frac{\omega(IJ)}{\omega(I)} : \frac{\nu_0(IJ)}{\nu_0(I)} \right| < \frac{1}{\gamma}. \tag{22}$$

Then we can construct another admissible Cantor set $\tilde{X}$ (not necessarily of the same dimension ρ), a ρ-conformal measure $\tilde{\nu}$ on $\tilde{X}$ and a bounded subharmonic function $F \in \mathcal{G}_{\tilde{X}}$ such that $\Delta F = \tilde{\nu}$.

Proof. The set $\tilde{X}$ will be constructed through the limit procedure described in Section 7. Let (γ_n) be a sequence of numbers in $(0,1)$, such that $\lim_{n\to\infty} \gamma_n = 1$. Under the hypothesis we can find a sequence $(I_n)_n$ of cylinders of size k_n, such that for every word J of length $\le n$

$$\gamma_n < \left| \frac{\omega(I_n J)}{\omega(I_n)} : \frac{\nu_0(I_n J)}{\nu_0(I_n)} \right| < \frac{1}{\gamma_n}. \tag{23}$$

For any cylinder I of length k, denote by f_I the linear map $f_{k-1} \circ \cdots \circ f_0$ mapping Q_I onto Q. Consider the functions G_{k_n} defined in Q by

$$G_{k_n}(x) = \frac{1}{\omega(I_n)} G(f_{I_n}^{-1}(x)).$$

Observe that $G_{k_n} \in \mathcal{G}_{X_{k_n}}$. Denote $\mu_{k_n} = \Delta G_{k_n}$. Thus, μ_{k_n} is a probability measure on X_{k_n}. Let J be a cylinder, identified, through the coding, with the appropriate subset of X_{k_n}. Then

$$\mu_{k_n}(J) = \frac{\omega(I_{k_n} J)}{\omega(I_{k_n})}.$$

The formula (23) can be now rewritten as follows: for every cylinder J of length $\le n$:

$$\gamma_n < |\mu_{k_n}(J) \colon \nu_{k_n}(J)| < \frac{1}{\gamma_n}. \tag{24}$$

We can now apply Propositions 27 and 28 to the sequence of admissible Cantor sets $X^{(n)} := X_{k_n}$, the associated ρ-conformal measures $\nu_0^{(n)} := \nu_{k_n}$ (and $\nu_m^{(n)} := \nu_{k_n+m}, m = 1, 2 \dots$) and the sequence of functions

$$F^{(n)} := G_{k_n} \in \mathcal{G}(X_{k_n}) = \mathcal{G}(X^{(n)}).$$

We obtain an admissible Cantor set $\tilde{X}$ and a function $\tilde{G} \in \mathcal{G}_{\tilde{X}}$ such that $\Delta \tilde{G} = \tilde{\mu}$, $\tilde{\mu}$ being the limit of (a subsequence of) the measures μ_{n_k}. Moreover, the measures ν_{k_n} converge weakly to the ρ- conformal measure $\tilde{\nu}$ on $\tilde{X}$.

On the other hand, the relation (24), implies that, for every cylinder J,

$$\frac{\mu_{k_n}(J)}{\nu_{k_n}(J)} \to 1$$

(where, again J is identified with an appropriate subset of X_{k_n}). This implies (cf. Proposition 27) that $\tilde{\mu}$ is a ρ-conformal measure on $\tilde{X}$, which completes the proof. $\square$

10. Rigidity argument

In this section we prove the following result which implies that the "alternative case" considered in the previous section cannot hold.

Proposition 31. *Let $X = X_0$ be an admissible Cantor set, and let $(\nu_k)_{k=0}^{\infty}$ be the collection of associated ρ conformal measures, where ρ is not necessarily equal to the Hausdorff dimension of the sets X_k. Further, let $\tilde{G} \in \mathcal{G}_X$ and let $\tilde{\omega} = \Delta\tilde{G}$. Then the measures $\tilde{\omega}$ and $\nu = \nu_0$ do not coincide.*

Proof. Consider, again, the sets

$$X = X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots, \tag{25}$$

the family of functions $\tilde{G}_j$ defined inductively by setting $\tilde{G}_0 = \tilde{G}$, $\tilde{G}_{k+1} = \mathcal{P}_k(\tilde{G}_k)$, and the corresponding measures $\tilde{\omega}_0 = \tilde{\omega} = \Delta\tilde{G}_0$, $\tilde{\omega}_k = \Delta\tilde{G}_k$.

The proof of Proposition 31 will be divided into two parts.

10.1. Non-real case.

Lemma 32. *Assume that none of the sets $X_0, X_1, X_2 \dots$ is contained in a set of zeros of a harmonic function defined in Q. If $\tilde{\omega} = \nu$ then for every cylinder $I \in \mathcal{E}_k$ there exists a constant α_I such that the equality*

$$\tilde{G}_k \circ f^k = \tilde{G}_0 \cdot \alpha_I \tag{26}$$

holds everywhere in Q_I.

Proof of the lemma. Since $\tilde{\omega}_k$ is the image of $\tilde{\omega}_0$ under the map f^k, ν_k is the image of ν_0 under f^k and also $\tilde{\omega}_0 = \nu_0$, we have: $\tilde{\omega}_k = \nu_k$.

Consider now two measures in Q_I: $(\tilde{\omega}_0)_{|Q_I}$ and $\tilde{\omega}_k \circ f^k_{Q_I}$. We have

$$\tilde{\omega}_k \circ f^k_{|Q_I} = \nu_k \circ f^k_{Q_I} = (\alpha_I \cdot \nu_0)_{|Q_I}$$

where $\alpha_I = |(f^k)'|^{\rho}_{|Q_I} \cdot \lambda_{0,\rho} \cdot \dots \cdot \lambda_{k-1,\rho}$. But $(\tilde{\omega}_0)_{|Q_I} = \Delta((\tilde{G}_0)_{|Q_I}$ and $(\tilde{\omega}_k \circ f^k)_{|Q_j} = \Delta((\tilde{G}_k \circ f^k)_{|Q_I})$. Since the measures are equal in Q_I, we get

$$(\tilde{G}_k \circ f^k)_{|Q_I} = (\tilde{G}_0)_{|Q_I} \cdot \alpha_I + H \tag{27}$$

where H is a harmonic function in Q_I. On the other hand, both $\tilde{G}_k \circ f^k$ and $\tilde{G}_0$ are equal to 0 in $Q_I \cap X = I$ and by assumption the set X_k (thus: also $X \cap Q_I = I$) is not contained in a set of zeros of a harmonic function. We deduce that H must be equal to 0 and the lemma follows. $\square$

We continue the proof of Proposition 31. We keep the assumption of Lemma 32. Consider two cylinders I, I' of the same length k. Then $f^k(I) = f^k(I') = X_k$.

Denote by $f_{I'}^{-k}$ the branch of f^{-k} mapping X_k to I' (and Q to $Q_{I'}$). Let $g = g_{II'} = f_{I'}^{-k} \circ f^k \colon Q_I \to Q_{I'}$. Then, by Lemma 32, everywhere in Q_I,

$$\frac{\alpha_I}{\alpha_{I'}} \tilde{G}_0 \circ g = \tilde{G}_0 \tag{28}$$

Now consider two cases.

(1) Case 1: There exists $D > 0$ such that for every $k \in \mathbf{N}$, for all $I, I' \in \mathcal{E}_k$,

$$\frac{\mathrm{diam} Q_I}{\mathrm{diam Q_{I'}}} < D.$$

(2) Case 2: The opposite.

First, we deal with Case 2. In this case, we can choose the cylinders I, I' so that g is a strong contraction; since it is a linear map, it is actually defined everywhere in $\mathbf{C}$ and we have $\mathrm{cl} g(Q) \subset Q$, so

$$\bigcup_k g^{-k}(Q) = \mathbf{C}.$$

Now, two functions: $\frac{\alpha_{I'}}{\alpha_I} \tilde{G}_0 \circ g$ and $\tilde{G}_0$ are defined and subharmonic in Q, harmonic in an open connected dense set $Q \setminus (X \cup g^{-1}(X))$. Since they coincide in an open set Q_I (see (28)), they coincide everywhere in Q. So, the formula

$$\frac{\alpha_{I'}}{\alpha_I} \tilde{G}_0 \circ g$$

gives an extension of $\tilde{G}_0$ to a subharmonic function defined in $g^{-1}(Q)$ and, in the same way, to a subharmonic function defined everywhere in $\mathbf{C}$.

Now, choosing another pair of cylinders, we can produce another relation of the type (28) and another extension of $\tilde{G}_0$, say

$$\frac{\alpha_{J'}}{\alpha_J} \tilde{G}_0 \cdot h = \tilde{G}_0.$$

By the same argument as above, these two extensions must coincide. We use the same letter $\tilde{G}_0$ to denote this, just described, extension.

In the reasoning below we use the following argument from Volberg's paper [Vol2]. Denote

$$Z = \{z \in \mathbf{C} \colon \tilde{G}_0(z) = 0\},$$

in particular,

$$Z \cap Q = X \tag{29}$$

The set Z is invariant under the action of both contractions h and g, and, consequently, the action of the group generated by them. It is easy to see that this group contains arbitrarily small translations. Thus, there exists such a small translation T that $T(X) \subset Q$. This would imply $T(X) \subset X$, a contradiction.

So, we are left with Case 1. Given $k \in \mathbf{N}$, we consider all cylinders of length k. There are N^k of them, and, by the assumption,

$$\frac{\mathrm{diam} Q_I}{\mathrm{diam Q_{I'}}} < D \tag{30}$$

for $I, I' \in \mathcal{E}_k$.

For $I, I' \in \mathcal{E}_k$ let, as above $g_{II'} = f_{I'}^{-k} \circ f^k \colon Q_I \to Q_{I'}$. Using (30) and the fact that $\mathrm{card}(\mathcal{E}_k) = N^k$ it is easy to see the following.

Claim. Let $\delta = \mathrm{dist}(X, \partial Q)$. There exists $0 < b_0 < \delta$ and a sequence $k_n \to \infty$ such that for every k_n one can find two cylinders $I, I' \in \mathcal{E}_{k_n}$ such that, putting

$$g_{II'} = \gamma_n z + b_n$$

we have

$$\gamma_n \to 1, \quad b_n \to b_0. \tag{31}$$

The functions $\tilde{G}_0$ and $\tilde{G}_0 \cap g_{II'}$ are continuous in $R := Q \cap g_{II'}^{-1}(Q)$ and harmonic in the open connected dense set $R \setminus (X \cup g_{II'}^{-1} X)$. Since they coincide in an open set Q_I, they coincide everywhere in R.

For n sufficiently large we have $X \subset R$ and $g_{II}^{-1}(X) \subset R$. Since both sets can be defined as sets of zeros of $\tilde{G}_0$ and $\tilde{G}_0 \circ g_{II'}$ respectively, they must coincide. Passing to a limit in (31), we see that X would be invariant under a (small) translation; again a contradiction. This ends the proof of Proposition 31 in the first case.

10.2. Real case. This case can be reduced to the previous one. We briefly describe the procedure: the previous proof goes through unchanged, until the formula (27). Now, we cannot conclude that $H = 0$. However, (27) implies that some X_k is contained in a set of zeros of a harmonic function H. Replacing X_0 by X_k, we can assume that $k = 0$.

Proposition 33. *Le $X = X_0$ be an admissible Cantor set. Assume that there exists a harmonic function H in Q such that $X \subset \{z\colon H(z) = 0\}$. Then there exists $k \geq 0$ such that X_k is contained in a straight line.*

Proof. Denote by $l = \{z \in Q : H(z) = 0\}$. Note that, after diminishing slightly the set Q so that it still contains the whole set X, we can assume that l is a union of finitely many real analytic arcs $l = l_1 \cup \cdots \cup l_r$, and that the set of intersections $l_j \cap l_j$ is finite. One can also assume that each such arc has infinitely many intersections with the set X. Let $x \in X$ be an intersection point of some arcs, say $x \in l_1 \cap l_2 \cap X$. Let I be a cylinder containing x, let I' be another cylinder of the same length and let $x' = g_{II'}(x)$.

We claim that x is an isolated point in either $l_1 \cap X$ or $l_2 \cap X$. Indeed, otherwise take $x' = g_{II'}(x)$ and observe that the set X in a neighborhood of x' (more precisely: the set $X \cap Q_{I'}$) would be contained in a union of two intersecting arcs, and not contained in one arc. Since the total number of intersections of the arcs $l_1, \ldots, l_r$ is finite, and the number of possible choices of x' is infinite, we get a contradiction.

Therefore, one can assume that X is contained in a union of a finite number of analytic arcs $l_1, \ldots, l_r$, which do not intersect. Pick a point $x \in X$ and a cylinder I containing x, of sufficiently high generation k so that the neighborhood Q_I of x intersects only one curve l_j. Then $f^k(Q_I) = Q$, $f^k(l_j \cap Q_I)$ is an analytic arc $L \subset Q$, and $X_k \subset L$.

The conclusion is that, replacing $X = X_0$ by some X_k, one can assume that X is contained in one analytic arc L. We claim that L is, actually, a straight line. To check it, first notice that $g_{II'}(L \cap I) = L \cap I'$, thus

$$g_{II'}(L \cap Q_I) = L \cap Q_{I'} \tag{32}$$

Assume first that there are arbitrarily strong contractions among the maps $g_{II'}$. Then, for such a strong contraction, (32) implies that $g_{II'}(L) \subset L$. If L is not a straight line then there are three points in L which are non-collinear. Applying the maps (contracting similitudies) $g_{II'}$ and using the fact $g_{II'}(L) \subset L$ we conclude that the curve L would not be differentiable, a contradiction.

If there are no strong contractions among the maps $g_{II'}$ (case one in the proof of part 1) then, as before, one can produce arbitrarily small translations τ such that $\tau(L) \cap Q \subset L$. Thus, L is a straight line. $\square$

Composing the maps f_k with rotations, we can assume that all the sets X_k are contained in the real line $\mathbf{R}$. Thus, since all the functions H in the formulas (27) must be equal to 0 in $\mathbf{R}$, $H(\overline{z}) = -H(z)$ and we can symmetrize all the formulas (27) by taking $\hat{G}_k(z) = \tilde{G}_k(z) + \tilde{G}(\overline{z})$. Then we get, instead of (27),

$$(\hat{G}_k \circ f^k)_{|Q_I} = (\hat{G}_0)_{|Q_I} \cdot \alpha_I$$

and the proof of the previous case applies. $\square$

Final conclusion—Proof of Theorem A.

Proof of Theorem A is now clear. Indeed, either harmonic and ρ-conformal measure of X satisfy relation (17) and hence $\dim_{\mathcal{H}} \omega < \dim_{\mathcal{H}} X$ by Proposition 29, or (17) fails and we get a contradiction by combining Propositions 31 and 30. $\square$

11. Further comments and remarks

In this paper the number of subdomains associated to an admissible map is fixed (equal to some N, cf section 1). Modulo some technical but small modifications the proofs can be carried out if we consider sequences of admissible funtions (f_n) with varying multiplicities $2 \leq N_n \leq N$, see Figure 1.

We can also easily modify the proof to get a uniform bound on $\dim_{\mathcal{H}} X - \dim_{\mathcal{H}} \omega$. To see this, observe that the difference $\dim_{\mathcal{H}} X - \dim_{\mathcal{H}} \omega$ depends only on γ and K in Proposition 29. Therefore, we need to show that γ and K can be chosen uniformly for $\underline{a}$, M and N fixed. But then, if the uniformity of (17) fails, for all $0 < \gamma < 1$ and $K > 0$ there exists a set X and a cylinder I as in Proposition 30. Using once again the diagonal argument (proposition 28) we return to the situation of section 10 and deduce the contradiction.

Nevertheless, the hypothesis on the upper bound of multiplicities (and hence lower bound $\underline{a}$ of contracting ratios) cannot be omitted as shows the following Proposition.

Proposition 34. *There exists a (unbounded) sequence N_n and a sequence of admissible functions (f_n) of multiplicities N_n such that the dimension of harmonic measure ω of the Cantor set X associated to (f_n) is equal to the Hausdorff dimension of the set.*

Let us give a sketch of the proof of this statement.

Proof. Consider, for instance, the self-similar triadic linear Cantor set X_0 that we identify with the symbolic dyadic tree. If σ is the left shift, $I \in \mathcal{E}_n$ a cylinder of length n and K any set, we will write IK for the set $\sigma^{-n}(K) \cap I$. So, IK is a subset of I.

It is well known that the dimension τ of the harmonic measure ω_{X_0} of $\mathbf{R}^2 \setminus X_0$ is strictly smaller than the Hausdorff dimension of the set X_0. Take $K_0 \subset X_0$ to be a compact set of dimension τ and of harmonic measure $\omega_{X_0}(K_0) > \frac{1}{2}$. Then, we can find a finite covering $\mathcal{J}_1$ of K_0 with cylinders $(I_j^1)_j$ with $I_j^1 \in \mathcal{J}_1 \subset \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{N_1}$ such that $\sum_j \operatorname{diam}(I_j^1)^{\tau+\frac{\tau}{2}} < \frac{1}{2}$.

Choose $K_1 \supset K_0$ compact of dimension τ and such that $\omega_{X_0}(K_1) > \frac{3}{4}$. Since $\dim_{\mathcal{H}}(I \cap K_0) \leq \tau$ for all cylinders I, we can augment K_1 with all images $\sigma^n(K_0)$, $n = 1, \ldots, N_1$. We can therefore assume that $I \cap K_0 \subset IK_1$ for all $I \in \mathcal{J}_1$ (but still $\dim_{\mathcal{H}}(K_1) = \tau$).

There is a finite collection $\mathcal{J}_2$ of cylinders $(I_j^2)_j$ with $I_j^2 \in \mathcal{J}_2 \subset \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{N_2}$ covering K_1 and verifying

$$\sum_j \operatorname{diam}(II_j^2)^{\tau+\frac{\tau}{4}} < \frac{1}{2^2} \operatorname{diam}(I)^{\tau+\frac{\tau}{2}},$$

for any cylinder $I \in \mathcal{J}_1$.

We proceed by induction. Assume we have constructed $\mathcal{J}_n \subset \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{N_n}$, a finite collection of cylinders covering a compact set K_{n-1} satisfying

- $K_0 \subset \cdots \subset K_{n-1}$ and $I \cap K_{n-2} \subset IK_{n-1}$ for all $I \in \mathcal{J}_{n-1}$,
- $\dim_{\mathcal{H}} K_{n-1} = \tau$,
- $\omega_{X_0}(K_{n-1}) > (1 - \frac{1}{2^{n-1}})$,
- $\sum_{J \in \mathcal{J}_n} \operatorname{diam}(IJ)^{\tau+\frac{\tau}{2^n}} < \frac{1}{2^n} \operatorname{diam}(I)^{\tau+\frac{\tau}{2^{n-1}}}$, for all $I \in \mathcal{J}_{n-1}$.

Take $K_n \supset K_{n-1}$, a compact set of dimension τ , such that $I \cap K_{n-1} \subset IK_n$, for all $I \in \mathcal{J}_n$ and verifying

$$\omega_{X_0}(K_n) > \left(1 - \frac{1}{2^n}\right).$$

There is a finite collection $\mathcal{J}_{n+1}$ of cylinders $(I_j^{n+1})_j$ with $I_j^{n+1} \in \mathcal{J}_{n+1} \subset \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{N_{n+1}}$ such that the sets $(I_j^{n+1})_j$ cover K_n and verify

$$\sum_j \operatorname{diam}(II_j^{n+1})^{\tau+\frac{\tau}{2^{n+1}}} < \frac{1}{2^{n+1}} \operatorname{diam}(I)^{\tau+\frac{\tau}{2^n}},$$

for every cylinder I from $\mathcal{J}_n$.

Note that by Harnack's principle there exist a constant $C > 0$ such that, for all cylinders I,

$$\omega_{X_0}(IK_n) > \left(1 - C\frac{1}{2^n}\right) \omega_{X_0}(I).$$

Consider the Cantor set

$$X = \bigcap_{n \in \mathbf{N}} \bigcup_{I_1 \in \mathcal{J}_1} \cdots \bigcup_{I_n \in \mathcal{J}_n} I_1 \ldots I_n.$$

Note that $K_0 \subset X \subset X_0$. Moreover, by construction, the Hausdorff dimension of X is less or equal to τ and since $K_0 \subset X_0$ it is equal to τ. On the other hand, by the monotonicity of the measure, $\omega_X(A) \geq \omega_{X_0}(A)$, for all $A \subset X$.

We only need to show that $\dim_{\mathcal{H}} \omega_X = \tau$. Suppose that $\dim_{\mathcal{H}} \omega_X < \tau$. Then, there exists $A \subset X$ such that $\dim_{\mathcal{H}} A < \tau$ and $\omega_X(A) = 1$. We deduce that $\omega_X(X \setminus A) = 0$ and a fortiori, $\omega_{X_0}(X \setminus A) = 0$. Therefore, $\omega_{X_0}(K_0) = \omega_{X_0}(K_0 \cap A)$ and $\dim(K_0 \cap A) < \tau$ which is absurd. □

Remark 35. A stated before, in the example the number N_k of subdomains $Q_{k,i}$ goes to infinity as $k \to \infty$ and, hence, there is no lower bound of the contraction ratios. Nevertheless, the annuli hypothesis remains valid.

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