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POSITIVE SOLUTIONS FOR PERTURBATIONS OF THE EIGENVALUE PROBLEM FOR THE ROBIN p-LAPLACIAN

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Abstract. We study perturbations of the eigenvalue problem for the Robin p-Laplacian. First we consider the case of a $(p-1)$-sublinear perturbation and prove existence, nonexistence and uniqueness of positive solutions. Then we deal with the case of a $(p-1)$-superlinear perturbation which need not satisfy the Ambrosetti–Rabinowitz condition and prove a multiplicity result for positive solutions. Our approach uses variational methods together with suitable truncation and perturbation techniques.

1. Introduction

Let $\Omega \subseteq \mathbf{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear parametric Robin problem:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u(z) = \lambda u(z)^{p-1} + f(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)u(z)^{p-1} = 0 & \text{on } \partial\Omega,\ u > 0,\ 1 < p < \infty. \end{cases}$$

Here Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}\left(|Du|^{p-2}Du\right) \quad \text{for all } u \in W^{1,p}(\Omega).$$

Also $\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du,n)_{\mathbf{R}^N}$ with $n(z)$ being the outward unit normal at $z \in \partial\Omega$. Moreover, $\lambda \in \mathbf{R}$ is a parameter and $f(z,x)$ is a Carathéodory perturbation (that is, for all $x \in \mathbf{R}$, the mapping $z \longmapsto f(z,x)$ is measurable on Ω and for a.a. $z \in \Omega$, $x \longmapsto f(z,x)$ is continuous).

We are interested in the existence, nonexistence and uniqueness of positive solutions for problem (P_λ) as the parameter $\lambda \in \mathbf{R}$ varies. We can view problem (P_λ) as a perturbation of the classical eigenvalue problem for the Robin p-Laplacian, investigated by Lê [12] and Papageorgiou and Rădulescu [15]. Similar studies concerning positive solutions, were conducted by Brezis and Oswald [5] (for problems driven by the Dirichlet Laplacian) and by Diaz and Saa [6] (for problems driven by the Dirichlet p-Laplacian). More recently, Gasinski and Papageorgiou [11] produced analogous results for the Neumann p-Laplacian. Multiplicity results concerning perturbed Robin problems involving the p-Laplacian were investigated recently by Winkert [18]. We

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also mention the recent work of Papageorgiou and Rădulescu [15], who studied a class of parametric equations driven by the Robin p-Laplacian and proved multiplicity results with precise sign information for all the solutions produced.

Here we first examine the case where $f(z,\cdot)$ is $(p-1)$-sublinear near $+\infty$, which leads to uniqueness results. Next, we consider the case where $f(z,\cdot)$ is $(p-1)$-superlinear (but without employing the Ambrosetti–Rabinowitz condition), which leads to multiplicity results.

2. Mathematical background

Our approach uses variational methods based on the critical point theory as well as suitable truncation and perturbation techniques. So, let X be a Banach space and X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). Given $\varphi\in C^1(X)$, we say that φ satisfies the Cerami condition (the C-condition for short), if the following is true: Every sequence $\{u_n\}_{n\geqslant 1}\subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1}\subseteq \mathbf{R}$ is bounded and

$$(1+\|u_n\|)\varphi'(u_n)\to 0 \ \text{ in } X^* \text{ as } n\to\infty,$$

admits a strongly convergent subsequence.

This is a compactness type condition on φ needed to offset the fact that the space X is not necessarily locally compact (being in general infinite dimensional). It is a basic tool in proving a deformation theorem which in turn leads to a minimax theory for the critical values of φ. Prominent in this theory, is the so-called "mountain pass theorem" due to Ambrosetti and Rabinowitz [3], stated here in a slightly more general form.

Theorem 1. *Assume that $\varphi\in C^1(X)$ satisfies the C-condition, $u_0,u_1\in X$, $\|u_1-u_0\|>\rho>0$,*

$$\max\{\varphi(u_0),\varphi(u_1)\}<\inf[\varphi(u)\colon \|u-u_0\|=\rho]=m_\rho$$

and $c=\inf\limits_{\gamma\in\Gamma}\max\limits_{0\leqslant t\leqslant 1}\varphi(\gamma(t))$ where $\Gamma=\{\gamma\in C([0,1],X)\colon \gamma(0)=u_0,\gamma(1)=u_1\}$. Then $c\geqslant m_\rho$ and c is a critical value of φ.

In the analysis of problem (P_λ), in addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive cone

$$C_+=\{u\in C^1(\overline{\Omega})\colon u(z)\geqslant 0 \text{ for all } z\in\overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+=\{u\in C_+\colon u(z)>0 \text{ for all } z\in\overline{\Omega}\}.$$

In the Sobolev space $W^{1,p}(\Omega)$, we consider the usual norm given by

$$\|u\|=\left[\|u\|_p^p+\|Du\|_p^p\right]^{1/p} \quad \text{for all } u\in W^{1,p}(\Omega).$$

To distinguish, we denote by $|\cdot|$ the Euclidean norm on $\mathbf{R}^N$. On $\partial\Omega$ we use the $(N-1)$-dimensional surface (Hausdorff) measure $\sigma(\,\cdot\,)$. So, we can define the Lebesgue spaces $L^q(\partial\Omega)$, $1\leqslant q\leqslant\infty$. We know that there is a unique, continuous linear map $\gamma_0\colon W^{1,p}(\Omega)\to L^p(\partial\Omega)$, known as the "trace map", such that $\gamma_0(u)=u|_{\partial\Omega}$ for all $u\in C^1(\overline{\Omega})$. We have $\gamma_0(W^{1,p}(\Omega))=W^{\frac{1}{p'},p}(\partial\Omega)$ $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$ and $\ker\gamma_0=W_0^{1,p}(\Omega)$. In the sequel, for the sake of notational simplicity, we drop the use of the trace map

γ_0 to denote the restriction of a Sobolev function on $\partial\Omega$. All such restrictions are understood in the sense of traces.

For every $x \in \mathbf{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$ we define $u^{\pm}(\cdot) = u(\cdot)^{\pm}$. We know that

$$u^{\pm} \in W^{1,p}(\Omega), \quad u = u^+ - u^- \text{ and } |u| = u^+ + u^-.$$

Given a measurable function $h : \Omega \times \mathbf{R} \to \mathbf{R}$ (for example, a Carathéodory function), we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \text{ for all } u \in W^{1,p}(\Omega)$$

and by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbf{R}^N$.

Let $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), v\rangle = \int_\Omega |Du|^{p-2}(Du, Dv)_{\mathbf{R}^N}\, dz \quad \text{for all } u, v \in W^{1,p}(\Omega). \tag{1}$$

Proposition 2. *The map $A\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by (1) is bounded (maps bounded sets to bounded sets), demicontinuous monotone (hence maximal monotone too) and of type $(S)_+$, that is, if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and*

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leqslant 0,$$

then $u_n \to u$ in $W^{1,p}(\Omega)$.

Suppose that $f_0\colon \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function such that

$$|f_0(z,x)| \leqslant a(z)(1 + |x|^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbf{R},$$

with $a \in L^\infty(\Omega)_+$ and $1 < r < p^* = \begin{cases} \frac{Np}{N-p} \text{ if } p < N, \\ +\infty \text{ if } p \geqslant N. \end{cases}$

We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0\colon W^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u(z)|^p\, d\sigma - \int_\Omega F_0(z, u(z))\, dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

We assume that $\beta \in C^{0,\tau}(\partial\Omega)$ with $0 < \tau < 1$ and $\beta \geqslant 0$, $\beta \neq 0$. From Papageorgiou and Rădulescu [15], we have the following result, which is a consequence of the nonlinear regularity theory.

Proposition 3. *Let $u_0 \in W^{1,p}(\Omega)$ be a local $C^1(\overline{\Omega})$-minimizer of φ_0, that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \textit{for all } h \in C^1(\overline{\Omega}),\ \|h\|_{C^1(\overline{\Omega})} \leqslant \rho_0.$$

Then $u_0 \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$ and it is also a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leqslant \varphi_0(u_0 + h) \quad \textit{for all } h \in W^{1,p}(\Omega),\ \|h\| \leqslant \rho_1.$$

Remark 1. We mention that the first such result was proved by Brezis and Nirenberg [4] for the space $H_0^1(\Omega)$.

Finally consider the nonlinear eigenvalue problem

$$(E_\lambda) \qquad \begin{cases} -\Delta_p u(z) = \lambda |u(z)|^{p-2} u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)|u(z)|^{p-2}u(z) = 0 & \text{on } \partial\Omega. \end{cases}$$

This eigenvalue problem was studied by Lê [12] and Papageorgiou and Rădulescu [15].

We say that $\lambda \in \mathbf{R}$ is an eigenvalue of the negative Robin p-Laplacian (denoted by $-\Delta_p^R$), if problem (E_λ) admits a nontrivial solution u, known as an eigenfunction corresponding to the eigenvalue λ.

Suppose that $\beta \in C^{0,\tau}(\partial\Omega)$, $0 < \tau < 1$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$, $\beta \neq 0$. Then we know that (E_λ) admits a smallest eigenvalue $\hat{\lambda}_1$ such that

- $\hat{\lambda}_1 > 0$;
- $\hat{\lambda}_1$ is simple and isolated (that is, if u, v are eigenfunctions corresponding to $\hat{\lambda}_1$, then $u = \xi v$ for some $\xi \in \mathbf{R} \setminus \{0\}$ and there exists $\varepsilon > 0$ such that $(\hat{\lambda}_1, \hat{\lambda}_1 + \varepsilon)$ contains no eigenvalue);
- we have

$$(2) \qquad \hat{\lambda}_1 = \inf \left[\frac{\|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p \, d\sigma}{\|u\|_p^p} : u \in W^{1,p}(\Omega),\ u \neq 0 \right].$$

The infimum in (2) is realized on the one dimensional eigenspace corresponding to $\hat{\lambda}_1$. From (2) it is clear that the elements of this eigenspace, do not change sign. Let $\hat{u}_1 \in W^{1,p}(\Omega)$ be the positive, L^p-normalized (that is, $\|\hat{u}_1\|_p = 1$) eigenfunction corresponding to $\hat{\lambda}_1$. The nonlinear regularity theory (see Lieberman [13]) and the nonlinear maximum principle (see Vazquez [16]), imply $\hat{u}_1 \in \operatorname{int} C_+$. We mention that $\hat{\lambda}_1$ is the only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have nodal (sign-changing) eigenfunctions. For more about the higher parts of the spectrum of $-\Delta_p^R$, we refer to Lê [12] and Papageorgiou and Rădulescu [15].

As an easy consequence of the above properties, we have the following result (see for example, Papageorgiou and Rădulescu [14]).

Proposition 4. *If $\vartheta \in L^\infty(\Omega)$, $\vartheta(z) \leqslant \hat{\lambda}_1$ a.e. in Ω, $\vartheta \neq \hat{\lambda}_1$, then there exists $\xi_0 > 0$ such that*

$$\|Du\|_p^p + \int_{\partial\Omega} \beta(z)|u|^p \, d\sigma - \int_\Omega \vartheta(z)|u|^p \, dz \geqslant \xi_0 \|u\|^p,$$

for all $u \in W^{1,p}(\Omega)$.

In the next section, we study the case in which the perturbation $f(z,\cdot)$ is $(p-1)$-sublinear.

3. Sublinear perturbations

Our hypotheses on the data of problem (P_λ), are the following:

$H(\beta)$: $\beta \in C^{0,\tau}(\partial\Omega)$, with $\tau \in (0,1)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$, $\beta \neq 0$.

$H(f)$: $f\colon \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,x) > 0$ for all $x > 0$ and

(i) $f(z,x) \leqslant a(z)(1 + x^{p-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)_+$;

(ii) $\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$;

(iii) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{p-1}} = +\infty$ uniformly for a.a. $z \in \Omega$.

Remark 2. Since we are interested in positive solutions and the above hypotheses concern the positive semiaxis $(0, +\infty)$, without any loss of generality, we assume that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$. Hypothesis $H(f)$(ii) implies that the perturbation $f(z, \cdot)$ is strictly $(p-1)$-sublinear near $+\infty$, while hypothesis $H(f)$(iii) dictates a similar polynomial growth near 0^+. A simple example illustrating such a perturbation, is given by the function $f(x) = x^{q-1}$ for all $x \geqslant 0$, with $q \in (1, p)$. In the sequel $F(z, x) = \int_0^x f(z, s)\, ds$.

We introduce the following two sets related to problem (P_λ):

$$\mathcal{P} = \{\lambda \in \mathbf{R}\colon \text{problem } (P_\lambda) \text{ admits a positive solution}\},$$
$$S(\lambda) = \text{the set of positive solutions for problem } (P_\lambda).$$

Note that as in Filippakis, Kristaly and Papageorgiou [8], exploiting the monotonicity of the operator A (see Proposition 2), we have that $S(\lambda)$ is downward directed, that is, if $u_1, u_2 \in S(\lambda)$, then we can find $u \in S(\lambda)$ such that $u \leqslant u_1$, $u \leqslant u_2$.

Proposition 5. *If hypotheses $H(\beta)$ and $H(f)$ hold, then $\mathcal{P} \neq \emptyset$ and for every $\lambda \in \mathcal{P}$, we have $S(\lambda) \subseteq \operatorname{int} C_+$.*

Proof. For every $\lambda \in \mathbf{R}$, we consider the C^1-functional $\hat{\varphi}_\lambda\colon W^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$\hat{\varphi}_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u^-\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p\, d\sigma - \frac{\lambda}{p}\|u^+\|_p^p - \int_\Omega F(z, u^+)\, dz$$

for all $u \in W^{1,p}(\Omega)$. Hypotheses $H(f)(i), (ii)$ imply that given $\varepsilon > 0$, we can find $c_1 = c_1(\varepsilon) > 0$ such that

$$F(z, x) \leqslant \frac{\varepsilon}{p} x^p + c_1 \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{3}$$

Let $\lambda < \hat{\lambda}_1$. Then for all $u \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}
\hat{\varphi}_\lambda(u) &\geqslant \frac{1}{p}\|Du^+\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p\, d\sigma - \frac{\lambda + \varepsilon}{p}\|u^+\|_p^p \\
&\quad + \frac{1}{p}\|Du^-\|_p^p + \frac{1}{p}\|u^-\|_p^p - c_1|\Omega|_N \quad (\text{see } (3)) \\
&\geqslant \frac{1}{p}[c_2 - \varepsilon]\|u^+\|^p + \frac{1}{p}\|u^-\|^p - c_1|\Omega|_N \quad (\text{see Prop. 4 and recall } \lambda < \hat{\lambda}_1).
\end{aligned}$$

Choosing $\varepsilon \in (0, c_2)$, we see that

$$\hat{\varphi}_\lambda(u) \geqslant \frac{c_3}{p}\|u\|^p - c_1|\Omega|_N \text{ with } c_3 = \min\{1, c_2 - \varepsilon\} > 0 \implies \hat{\varphi}_\lambda \text{ is coercive.}$$

Also, using the Sobolev embedding theorem and the continuity of the trace map, we see that $\hat{\varphi}_\lambda$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $\hat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\lambda(\hat{u}_\lambda) = \inf\left[\hat{\varphi}_\lambda(u)\colon u \in W^{1,p}(\Omega)\right]. \tag{4}$$

By virtue of hypothesis $H(f)$(iii), given any $\xi > \hat{\lambda}_1 - \lambda$, we can find $\delta = \delta(\xi) > 0$ such that

$$F(z,x) \geqslant \frac{\xi}{p} x^p \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [0,\delta]. \tag{5}$$

Choose $t \in (0,1)$ small such that $t\hat{u}_1(z) \in (0,\delta]$ for all $z \in \overline{\Omega}$ (recall that $\hat{u}_1 \in \operatorname{int} C_+$). We have

$$\begin{aligned}\hat{\varphi}_\lambda(t\hat{u}_1) &\leqslant \frac{t^p}{p}\|D\hat{u}_1\|_p^p + \frac{t^p}{p}\int_{\partial\Omega}\beta(z)\hat{u}_1^p\,d\sigma - \frac{\lambda t^p}{p}\|\hat{u}_1\|_p^p - \frac{\xi t^p}{p}\|\hat{u}_1\|_p^p \quad (\text{see } (5))\\ &= \frac{t^p}{p}[\hat{\lambda}_1 - \lambda - \xi] \quad (\text{recall } \|\hat{u}_1\|_p = 1).\end{aligned}$$

Since $\xi > \hat{\lambda}_1 - \lambda$, it follows that

$$\hat{\varphi}_\lambda(t\hat{u}_1) < 0 \implies \hat{\varphi}_\lambda(\hat{u}_\lambda) < 0 = \hat{\varphi}_\lambda(0) \text{ (see (4)), hence } \hat{u}_\lambda \neq 0.$$

From (4), we have

$$\begin{aligned}&\hat{\varphi}'_\lambda(\hat{u}_\lambda) = 0 \implies\\ &\langle A(\hat{u}_\lambda), h\rangle + \int_{\partial\Omega}\beta(z)(\hat{u}_\lambda^+)^{p-1}h\,d\sigma - \int_\Omega (\hat{u}_\lambda^-)^{p-1}h\,dz\\ &= \lambda\int_\Omega (\hat{u}_\lambda^+)^{p-1}h\,dz + \int_\Omega f(z,\hat{u}_\lambda^+)h\,dz \quad \text{for all } h \in W^{1,p}(\Omega).\end{aligned} \tag{6}$$

In (6) we choose $h = -\hat{u}_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\|D\hat{u}_\lambda^-\|_p^p + \|\hat{u}_\lambda^-\|_p^p = 0 \implies \hat{u}_\lambda \geqslant 0,\ \hat{u}_\lambda \neq 0.$$

Therefore (6) becomes

$$\langle A(\hat{u}_\lambda), h\rangle + \int_{\partial\Omega}\beta(z)\hat{u}_\lambda^{p-1}h\,d\sigma = \lambda\int_\Omega \hat{u}_\lambda^{p-1}h\,dz + \int_\Omega f(z,\hat{u}_\lambda)h\,dz \tag{7}$$

for all $h \in W^{1,p}(\Omega)$.

By $\langle\cdot,\cdot\rangle_0$ we denote the duality brackets for the pair $(W^{-1,p'}(\Omega), W_0^{1,p}(\Omega))$. From the representation theorem for the elements of $W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ (see, for example, Gasinski and Papageorgiou [9, p. 212]), we have

$$\operatorname{div}\left(|D\hat{u}_\lambda|^{p-2}D\hat{u}_\lambda\right) \in W^{-1,p'}(\Omega) \quad \left(\tfrac{1}{p} + \tfrac{1}{p'} = 1\right).$$

Integrating by parts, we have

$$\langle A(\hat{u}_\lambda), h\rangle = \langle -\operatorname{div}\left(|D\hat{u}_\lambda|^{p-2}D\hat{u}_\lambda\right), h\rangle_0 \quad \text{for all } h \in W_0^{1,p}(\Omega) \subseteq W^{1,p}(\Omega).$$

We use this in (7) and recall that $h|_{\partial\Omega} = 0$ for all $h \in W_0^{1,p}(\Omega)$. We obtain

$$\begin{aligned}&\langle -\operatorname{div}\left(|D\hat{u}_\lambda|^{p-2}D\hat{u}_\lambda\right), h\rangle_0 = \lambda\int_\Omega \hat{u}_\lambda^{p-1}h\,dz + \int_\Omega f(z,\hat{u}_\lambda)h\,dz\\ &\text{for all } h \in W_0^{1,p}(\Omega) \implies -\Delta_p\hat{u}_\lambda(z) = \lambda\hat{u}_\lambda(z)^{p-1} + f(z,\hat{u}_\lambda(z)) \ \text{ a.e. in } \Omega.\end{aligned} \tag{8}$$

From the nonlinear Green's identity (see, for example, Gasinski and Papageorgiou [9, p. 210]), we have

$$\langle A(\hat{u}_\lambda), h\rangle + \int_\Omega (\Delta_p\hat{u}_\lambda)h\,dz = \left\langle \frac{\partial\hat{u}_\lambda}{\partial n_p}, h\right\rangle_{\partial\Omega} \quad \text{for all } h \in W^{1,p}(\Omega) \quad (\text{see } (8))$$

where by $\langle \cdot, \cdot \rangle_{\partial\Omega}$ we denote the duality brackets for the pair

$$\left(W^{-\frac{1}{p'},p'}(\partial\Omega), W^{\frac{1}{p},p}(\partial\Omega)\right) \quad \left(\tfrac{1}{p} + \tfrac{1}{p'} = 1\right). \tag{9}$$

We return to (7) and use (9) above. We obtain

$$\begin{aligned}
&\int_\Omega (-\Delta_p \hat{u}_\lambda) h \, dz + \left\langle \frac{\partial \hat{u}_\lambda}{\partial n_p}, h \right\rangle_{\partial\Omega} + \int_{\partial\Omega} \beta(z) \hat{u}_\lambda^{p-1} h \, d\sigma \\
&= \lambda \int_\Omega \hat{u}_\lambda^{p-1} h \, dz + \int_\Omega f(z, \hat{u}_\lambda) h \, dz \ \text{ for all } h \in W^{1,p}(\Omega) \\
&\implies \left\langle \frac{\partial \hat{u}_\lambda}{\partial n_p}, h \right\rangle_{\partial\Omega} + \int_{\partial\Omega} \beta(z) \hat{u}_\lambda^{p-1} h \, d\sigma = 0 \ \text{ for all } h \in W^{1,p}(\Omega) \quad \text{(see (8))} \\
&\implies \frac{\partial \hat{u}_\lambda}{\partial n_p} + \beta(z) \hat{u}_\lambda^{p-1} = 0 \ \text{ on } \partial\Omega.
\end{aligned} \tag{10}$$

From (8) and (10) it follows that $\hat{u}_\lambda \in S(\lambda)$ and so $\lambda \in \mathcal{P}$ for every $\lambda < \hat{\lambda}_1$. From Winkert [17], we have that $\hat{u}_\lambda \in L^\infty(\Omega)$. So, we can apply Theorem 2 of Lieberman [13] and obtain that $\hat{u}_\lambda \in C_+ \setminus \{0\}$.

Hypotheses $H(f)$(i),(iii) imply that given $\rho > 0$, we can find $\xi_\rho > 0$ such that

$$f(z, x) + \xi_\rho x^{p-1} \geqslant 0 \ \text{ for a.a. } z \in \Omega, \ \text{ all } x \in [0, \rho]. \tag{11}$$

Let $\rho = \|\hat{u}_\lambda\|_\infty$ and let $\xi_\rho > 0$ be as in (11) above. Then

$$\begin{aligned}
&-\Delta_p \hat{u}_\lambda(z) + \xi_\rho \hat{u}_\lambda(z)^{p-1} \\
&= \lambda \hat{u}_\lambda(z)^{p-1} + f(z, \hat{u}_\lambda(z)) + \xi_\rho \hat{u}_\lambda(z)^{p-1} \geqslant 0 \ \text{ a.e. in } \Omega \quad \text{(see (11))} \\
&\implies \Delta_p \hat{u}_\lambda(z) \leqslant \xi_\rho \hat{u}_\lambda(z)^{p-1} \ \text{ a.e. in } \Omega, \\
&\implies \hat{u}_\lambda \in \operatorname{int} C_+ \quad \text{(see Vazquez [16]).}
\end{aligned}$$

So, we have proved that $S(\lambda) \subseteq \operatorname{int} C_+$. ☐

Proposition 6. *If hypotheses $H(\beta)$ and $H(f)$ hold and $\lambda \in \mathcal{P}$, then $(-\infty, \lambda] \subseteq \mathcal{P}$.*

Proof. Since $\lambda \in \mathcal{P}$, we can find $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$ (see Proposition 5). Let $\mu \in (-\infty, \lambda]$. Using $u_\lambda \in \operatorname{int} C_+$, we introduce the following truncation-perturbation of the reaction in problem (P_μ):

$$e_\mu(z, x) = \begin{cases} 0 & \text{if } x < 0, \\ (\mu + 1) x^{p-1} + f(z, x) & \text{if } 0 \leqslant x \leqslant u_\lambda(z), \\ (\mu + 1) u_\lambda(z)^{p-1} + f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{12}$$

This is a Carathéodory function. We set $E_\mu(z, x) = \int_0^x e_\mu(z, s) \, ds$ and consider the C^1-functional $\tau_\mu \colon W^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$\begin{aligned}
&\tau_\mu(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{p} \|u\|_p^p + \frac{1}{p} \int_{\partial\Omega} \beta(z) (u^+)^p \, d\sigma - \int_\Omega E_\mu(z, u) \, dz \ \text{ for all } u \in W^{1,p}(\Omega), \\
&\implies \tau_\mu(u) \geqslant \frac{1}{p} \|u\|^p - c_4 \ \text{ for some } c_4 > 0 \quad \text{(see } H(\beta) \text{ and (12))} \\
&\implies \tau_\mu \text{ is coercive.}
\end{aligned}$$

Also τ_μ is sequentially weakly lower semicontinuous. Hence we can find $u_\mu \in W^{1,p}(\Omega)$ such that

$$\tau_\mu(u_\mu) = \inf[\tau_\mu(u) \colon u \in W^{1,p}(\Omega)]. \tag{13}$$

As in the proof of Proposition 5 for $t \in (0,1)$ small (at least such that $t\hat{u}_1(z) \leqslant \min_{\overline{\Omega}} u_\lambda$ for all $z \in \overline{\Omega}$; recall that $\hat{u}_\lambda \in \operatorname{int} C_+$), we have

$$\tau_\mu(t\hat{u}_1) < 0 \implies \tau_\mu(u_\mu) < 0 = \tau_\mu(0) \quad \text{(see (13))}, \quad \text{hence } u_\mu \neq 0.$$

From (13) we have

$$\begin{aligned} &\tau_\mu'(u_\mu) = 0 \implies \\ &\langle A(u_\mu), h\rangle + \int_\Omega |u_\mu|^{p-2}u_\mu h\,dz + \int_{\partial\Omega} \beta(z)(u_\mu^+)^{p-1}h\,d\sigma = \int_\Omega e_\mu(z,u_\mu)h\,dz \\ &\text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{14}$$

In (14) we choose $h = -u_\mu^- \in W^{1,p}(\Omega)$. Then

$$\|Du_\mu^-\|_p^p + \|u_\mu^-\|_p^p = 0 \quad \text{(see (12))} \implies u_\mu \geqslant 0,\ u_\mu \neq 0.$$

Next in (14) we choose $(u_\mu - u_\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A(u_\mu), (u_\mu - u_\lambda)^+\rangle + \int_\Omega u_\mu^{p-1}(u_\mu - u_\lambda)^+\,dz + \int_{\partial\Omega} \beta(z)u_\mu^{p-1}(u_\mu - u_\lambda)^+\,d\sigma \\ &= \int_\Omega e_\mu(z,u_\mu)(u_\mu - u_\lambda)^+\,dz \\ &= \int_\Omega \left[\mu u_\lambda^{p-1} + f(z,u_\lambda)\right](u_\mu - u_\lambda)^+\,dz + \int_\Omega u_\lambda^{p-1}(u_\mu - u_\lambda)^+\,dz \\ &\leqslant \int_\Omega \left[\lambda u_\lambda^{p-1} + f(z,u_\lambda)\right](u_\mu - u_\lambda)^+\,dz + \int_\Omega u_\lambda^{p-1}(u_\mu - u_\lambda)^+\,dz \\ &= \langle A(u_\lambda), (u_\mu - u_\lambda)^+\rangle + \int_\Omega u_\lambda^{p-1}(u_\mu - u_\lambda)^+\,dz + \int_{\partial\Omega} \beta(z)u_\lambda^{p-1}(u_\mu - u_\lambda)^+\,d\sigma, \\ &\implies \langle A(u_\mu) - A(u_\lambda), (u_\mu - u_\lambda)^+\rangle + \int_\Omega (u_\mu^{p-1} - u_\lambda^{p-1})(u_\mu - u_\lambda)^+\,dz \\ &\qquad + \int_{\partial\Omega} \beta(z)(u_\mu^{p-1} - u_\lambda^{p-1})(u_\mu - u_\lambda)^+\,d\sigma \leqslant 0, \\ &\implies |\{u_\mu > u_\lambda\}|_N = 0, \text{ hence } u_\mu \leqslant u_\lambda. \end{aligned}$$

So, we have proved that

$$u_\mu \in [0, u_\lambda] \setminus \{0\},$$

where $[0,u_\lambda] = \{u \in W^{1,p}(\Omega) \colon 0 \leqslant u(z) \leqslant u_\lambda(z) \text{ a.e. in } \Omega\}$. Then (14) becomes

$$\langle A(u_\mu), h\rangle + \int_\Omega u_\mu^{p-1}h\,dz + \int_{\partial\Omega} \beta(z)u_\mu^{p-1}h\,d\sigma = (\mu+1)\int_\Omega u_\mu^{p-1}h\,dz + \int_\Omega f(z,u_\mu)h\,dz$$

for all $h \in W^{1,p}(\Omega)$. As in the proof of Proposition 5, using the nonlinear Green's identity, we obtain

$$u_\mu \in S(\mu) \subseteq \operatorname{int} C_+ \quad \text{and so } \mu \in \mathcal{P}.$$

Therefore $(-\infty, \lambda] \subseteq \mathcal{P}$. $\square$

Hypotheses $H(f)$(i),(iii) imply that given any $\xi > 0$ and $r \in (p, p^*)$, we can find $c_5 = c_5(\xi, r) > 0$ such that

$$f(z,x) \geqslant \xi x^{p-1} - c_5 x^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{15}$$

This unilateral growth constraint on the perturbation $f(z,x)$, leads to the following auxiliary Robin problem:

$$\begin{cases} -\Delta_p u(z) = \xi u(z)^{p-1} - c_5 u(z)^{r-1} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z) u(z)^{p-1} = 0 & \text{on } \partial\Omega, \ u > 0. \end{cases} \tag{16}$$

Proposition 7. *If hypotheses $H(\beta)$ hold, then for $\xi > 0$ big problem (16) has a unique positive solution $\overline{u} \in \operatorname{int} C_+$.*

Proof. First we establish the existence of a positive solution for problem (16). To this end, we consider the C^1-functional $\psi\colon W^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$\psi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u^-\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p \, d\sigma + \frac{c_5}{r}\|u^+\|_r^r - \frac{\xi}{p}\|u^+\|_p^p$$

for all $u \in W^{1,p}(\Omega)$. We have

$$\psi(u) \geqslant \frac{1}{p}\|u\|^p + \left[\frac{c_5}{r}\|u^+\|_r^{r-p} - \left(\frac{\xi}{p} + 1\right) c_6\right] \|u^+\|_r^p \ \text{ for some } c_6 > 0. \tag{17}$$

Since $r > p$, from (17) it follows that ψ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u} \in W^{1,p}(\Omega)$ such that

$$\psi(\overline{u}) = \inf[\psi(u)\colon u \in W^{1,p}(\Omega)]. \tag{18}$$

Choosing $\xi > \hat{\lambda}_1$ and since $r > p$, we see that for $t \in (0,1)$ small, we have

$$\psi(t\hat{u}_1) < 0 \implies \psi(\overline{u}) < 0 = \psi(0) \quad \text{(see (18)),} \quad \text{hence } \overline{u} \neq 0.$$

From (18) we have

$$\begin{aligned} &\psi'(\overline{u}) = 0 \implies \langle A(\overline{u}), h\rangle - \int_\Omega (\overline{u}^-)^{p-1} h \, dz + \int_{\partial\Omega} \beta(z)(\overline{u}^+)^{p-1} h \, d\sigma \\ &= \xi \int_\Omega (\overline{u}^+)^{p-1} h \, dz - c_5 \int_\Omega (\overline{u}^+)^{r-1} h \, dz \ \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{19}$$

Choose $h = -\overline{u}^- \in W^{1,p}(\Omega)$. Then we obtain $\overline{u} \geqslant 0$, $\overline{u} \neq 0$ and so (19) becomes

$$\begin{aligned} &\langle A(\overline{u}), h\rangle + \int_{\partial\Omega} \beta(z) \overline{u}^{p-1} h \, d\sigma = \xi \int_\Omega \overline{u}^{p-1} h \, dz - c_5 \int_\Omega \overline{u}^{r-1} h \, dz \ \text{ for all } h \in W^{1,p}(\Omega) \\ &\implies \overline{u} \text{ is a positive solution of (16)} \quad \text{(as in the proof of Proposition 5).} \end{aligned}$$

The nonlinear regularity theory (see [17], [13]) implies that $\overline{u} \in C_+ \setminus \{0\}$. We have

$$\begin{aligned} -\Delta_p \overline{u}(z) \geqslant -c_5 \overline{u}(z)^{r-1} \ \text{ a.e. in } \Omega &\implies \Delta_p \overline{u}(z) \leqslant c_5 \|\overline{u}\|_\infty^{r-p} \overline{u}(z)^{p-1} \ \text{ a.e. in } \Omega \\ &\implies \overline{u} \in \operatorname{int} C_+ \quad \text{(see Vazquez [16]).} \end{aligned}$$

Next we show the uniqueness of this positive solution. For this purpose, we introduce the integral functional $\vartheta\colon L^p(\Omega) \to \overline{\mathbf{R}} = \mathbf{R} \cup \{+\infty\}$ defined by

$$\vartheta(u) = \begin{cases} \dfrac{1}{p}\|Du^{1/p}\|_p^p + \dfrac{1}{p}\displaystyle\int_{\partial\Omega} \beta(z) u \, d\sigma & \text{if } u \geqslant 0, \ u^{1/p} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Lemma 1 of Diaz and Saa [6] implies that ϑ is convex and lower semicontinuous. Suppose that $\overline{u}$, v are two positive solutions of the auxiliary problem (16). From the first part of the proof, we have

$$\begin{aligned}&\overline{u}, v \in \operatorname{int} C_+\\ &\quad\Longrightarrow \overline{u}^p, v^p \in \operatorname{dom}\vartheta = \{y \in W^{1,p}(\Omega) \colon \vartheta(y) < \infty\} \quad \text{(the effective domain of } \vartheta).\end{aligned}$$

Then for every $h \in C^1(\overline{\Omega})$ and for $|t| \leqslant 1$ small, we have

$$\overline{u}^p + th,\ v + th \in \operatorname{dom}\vartheta.$$

It follows that ϑ is Gâteaux differentiable at $\overline{u}^p$ and at v^p in the direction h. Using the chain rule, we have

$$\begin{aligned}\vartheta'(\overline{u}^p)(h) &= \frac{1}{p}\int_\Omega \frac{-\Delta_p \overline{u}}{\overline{u}^{p-1}} h\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z) h\,d\sigma\\ \vartheta'(v^p)(h) &= \frac{1}{p}\int_\Omega \frac{-\Delta_p v}{v^{p-1}} h\,dz + \frac{1}{p}\int_{\partial\Omega} \beta(z) h\,d\sigma \quad \text{for all } h \in W^{1,p}(\Omega)\end{aligned}$$

(recall that $C^1(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$). The convexity of ϑ implies the monotonicity of ϑ'. So, we have

$$\begin{aligned}0 &\leqslant \frac{1}{p}\int_\Omega \left[\frac{-\Delta_p \overline{u}}{\overline{u}^{p-1}} - \frac{-\Delta_p v}{v^{p-1}}\right](\overline{u}^p - v^p)\,dz\\ &\leqslant \frac{1}{p}\int_\Omega c_5(v^{r-p} - \overline{u}^{r-p})(\overline{u}^p - v^p)\,dz \leqslant 0 \quad \text{(see (16))}\\ &\Longrightarrow \overline{u} = v \Longrightarrow \overline{u} \in \operatorname{int} C_+ \text{ is the unique positive solution of (16).} \qquad \square\end{aligned}$$

Proposition 8. *If hypotheses $H(\beta)$ and $H(f)$ hold and $\lambda \in \mathcal{P}$, then $\overline{u} \leqslant u$ for all $u \in S(\lambda)$.*

Proof. Let $u \in S(\lambda)$. We introduce the following Carathéodory function

$$\gamma(z,x) = \begin{cases} 0 & \text{if } x < 0,\\ (\xi+1)x^{p-1} - c_5 x^{r-1} & \text{if } 0 \leqslant x \leqslant u(z),\\ (\xi+1)u(z)^{p-1} - c_5 u(z)^{r-1} & \text{if } u(z) < x.\end{cases} \tag{20}$$

Let $\Gamma(z,x) = \int_0^x \gamma(z,s)\,ds$ and consider the C^1-functional $\chi \colon W^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$\chi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p\,d\sigma - \int_\Omega \Gamma(z,u)\,dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Using hypothesis $H(\beta)$ and (20), we see that

$$\chi(u) \geqslant \frac{1}{p}\|u\|^p - c_6 \quad \text{for some } c_6 > 0 \Longrightarrow \chi \text{ is coercive.}$$

In addition, χ is sequentially weakly lower semicontinuous. So, we can find $\overline{u}_* \in W^{1,p}(\Omega)$ such that

$$\chi(\overline{u}_*) = \inf[\chi(u) \colon u \in W^{1,p}(\Omega)]. \tag{21}$$

As before, since $r > p$, for $t \in (0,1)$ small, we have

$$\chi(t\hat{u}_1) < 0 \Longrightarrow \chi(\overline{u}_*) < 0 = \chi(0) \quad \text{(see (21)), hence } \overline{u}_* \neq 0.$$

From (21) we have

$$\chi'(\overline{u}_*) = 0 \implies$$
$$\langle A(\overline{u}_*), h\rangle + \int_\Omega |\overline{u}_*|^{p-2}\overline{u}_* h\,dz + \int_{\partial\Omega} \beta(z)(\overline{u}_*^+)^{p-1} h\,d\sigma = \int_\Omega \gamma(z,\overline{u}_*)h\,dz \tag{22}$$
$$\text{for all } h \in W^{1,p}(\Omega).$$

In (22) we choose $h = -\overline{u}_*^- \in W^{1,p}(\Omega)$. Then

$$\|D\overline{u}_*^-\|_p^p + \|\overline{u}_*^-\|_p^p = 0 \quad \text{(see (20))} \implies \overline{u}_* \geqslant 0,\ \overline{u}_* \neq 0.$$

Next in (22) we choose $h = (\overline{u}_* - u)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A(\overline{u}_*), (\overline{u}_* - u)^+\rangle + \int_\Omega \overline{u}_*^{p-1}(\overline{u}_* - u)^+\,dz + \int_{\partial\Omega} \beta(z)\overline{u}_*^{p-1}(\overline{u}_* - u)^+\,d\sigma \\
&= \int_\Omega [\xi u^{p-1} - c_5 u^{r-1}](\overline{u}_* - u)^+\,dz + \int_\Omega u^{p-1}(\overline{u}_* - u)^+\,dz \quad \text{(see (20))} \\
&\leqslant \int_\Omega [\lambda u^{p-1} + f(z,u)](\overline{u}_* - u)^+\,dz + \int_\Omega u^{p-1}(\overline{u}_* - u)^+\,dz \quad \text{(see (15))} \\
&= \langle A(u), (\overline{u}_* - u)^+\rangle + \int_\Omega u^{p-1}(\overline{u}_* - u)^+\,dz + \int_{\partial\Omega} \beta(z)u^{p-1}(\overline{u}_* - u)^+\,d\sigma \\
&\quad \text{(since } u \in S(\lambda)) \\
&\implies |\{\overline{u}_* > u\}|_N = 0 \text{ (as before), hence } \overline{u}_* \leqslant u.
\end{aligned}$$

So, we have proved that

$$\overline{u}_* \in [0,u] \setminus \{0\}.$$

Then from (20) and (22) it follows that $\overline{u}_* \in \operatorname{int} C_+$ is a positive solution of (16) and so by virtue of Proposition 7, we have

$$\overline{u}_* = \overline{u} \implies \overline{u} \leqslant u \quad \text{for all } u \in S(\lambda). \qquad \square$$

In the proof of Proposition 5 we have seen that $(-\infty, \hat{\lambda}_1) \subseteq \mathcal{P}$. Next we show that in fact we have $\mathcal{P} = (-\infty, \hat{\lambda}_1)$.

Proposition 9. *If hypotheses $H(\beta)$ and $H(f)$ hold, then $\hat{\lambda}_1 \notin \mathcal{P}$.*

Proof. Arguing by contradiction, suppose that $\hat{\lambda}_1 \in \mathcal{P}$. Then we can find $u_0 \in S(\hat{\lambda}_1) \subseteq \operatorname{int} C_+$. Recall that $\hat{u}_1 \in \operatorname{int} C_+$ too. Invoking Lemma 3.3 of Filippakis, Kristaly and Papageorgiou [8] we can find $c_7, c_8 > 0$ such that

$$\begin{aligned}
c_7 u_0 \leqslant \hat{u}_1 \leqslant c_8 u_0 &\implies c_7 \leqslant \frac{\hat{u}_1}{u_0} \leqslant c_8 \ \text{ and } \ \frac{1}{c_8} \leqslant \frac{u_0}{\hat{u}_1} \leqslant \frac{1}{c_7} \\
&\implies \frac{\hat{u}_1}{u_0} \ \text{ and } \ \frac{u_0}{\hat{u}_1} \ \text{ belong in } L^\infty(\Omega).
\end{aligned} \tag{23}$$

We have

$$-\Delta_p u_0(z) = \hat{\lambda}_1 u_0(z)^{p-1} + f(z,u_0(z)) \ \text{ a.e. in } \Omega, \quad \frac{\partial u_0}{\partial n_p} + \beta(z)u_0^{p-1} = 0 \ \text{ on } \partial\Omega. \tag{24}$$

Let

$$R(\hat{u}_1, u_0)(z) = |D\hat{u}_1(z)|^p - |Du_0(z)|^{p-2}\left(Du_0(z), D\left(\frac{\hat{u}_1^p}{u_0^{p-1}}\right)(z)\right)_{\mathbf{R}^N}. \tag{25}$$

From the nonlinear Picone's identity of Allegretto and Huang [2], we have

$$0 \leqslant \int_\Omega R(\hat{u}_1, u_0)\, dz = \|D\hat{u}_1\|_p^p - \int_\Omega |Du_0|^{p-2} \left(Du_0, D\left(\frac{\hat{u}_1^p}{u_0^{p-1}}\right)\right)_{\mathbf{R}^N} dz. \tag{26}$$

From (23), (24) and the nonlinear Green's identity (see, for example, Gasinski and Papageorgiou [9, p. 211]), we have

$$\begin{aligned} &\int_\Omega |Du_0|^{p-2} \left(Du_0, D\left(\frac{\hat{u}_1^p}{u_0^{p-1}}\right)\right)_{\mathbf{R}^N} dz \\ &= \int_\Omega (-\Delta_p u_0)\left(\frac{\hat{u}_1^p}{u_0^{p-1}}\right) dz + \left\langle \frac{\partial u_0}{\partial n_p}, \frac{\hat{u}_1^p}{u_0^{p-1}}\right\rangle_{\partial\Omega}. \end{aligned} \tag{27}$$

Returning to (26) and using (24) and (27), we obtain

$$\begin{aligned} 0 &\leqslant \|D\hat{u}_1\|_p^p - \hat{\lambda}_1\|\hat{u}_1\|_p^p - \int_\Omega f(z,u_0)\frac{\hat{u}_1^p}{u_0^{p-1}}\, dz + \int_{\partial\Omega} \beta(z)\hat{u}_1^p\, d\sigma \\ &= -\int_\Omega f(z,u_0)\frac{\hat{u}_1^p}{u_0^{p-1}}\, dz < 0 \quad (\text{see } H(f)), \end{aligned}$$

a contradiction. So, $\hat{\lambda}_1 \notin \mathcal{P}$. ☐

From Propositions 6 and 9 it follows that

$$\mathcal{P} = (-\infty, \hat{\lambda}_1)$$

(recall that in the proof of Proposition 5 we established that $(-\infty, \hat{\lambda}_1) \subseteq \mathcal{P}$).

Proposition 10. *If hypotheses $H(\beta)$ and $H(f)$ hold, $\lambda \in \mathcal{P}$ and $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$, then for every $\mu < \lambda$, we can find $u_\mu \in S(\mu) \subseteq \operatorname{int} C_+$ such that $u_\mu \leqslant u_\lambda$.*

Proof. We consider the following truncation-perturbation of the reaction in problem (P_μ):

$$\gamma_\mu(z,x) = \begin{cases} 0 & \text{if} x < 0, \\ (\mu+1)x^{p-1} + f(z,x) & \text{if } 0 \leqslant x \leqslant u_\lambda(z), \\ (\mu+1)u_\lambda(z)^{p-1} + f(z,u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{28}$$

This is a Carathéodory function. We set $\Gamma_\mu(z,x) = \int_0^x \gamma_\mu(z,s)ds$ and consider the C^1-functional $\eta\colon W^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$\eta(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p}\|u\|_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p\, d\sigma - \int_\Omega \Gamma_\mu(z,u)\, dz \ \text{ for all } u \in W^{1,p}(\Omega).$$

From hypothesis $H(\beta)$ and (28) it is clear that η is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\mu \in W^{1,p}(\Omega)$ such that

$$\eta(u_\mu) = \inf[\eta(u)\colon u \in W^{1,p}(\Omega)]. \tag{29}$$

As before (see the proof of Proposition 5), using hypothesis $H(f)$(iii), we show that for $t \in (0,1)$ small (at least such that $t\hat{u}_1(z) \leqslant \min_{\overline{\Omega}} u_\lambda$, recall $u_\lambda \in \operatorname{int} C_+$), we have

$$\eta(t\hat{u}_1) < 0 \implies \eta(u_\mu) < 0 = \eta(0) \quad (\text{see (29)}), \quad \text{hence } u_\mu \neq 0.$$

From (29), we have

$$\eta'(u_\mu) = 0 \implies$$
$$\langle A(u_\mu), h\rangle + \int_\Omega |u_\mu|^{p-2}u_\mu h\,dz + \int_{\partial\Omega} \beta(z)(u_\mu^+)^{p-1}h\,d\sigma = \int_\Omega \gamma_\mu(z, u_\mu)h\,dz$$

for all $h \in W^{1,p}(\Omega)$. As in the proof of Proposition 8, choosing first $h = -u_\mu^- \in W^{1,p}(\Omega)$ and then $h = (u_\mu - u_\lambda)^+ \in W^{1,p}(\Omega)$, we show that

$$u_\mu \in [0, u_\lambda] \setminus \{0\}.$$

From (28) it follows that $u_\mu \in S(\mu) \subseteq \operatorname{int} C_+$ and $u_\mu \leqslant u_\lambda$. □

Proposition 11. *If hypotheses $H(\beta)$ and $H(f)$ hold, $\lambda \in \mathcal{P} = (-\infty, \hat{\lambda}_1)$, then problem (P_λ) admits a smallest positive solution $u_\lambda^* \in S(\lambda) \subseteq int\, C_+$.*

Proof. From Dunford and Schwartz [7, p. 336], we know that we can find $\{u_n\}_{n\geqslant 1} \subseteq S(\lambda)$ such that

$$\inf S(\lambda) = \inf_{n\geqslant 1} u_n.$$

From Proposition 10 and since $S(\lambda)$ is downward directed, we may assume that

$$u_n \leqslant \hat{u} \text{ for all } n \geqslant 1, \quad \text{with } \hat{u} \in S(\hat{\lambda}) \subseteq \operatorname{int} C_+,\ \hat{\lambda} \in \mathcal{P},\ \lambda_n < \hat{\lambda},\ n \geqslant 1. \tag{30}$$

We have

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z)u_n^{p-1}h\,d\sigma = \lambda \int_\Omega u_n^{p-1}h\,dz + \int_\Omega f(z, u_n)h\,dz \tag{31}$$

for all $h \in W^{1,p}(\Omega)$, all $n \geqslant 1$. In (31) we choose $h = u_n \in W^{1,p}(\Omega)$. Then using hypotheses $H(\beta), H(f)(i)$ and (30) we see that

$$\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_\lambda^* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{32}$$

In (31) we choose $h = u_n - u_\lambda^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (32). We obtain

$$\begin{aligned} &\lim_{n\to\infty} \langle A(u_n), u_n - u_\lambda^*\rangle = 0 \implies \\ &u_n \to u_\lambda^* \text{ in } W^{1,p}(\Omega) \quad \text{(see Proposition 2 and (32)).} \end{aligned} \tag{33}$$

So, if in (31) we pass to the limit as $n \to \infty$ and use (33) and Proposition 2, then

$$\langle A(u_\lambda^*), h\rangle + \int_{\partial\Omega} \beta(z)(u_\lambda^*)^{p-1}hd\sigma = \lambda \int_\Omega (u_\lambda^*)^{p-1}hdz + \int_\Omega f(z, u_\lambda^*)hdz$$

for all $h \in W^{1,p}(\Omega)$ which implies

$$-\Delta_p u_\lambda^*(z) = \lambda(u_\lambda^*)(z)^{p-1} + f(z, u_\lambda^*(z)) \text{ a.e. in } \Omega,\ \frac{\partial u_\lambda^*}{\partial n_p} + \beta(z)(u_\lambda^*)^{p-1} = 0 \text{ on } \partial\Omega \tag{34}$$

(as in the proof of Proposition 5). Moreover, from Proposition 8, we have

$$\overline{u} \leqslant u_n \text{ for all } n \geqslant 1 \implies \overline{u} \leqslant u_\lambda^* \quad \text{(see (33)).} \tag{35}$$

Then (34) and (35) imply that

$$u_\lambda^* \in S(\lambda) \text{ and } u_\lambda^* = \inf S(\lambda).$$ □

If we strengthen the conditions on the perturbation $f(z,\cdot)$, we can guarantee the uniqueness of the positive solution of problem (P_λ).

The new stronger conditions on $f(z,x)$ are the following:

$H(f)'$: $f\colon \Omega\times\mathbf{R}\to\mathbf{R}$ is a Carathéodory function such that for a.a. $z\in\Omega$, $f(z,0)=0$, $f(z,x)>0$ for all $x>0$, hypotheses $H(f)'$(i),(ii),(iii) are the same as the corresponding hypotheses $H(f)$(i),(ii),(iii) and

(iv) for a.a. $z\in\Omega$, $x\to\frac{f(z,x)}{x^{p-1}}$ is decreasing, strictly for all $z\in\Omega_0\subseteq\Omega$ with $|\Omega_0|_N>0$.

Proposition 12. *If hypotheses $H(\beta)$ and $H(f)'$ hold and $\lambda\in\mathcal{P}=(-\infty,\hat{\lambda}_1)$, then $S(\lambda)$ is a singleton $\{u_\lambda\}$ and the map $\lambda\longmapsto u_\lambda$ is continuous from $(-\infty,\hat{\lambda}_1)$ into $C^1(\overline{\Omega})$ and increasing (that is, if $\mu<\lambda$, then $u_\lambda-u_\mu\in C_+$).*

Proof. We already know that for all $\lambda\in(-\infty,\hat{\lambda}_1)$, $S(\lambda)\neq\emptyset$.

Let $u,v\in S(\lambda)\subseteq\operatorname{int}C_+$. Then as in the proof of Proposition 7, we have

$$\begin{aligned}0&\leqslant\frac{1}{p}\int_\Omega\left[\frac{-\Delta_p u}{u^{p-1}}-\frac{-\Delta_p v}{v^{p-1}}\right](u^p-v^p)\,dz\\&=\frac{1}{p}\int_\Omega\left[\frac{f(z,u)}{u^{p-1}}-\frac{f(z,v)}{v^{p-1}}\right](u^p-v^p)\,dz\leqslant 0,\\&\implies u=v \text{ (see hypothesis } H(f)'\text{(iv))},\\&\implies S(\lambda)=\{u_\lambda\}\quad\text{(a singleton)}.\end{aligned}$$

Next we show the continuity of $\lambda\longmapsto u_\lambda$. To this end, suppose $\{\lambda_n\}_{n\geqslant1}\subseteq(-\infty,\hat{\lambda}_1)$ and assume that $\lambda_n\to\lambda\in(-\infty,\hat{\lambda}_1)$. Let $u_n=u_{\lambda_n}\in S(\lambda_n)\subseteq\operatorname{int}C_+$, $n\geqslant1$. We can find $\hat{\lambda}\in(-\infty,\hat{\lambda}_1)$ such that $\lambda_n\leqslant\hat{\lambda}$ for all $n\geqslant1$. Let $\hat{u}\in S(\hat{\lambda})\subseteq\operatorname{int}C_+$. Proposition 8 and 10 imply that

$$\overline{u}\leqslant u_n\leqslant\hat{u}\quad\text{for all } n\geqslant1. \tag{36}$$

Also, we have

$$\langle A(u_n),h\rangle+\int_{\partial\Omega}\beta(z)u_n^{p-1}h\,d\sigma=\lambda\int_\Omega u_n^{p-1}h\,dz+\int_\Omega f(z,u_n)h\,dz \tag{37}$$

for all $h\in W^{1,p}(\Omega)$. Choosing $h=u_n\in W^{1,p}(\Omega)$ and using hypotheses $H(\beta)$, $H(f)$(i) and (36), we see that

$$\{u_n\}_{n\geqslant1}\subseteq W^{1,p}(\Omega)\ \text{ is bounded.}$$

So, we may assume that

$$u_n\xrightarrow{w}u_\lambda\ \text{ in } W^{1,p}(\Omega)\quad\text{and}\quad u_n\to u_\lambda\ \text{ in } L^p(\Omega)\ \text{and in}\ L^p(\partial\Omega). \tag{38}$$

If in (37) we choose $h=u_n-u_\lambda\in W^{1,p}(\Omega)$, pass to the limit as $n\to\infty$ and use (38), then

$$\lim_{n\to\infty}\langle A(u_n),u_n-u_\lambda\rangle=0\implies u_n\to u_\lambda\text{ in } W^{1,p}(\Omega). \tag{39}$$

So, if in (37) we pass to the limit as $n\to\infty$ and use (39) and Proposition 2, then

$$\begin{aligned}&\langle A(u_\lambda),h\rangle+\int_{\partial\Omega}\beta(z)u_\lambda^{p-1}h\,d\sigma=\lambda\int_\Omega u_\lambda^{p-1}h\,dz+\int_\Omega f(z,u_\lambda)h\,dz\ \text{ for all } h\in W^{1,p}(\Omega),\\&\implies u_\lambda\in S(\lambda)\subseteq\operatorname{int}C_+.\end{aligned}$$

Since $S(\lambda)$ is a singleton, we have

$$u_n \to u_\lambda \ \text{ in } W^{1,p}(\Omega) \text{ for the original sequence.} \tag{40}$$

From Theorem 2 of Lieberman [13], we know that we can find $\alpha \in (0,1)$ and $c_9 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant c_9 \ \text{ for all } n \geqslant 1. \tag{41}$$

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, from (40) and (41) it follows that

$$u_n \to u_\lambda \ \text{ in } C^1(\overline{\Omega}) \implies \lambda \longmapsto u_\lambda \text{ is continuous from } (-\infty, \hat{\lambda}_1) \text{ into } C^1(\overline{\Omega}).$$

Finally the monotonicity of $\lambda \longmapsto u_\lambda$ follows from Proposition 10. $\square$

In fact the monotonicity conclusion in the above proposition, can be improved provided we strengthen further the conditions on $f(z,\cdot)$.

The new stronger conditions on the perturbation $f(z,x)$ are the following:

$H(f)''$: $f\colon \Omega\times\mathbf{R} \to \mathbf{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,x) > 0$ for all $x > 0$, hypotheses $H(f)''$(i),(ii),(iii),(iv) are the same as the corresponding hypotheses $H(f)'$(i),(ii),(iii),(iv) and

(v) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$, the mapping $x \longmapsto f(z,x) + \xi_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Under these new conditions on the perturbation $f(z,x)$, we have the following result.

Proposition 13. *If hypotheses $H(\beta)$ and $H(f)''$ hold, then the mapping $\lambda \longmapsto u_\lambda$ from $(-\infty, \hat{\lambda}_1)$ into $C^1(\overline{\Omega})$ is strictly increasing, that is, if $\lambda < \vartheta \in (-\infty, \hat{\lambda}_1)$, then $u_\vartheta - u_\lambda \in \operatorname{int} C_+$.*

Proof. From Proposition 12, we know that $u_\vartheta - u_\lambda \in C_+$. Let $\rho = \|u_\vartheta\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H(f)''$(v). Also, for $\delta > 0$, let $u_\lambda^\delta = u_\lambda + \delta \in \operatorname{int} C_+$. We have

$$\begin{aligned}
&-\Delta_p u_\lambda^\delta(z) + \xi_\rho u_\lambda^\delta(z)^{p-1}\\
&\leqslant -\Delta_p u_\lambda(z) + \xi_\rho u_\lambda(z)^{p-1} + \gamma(\delta) \ \text{ with } \gamma(\delta) \to 0^+ \text{ as } \delta \to 0^+\\
&= \lambda u_\lambda(z)^{p-1} + f(z, u_\lambda(z)) + \xi_\rho u_\lambda(z)^{p-1} + \gamma(\delta)\\
&\leqslant \lambda u_\vartheta(z)^{p-1} + f(z, u_\vartheta(z)) + \xi_\rho u_\vartheta(z)^{p-1} + \gamma(\delta)\\
&\qquad\qquad\qquad\qquad\qquad\qquad (\text{see } H(f)''(v) \text{ and recall } u_\lambda \leqslant u_\vartheta)\\
&= \vartheta u_\vartheta(z)^{p-1} + f(z, u_\vartheta(z)) + \xi_\rho u_\vartheta(z)^{p-1} - (\vartheta - \lambda) u_\vartheta(z)^{p-1} + \gamma(\delta)\\
&\leqslant \vartheta u_\vartheta(z)^{p-1} + f(z, u_\vartheta(z)) + \xi_\rho u_\vartheta(z)^{p-1} - (\vartheta - \lambda)\hat{m}_\vartheta^{p-1} + \gamma(\delta)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{with } \hat{m}_\vartheta = \min_{\overline{\Omega}} u_\vartheta > 0\\
&\leqslant -\Delta_p u_\vartheta(z) + \xi_\rho u_\vartheta(z)^{p-1} \ \text{ for a.a. } z \in \Omega \text{ and for } \delta > 0 \text{ small}\\
&\implies u_\lambda^\delta \leqslant u_\vartheta \implies u_\vartheta - u_\lambda \in \operatorname{int} C_+.
\end{aligned}$$

$\square$

The next theorem summarizes the situation for problem (P_λ) when the perturbation $f(z,x)$ is $(p-1)$-sublinear in $x \in \mathbf{R}$.

Theorem 14. (a) *If hypotheses $H(\beta)$ and $H(f)$ hold, then for all $\lambda \in (-\infty, \hat{\lambda}_1)$, $S(\lambda) \neq \emptyset$, $S(\lambda) \subseteq \operatorname{int} C_+$ and $S(\lambda)$ admits a smallest element $u_\lambda^* \in \operatorname{int} C_+$; if $\lambda \geqslant \hat{\lambda}_1$, then $S(\lambda) = \emptyset$.*

(b) *If hypotheses $H(\beta)$ and $H(f)'$ hold, then for all $\lambda \in (-\infty, \hat{\lambda}_1)$, $S(\lambda) = \{u_\lambda\}$ and the map $\lambda \longmapsto u_\lambda$ is continuous and increasing (that is, $\lambda \leqslant \vartheta \Rightarrow u_\vartheta - u_\lambda \in C_+$).*

(c) *If hypotheses $H(\beta)$ and $H(f)''$ hold, then the map $\lambda \longmapsto u_\lambda$ is strictly increasing (that is, $\lambda < \vartheta \in (-\infty, \hat{\lambda}_1) \Rightarrow u_\vartheta - u_\lambda \in \operatorname{int} C_+$).*

4. Superlinear perturbation

In this section, we examine problem (P_λ) when the perturbation $f(z,\cdot)$ is $(p-1)$-superlinear, but without satisfying the usual in such cases Ambrosetti–Rabinowitz condition (*AR*-condition for short). Now we can not hope for uniqueness and we have multiplicity of positive solutions.

The hypotheses on the perturbation $f(z,x)$, are the following:

$H(f)_1$: $f\colon \Omega\times\mathbf{R} \to \mathbf{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,x) > 0$ for all $x > 0$ and

(i) $f(z,x) \leqslant a(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)_+$ and $p < r < p^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)\,ds$, then $\lim\limits_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) there exists $\mu \in \left((r-p)\max\left\{1, \frac{N}{p}\right\}, p^*\right)$ such that
$$0 < \eta_0 \leqslant \liminf_{x\to+\infty} \frac{f(z,x)x - pF(z,x)}{x^\mu} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) $\lim\limits_{x\to0^+} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$;

(v) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$, the map $x \longmapsto f(z,x) + \xi_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 3. As before, since we are interested on positive solutions and the above hypotheses concern the positive semiaxis $\mathbf{R}_+ = [0,+\infty)$, without any loss of generality, we assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leqslant 0$. From hypotheses $H(f)_1$(ii),(iii) it follows that
$$\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \text{ uniformly for a.a. } z \in \Omega.$$

So, for a.a. $z \in \Omega$, $f(z,\cdot)$ is $(p-1)$-superlinear. However, we do not employ the usual in such cases *AR*-condition (unilateral version) which says that there exist $q > p$ and $M > 0$ such that
$$0 < qF(z,x) \leqslant f(z,x)x \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant M \quad \text{(see [3])}.$$

This implies that
$$c_{10}x^q \leqslant F(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant M, \text{ some } c_{10} > 0.$$

Here instead, we employ the weaker condition $H(f)$(iii) which incorporates in our framework $(p-1)$-superlinear perturbations, with "slower" growth near $+\infty$ (see the examples below). A similar polynomial growth is assumed near 0^+ by virtue of hypothesis $H(f)_1$(iv).

Example 1. The following functions satisfy hypotheses $H(f)_1$. For the sake of simplicity, we drop the z-dependence:

$$f_1(x) = x^{r-1} \text{ for all } x \geqslant 0 \text{ with } p < r < p^*,$$
$$f_2(x) = x^{p-1}\left(\ln x + \frac{1}{p}\right) \text{ for all } x \geqslant 0.$$

Note that f_2 does not satisfy the AR-condition.

The sets $\mathcal{P}$ and $S(\lambda)$ have the same meaning as in Section 3.

Proposition 15. *If hypotheses $H(\beta)$ and $H(f)_1$ hold, then $\mathcal{P} \neq \emptyset$ and $S(\lambda) \subseteq \operatorname{int} C_+$.*

Proof. For $\lambda \in \mathbf{R}$, we consider the C^1-functional $\varphi_\lambda \colon W^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p \, d\sigma + \frac{1}{p}||u^-||_p^p - \frac{\lambda}{p}||u^+||_p^p - \int_\Omega F(z,u^+)\,dz$$

for all $u \in W^{1,p}(\Omega)$. Hypotheses $H(f)_1$(i),(iv) imply that given $\varepsilon > 0$, we can find $c_{11} = c_{11}(\varepsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\varepsilon}{p}x^p + c_{11}x^r \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{42}$$

Let $\lambda < \hat{\lambda}_1$. Then for any $u \in W^{1,p}(\Omega)$ we have

$$\begin{aligned}\varphi_\lambda(u) &= \frac{1}{p}||Du^+||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z)(u^+)^p \, d\sigma - \frac{\lambda}{p}||u^+||_p^p + \frac{1}{p}||Du^-||_p^p \\ &\quad + \frac{1}{p}||u^-||_p^p - \frac{\varepsilon}{p}||u^+||_p^p - c_{12}||u||^r \text{ for some } c_{12} > 0 \quad \text{(see (42))} \\ &\geqslant \left(c_{13} - \frac{\varepsilon}{p}\right)||u^+||^p + \frac{1}{p}||u^-||^p - c_{12}||u||^r \text{ for some } c_{13} > 0\end{aligned}$$

(see Proposition 4 and recall $\lambda < \hat{\lambda}_1$). Choosing $\varepsilon \in (0, p\, c_{13})$, we have

$$\varphi_\lambda(u) \geqslant c_{14}||u||^p - c_{12}||u||^r \text{ for some } c_{14} > 0.$$

Since $r > p$, if we choose $\rho \in (0,1)$ small, we have

$$\begin{aligned}&\varphi_\lambda(u) > 0 = \varphi_\lambda(0) \text{ for all } u \in W^{1,p}(\Omega) \text{ with } 0 < ||u|| \leqslant \rho \\ &\implies u = 0 \text{ is a (strict) local minimizer of } \varphi_\lambda.\end{aligned}$$

So, we can find $\rho \in (0,1)$ small such that

$$\varphi_\lambda(0) = 0 < \inf\,[\varphi_\lambda(u) \colon ||u|| = \rho] = m_\rho \tag{43}$$

(see, for example, Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 29).

By virtue of hypothesis $H(f)_1$(ii), we see that for every $u \in \operatorname{int} C_+$, we have

$$\varphi_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{44}$$

Moreover, as in Gasinski and Papageorgiou [10], we can check that

$$\varphi_\lambda \text{ satisfies the } C\text{-condition.} \tag{45}$$

Because of (43), (44) and (45), we can apply Theorem 1 (the mountain pass theorem) and obtain $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda(0) = 0 < m_\rho \leqslant \varphi_\lambda(u_\lambda) \quad \text{and} \quad \varphi'_\lambda(u_\lambda) = 0. \tag{46}$$

From (46) we have $u_\lambda \neq 0$ and

$$\begin{aligned} &\langle A(u_\lambda), h\rangle + \int_{\partial\Omega} \beta(z)(u_\lambda^+)^{p-1} h\, d\sigma - \int_\Omega (u_\lambda^-)^{p-1} h\, dz \\ &= \lambda \int_\Omega (u_\lambda^+)^{p-1} h\, dz + \int_\Omega f(z, u^+) h\, dz \ \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{47}$$

In (47) we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$ and we infer that $u_\lambda \geqslant 0$, $u_\lambda \neq 0$. So, we have

$$\langle A(u_\lambda), h\rangle + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1} h\, d\sigma = \lambda \int_\Omega u_\lambda^{p-1} h\, dz + \int_\Omega f(z, u) h\, dz \ \text{ for all } h \in W^{1,p}(\Omega)$$

$$\implies u_\lambda \in S(\lambda) \quad \text{(see the proof of Proposition 5).}$$

The nonlinear regularity theory implies $u_\lambda \in C_+ \backslash \{0\}$. Let $\rho = ||u_\lambda||_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H(f)_1(v)$. Then

$$\begin{aligned} &-\Delta_p u_\lambda(z) + \xi_\rho u_\lambda(z)^{p-1} = \lambda u_\lambda(z)^{p-1} + f(z, u_\lambda(z)) + \xi_\rho u_\lambda(z)^{p-1} \geqslant 0 \ \text{ a.e. in } \Omega \\ &\implies \Delta_p u_\lambda(z) \leqslant \xi_\rho u_\lambda(z)^{p-1} \ \text{ a.e. in } \Omega \implies u_\lambda \in \operatorname{int} C_+ \quad \text{(see Vazquez [16]).} \end{aligned}$$

Therefore we have proved that $\mathcal{P} \neq \varnothing$ (in fact $(-\infty, \hat{\lambda}_1) \subseteq \mathcal{P}$) and that $S(\lambda) \subseteq \operatorname{int} C_+$. $\square$

The proof of the next proposition is identical to the proof of Proposition 6.

Proposition 16. *If hypotheses $H(\beta)$ and $H(f)_1$ hold and $\lambda \in \mathcal{P}$, then $(-\infty, \lambda] \subseteq \mathcal{P}$.*

Moreover, as in the proof of Proposition 9, using the nonlinear Picone's identity (see [2]), we have:

Proposition 17. *If hypotheses $H(\beta)$ and $H(f)_1$ hold, then $\hat{\lambda}_1 \notin \mathcal{P}$ and so $\mathcal{P} = (-\infty, \hat{\lambda}_1)$.*

In fact, as we already mentioned, in this case for every $\lambda \in \mathcal{P} = (-\infty, \hat{\lambda}_1)$ problem (P_λ) has at least two positive solutions.

Proposition 18. *If hypotheses $H(\beta)$ and $H(f)_1$ hold and $\lambda \in \mathcal{P} = (-\infty, \hat{\lambda}_1)$, then problem (P_λ) has at least two positive solutions*

$$u_\lambda, v_\lambda \in \operatorname{int} C_+, \quad u_\lambda \leqslant v_\lambda, \quad u_\lambda \neq v_\lambda.$$

Proof. Since $\lambda \in \mathcal{P}$, we can find $u_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+$. We introduce the following Carathéodory function:

$$k_\lambda(z, x) = \begin{cases} (\lambda + 1) u_\lambda(z)^{p-1} + f(z, u_\lambda(z)) & \text{if } x \leqslant u_\lambda(z), \\ (\lambda + 1) x^{p-1} + f(z, x) & \text{if } u_\lambda(z) < x. \end{cases} \tag{48}$$

In addition we consider the following truncation of the boundary term (recall that $u_\lambda \in \operatorname{int} C_+$):

$$d_\lambda(z, x) = \begin{cases} u_\lambda(z)^{p-1} & \text{if } x \leqslant u_\lambda(z), \\ x^{p-1} & \text{if } u_\lambda(z) < x, \end{cases} \quad \text{for all } (z, x) \in \partial\Omega \times \mathbf{R}. \tag{49}$$

This is also a Carathéodory function on $\partial\Omega \times \mathbf{R}$.

Let $K_\lambda(z,x) = \int_0^x k_\lambda(z,s)\,ds$ and $D_\lambda(z,x) = \int_0^x d_\lambda(z,s)\,ds$ and consider the C^1-functional $\psi_\lambda\colon W^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p + \frac{1}{p}\int_{\partial\Omega} \beta(z) D_\lambda(z,u)\,d\sigma - \int_\Omega K_\lambda(z,u)\,dz$$

for all $u \in W^{1,p}(\Omega)$.

Claim 1. *We have*

$$K_{\psi_\lambda} = \{u \in W^{1,p}(\Omega)\colon \psi_\lambda'(u) = 0\} \subseteq [u_\lambda) = \{u \in W^{1,p}(\Omega)\colon u_\lambda(z) \leqslant u(z) \text{ a.e. in } \Omega\}.$$

To this end, let $u \in K_{\psi_\lambda}$. Then

$$\langle A(u), h\rangle + \int_\Omega |u|^{p-2}uh\,dz + \int_{\partial\Omega} \beta(z) d_\lambda(z,x)\,d\sigma = \int_\Omega k_\lambda(z,u)h\,dz \tag{50}$$

for all $h \in W^{1,p}(\Omega)$. In (50) we choose $h = (u_\lambda - u)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\big\langle A(u), (u_\lambda - u)^+\big\rangle + \int_\Omega |u|^{p-2}u(u_\lambda - u)^+\,dz + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1}(u_\lambda - u)^+\,d\sigma \\
&= \int_\Omega [\lambda u_\lambda^{p-1} + f(z,u_\lambda)](u_\lambda - u)^+\,dz + \int_\Omega u_\lambda^{p-1}(u_\lambda - u)^+\,dz \quad \text{(see (48) and (49))} \\
&= \big\langle A(u_\lambda), (u_\lambda - u)^+\big\rangle + \int_\Omega u_\lambda^{p-1}(u_\lambda - u)^+\,dz + \int_{\partial\Omega} \beta(z) u_\lambda^{p-1}(u_\lambda - u)^+\,d\sigma, \\
&\implies \big\langle A(u_\lambda) - A(u), (u_\lambda - u)^+\big\rangle + \int_\Omega (u_\lambda^{p-1} - |u|^{p-2}u)(u_\lambda - u)^+\,dz \leqslant 0 \quad \text{(see } H(\beta)), \\
&\implies |\{u_\lambda > u\}|_N = 0, \text{ hence } u_\lambda \leqslant u \text{ and so } u \in [u_\lambda)\,.
\end{aligned}$$

This proves Claim 1.

Claim 2. *Every $u \in K_{\psi_\lambda}$ belongs in $S(\lambda)$.*

From (50) and Claim 1, we have

$$\langle A(u), h\rangle + \int_\Omega u^{p-1}h\,dz + \int_{\partial\Omega} \beta(z) u^{p-1}h\,d\sigma = \int_\Omega [\lambda u^{p-1} + f(z,u)]h\,dz + \int_\Omega u^{p-1}h\,dz$$

for all $h \in W^{1,p}(\Omega)$ (see (48) and (49)) which implies

$$\langle A(u), h\rangle + \int_{\partial\Omega} \beta(z) u^{p-1}h\,d\sigma = \int_\Omega [\lambda u^{p-1} + f(z,u)]h\,dz \text{ for all } h \in W^{1,p}(\Omega).$$

From this as in the proof of Proposition 5, we infer that $u \in S(\lambda)$. This proves Claim 2.

Claim 3. *We may assume that $u_\lambda \in \operatorname{int} C_+$ is a local minimizer of ψ_λ.*

Let $\vartheta \in (\lambda, \hat{\lambda}_1) \subseteq \mathcal{P}$. We can find $u_\vartheta \in S(\vartheta)$. In fact as in the proof of Proposition 6 we can have $u_\lambda \leqslant u_\vartheta$. Then we introduce the following truncation of $k_\lambda(z,\cdot)$:

$$\hat{k}_\lambda(z,x) = \begin{cases} k_\lambda(z,x) & \text{if } x < u_\vartheta(z), \\ k_\lambda(z,u_\vartheta(z)) & \text{if } u_\vartheta(z) \leqslant x. \end{cases} \tag{51}$$

We also consider the corresponding truncation of the boundary term $d_\lambda(z,\cdot)$:

$$\hat{d}_\lambda(z,x) = \begin{cases} d_\lambda(z,x) & \text{if } x < u_\vartheta(z), \\ d_\lambda(z,u_\vartheta(z)) & \text{if } u_\vartheta(z) \leqslant x, \end{cases} \quad \text{for all } (z,x) \in \partial\Omega \times \mathbf{R}. \tag{52}$$

Both are Carathéodory functions. We set

$$\hat{K}_\lambda(z,x) = \int_0^x \hat{k}_\lambda(z,s)\,ds \quad \text{and} \quad \hat{D}_\lambda(z,x) = \int_0^x \hat{d}_\lambda(z,s)\,ds$$

and consider the C^1-functional $\hat{\psi}_\lambda \colon W^{1,p}(\Omega) \to \mathbf{R}$ defined by

$$\hat{\psi}_\lambda(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p}||u||_p^p + \int_{\partial\Omega} \beta(z)\hat{D}_\lambda(z,u)\,d\sigma - \int_\Omega \hat{K}_\lambda(z,u)\,dz$$

for all $u \in W^{1,p}(\Omega)$. From (51) and (52) it is clear that $\hat{\psi}_\lambda$ is coercive. Also, is is sequentially weakly lower semicontinuous. So, we can find $\hat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\hat{\psi}_\lambda(\hat{u}_\lambda) = \inf\left[\hat{\psi}_\lambda(u) \colon W^{1,p}(\Omega)\right] \implies \hat{\psi}'_\lambda(\hat{u}_\lambda) = 0 \implies \\ &\langle A(\hat{u}_\lambda), h\rangle + \int_\Omega |\hat{u}_\lambda|^{p-2}\hat{u}_\lambda h\,dz + \int_{\partial\Omega} \beta(z)\hat{d}_\lambda(z,u_\lambda)h\,d\sigma = \int_\Omega \hat{k}_\lambda(z,u_\lambda)h\,dz \end{aligned} \tag{53}$$

for all $h \in W^{1,p}(\Omega)$. As in the proof of Claim 1 earlier, choosing $h = (u_\lambda - \hat{\lambda}_\lambda)^+ \in W^{1,p}(\Omega)$ in (53), we obtain

$$u_\lambda \leqslant \hat{u}_\lambda.$$

Next in (53), we choose $h = (\hat{u}_\lambda - u_\vartheta)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\left\langle A(\hat{u}_\lambda), (\hat{u}_\lambda - u_\vartheta)^+\right\rangle + \int_\Omega \hat{u}_\lambda^{p-1}(\hat{u}_\lambda - u_\vartheta)^+\,dz + \int_{\partial\Omega} \beta(z)u_\vartheta^{p-1}(\hat{u}_\lambda - u_\vartheta)^+\,d\sigma \\ &= \int_\Omega [\lambda u_\vartheta^{p-1} + f(z,u_\vartheta)](\hat{u}_\lambda - u_\vartheta)^+\,dz + \int_\Omega u_\vartheta^{p-1}(\hat{u}_\lambda - u_\vartheta)^+\,dz \quad \text{(see (51) and (52))} \\ &\leqslant \int_\Omega [\vartheta u_\vartheta^{p-1} + f(z,u_\vartheta)](\hat{u}_\lambda - u_\vartheta)^+\,dz + \int_\Omega u_\vartheta^{p-1}(\hat{u}_\lambda - u_\vartheta)^+\,dz \text{ (since } \lambda < \vartheta) \\ &= \left\langle A(u_\vartheta), (\hat{u}_\lambda - u_\vartheta)^+\right\rangle + \int_\Omega u_\vartheta^{p-1}(\hat{u}_\lambda - u_\vartheta)^+\,dz + \int_{\partial\Omega} \beta(z)u_\vartheta^{p-1}(\hat{u}_\lambda - u_\vartheta)^+\,d\sigma \\ &\implies \left\langle A(\hat{u}_\lambda) - A(u_\vartheta), (\hat{u}_\lambda - u_\vartheta)^+\right\rangle + \int_\Omega (\hat{u}_\lambda^{p-1} - u_\vartheta^{p-1})(\hat{u}_\lambda - u_\vartheta)^+\,dz \leqslant 0 \\ &\implies |\{\hat{u}_\lambda > u_\vartheta\}|_N = 0, \text{ hence } \hat{u}_\lambda \leqslant u_\vartheta. \end{aligned}$$

So, we have proved that

$$\hat{u}_\lambda \in [u_\lambda, u_\vartheta] = \{u \in W^{1,p}(\Omega) \colon u_\lambda(z) \leqslant u(z) \leqslant u_\vartheta(z) \text{ a.e. in } \Omega\}.$$

Then from (51), (52) and Claim 2, it follows that $\hat{u}_\lambda \in S(\lambda)$. If $\hat{u}_\lambda \neq u_\lambda$, then this is desired second positive solution of problem (P_λ) and so we are done. Therefore, we may assume that $\hat{u}_\lambda = u_\lambda$.

Note that $\hat{\psi}_\lambda|_{[0,u_\vartheta]} = \psi_\lambda|_{[0,u_\vartheta]}$ (see (51) and (52)). Also as in the proof of Proposition 13, using $u_\lambda^\delta = u_\lambda + \delta \in \operatorname{int} C_+$ $(\delta > 0)$ and hypothesis $H(f)_1(v)$, we show that

$$\begin{aligned} u_\vartheta - u_\lambda \in \operatorname{int} C_+ &\implies u_\lambda \in \operatorname{int}_{C^1(\overline{\Omega})}[0, u_\vartheta] \\ &\implies u_\lambda \text{ is a local } C^1(\overline{\Omega}) - \text{minimizer of } \psi_\lambda \\ &\implies u_\lambda \text{ is a local } W^{1,p}(\Omega) - \text{minimizer of } \psi_\lambda \text{ (see Proposition 3).} \end{aligned}$$

This proves Claim 3.

We assume that K_{ψ_λ} is finite (or otherwise we are done since we already have an infinity of solutions (see Claim 1 and (48), (49))). By virtue of Claim 3, we can find $\rho > 0$ small such that

$$\psi_\lambda(u_0) < \inf\,[\psi_\lambda(u)\colon ||u - u_0|| = \rho] = m_\lambda \tag{54}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 29). If φ_λ is as in the proof of Proposition 15, then

$$\varphi_\lambda = \psi_\lambda + \xi^*_\lambda \quad \text{with } \xi^*_\lambda \in \mathbf{R}.$$

So, if $u \in \operatorname{int} C_+$, then

$$\begin{aligned} &\psi_\lambda(tu) \to -\infty \quad \text{as } t \to -\infty \quad \text{(see (44))}, \\ &\psi_\lambda \text{ satisfies the } C - \text{condition} \quad \text{(see (45))}. \end{aligned}$$

These two facts and (54), permit the use of Theorem 1 (the mountain pass theorem). Hence we obtain $v_\lambda \in W^{1,p}(\Omega)$ such that

$$m_\lambda \leqslant \psi_\lambda(v_\lambda) \quad \text{and} \quad v_\lambda \in K_{\psi_\lambda}. \tag{55}$$

From (54), (55) and Claims 1 and 2, we infer that

$$v_\lambda \in S(\lambda) \subseteq \operatorname{int} C_+, \quad u_\lambda \leqslant v_\lambda, \quad u_\lambda \neq v_\lambda. \qquad \square$$

We can also establish the existence of a smallest positive solution.

Proposition 19. *If hypotheses $H(\beta)$ and $H(f)_1$ hold and $\lambda \in \mathcal{P} = (-\infty, \hat{\lambda}_1)$, then problem (P_λ) admits a smallest positive solution $u^*_\lambda \in \operatorname{int} C_+$ and the map $\lambda \longmapsto u^*_\lambda$ is strictly increasing (that is, $\lambda < \vartheta \in (-\infty, \hat{\lambda}_1) \Rightarrow u^*_\vartheta - u^*_\lambda \in \operatorname{int} C_+$).*

Proof. As in the proof of Proposition 11, we can find $\{u_n\}_{n\geqslant 1} \subseteq S(\lambda)$ such that

$$\inf S(\lambda) = \inf_{n\geqslant 1} u_n.$$

Since $S(\lambda)$ is downward directed, we may assume that $\{u_n\}_{n\geqslant 1}$ is decreasing. So, we have

$$u_n \leqslant u_1 \in \operatorname{int} C_+ \quad \text{for all } n \geqslant 1. \tag{56}$$

We have

$$\langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n^{p-1} h\, d\sigma = \lambda \int_\Omega u_n^{p-1} h\, dz + \int_\Omega f(z, u_n) h\, dz \tag{57}$$

for all $h \in W^{1,p}(\Omega)$. Choosing $h = u_n \in W^{1,p}(\Omega)$ in (57) and using (56), we infer that

$$\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \quad \text{is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u^*_\lambda \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u^*_\lambda \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{58}$$

Suppose that $u_\lambda^* \equiv 0$. Let $y_n = \frac{u_n}{||u_n||}$ $n \geqslant 1$. Then $||y_n|| = 1$ for all $n \geqslant 1$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{59}$$

From (57) we have

$$\langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z) y_n^{p-1} h \, d\sigma = \lambda \int_\Omega y_n^{p-1} h \, dz + \int_\Omega \frac{f(z, u_n)}{||u_n||^{p-1}} h \, dz \tag{60}$$

for all $h \in W^{1,p}(\Omega)$. In (60) we choose $h = y_n - y \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (59). Then

$$\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0 \implies y_n \to y \text{ in } W^{1,p}(\Omega), \quad \text{and so } ||y|| = 1, \ y \geqslant 0. \tag{61}$$

Note that since we have assumed that $u_\lambda^* \equiv 0$, by virtue of hypothesis $H(f)_1$(iv), we have (at least for a subsequence) that

$$\frac{N_f(u_n)}{||u_n||^{p-1}} \xrightarrow{w} 0 \text{ in } L^{r'}(\Omega). \tag{62}$$

So, if in (60) we pass to the limit as $n \to \infty$ and use (61) and (62), then

$$\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y^{p-1} h \, d\sigma = \lambda \int_\Omega y^{p-1} h \, dz \ \text{ for all } h \in W^{1,p}(\Omega)$$
$$\implies -\Delta_p y(z) = \lambda y(z)^{p-1} \text{ a.e. in } \Omega, \quad \frac{\partial y}{\partial n_p} + \beta(z) y^{p-1} = 0 \text{ on } \partial\Omega.$$

Since $\lambda < \hat\lambda_1$, it follows that $y \equiv 0$, a contradiction to (61). Therefore $u_\lambda^* \neq 0$.

In (57) we choose $h = u_n - u_\lambda^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (58). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_\lambda^*\rangle = 0 \implies u_n \to u_\lambda^* \text{ in } W^{1,p}(\Omega) \quad \text{(see Proposition 2)}. \tag{63}$$

So, if in (57) we pass to the limit as $n \to \infty$ and use (63), then

$$\langle A(u_\lambda^*), h\rangle + \int_{\partial\Omega} \beta(z) (u_\lambda^*)^{p-1} h \, d\sigma = \lambda \int_\Omega (u_\lambda^*)^{p-1} h \, dz + \int_\Omega f(z, u_\lambda^*) h \, dz$$

for all $h \in W^{1,p}(\Omega)$ which implies

$$u_\lambda^* \in S(\lambda) \quad \text{and} \quad u_\lambda^* = \inf S(\lambda).$$

Therefore $u_\lambda^* \in \operatorname{int} C_+$ is the smallest positive solution of problem (P_λ).

Suppose that $\lambda < \vartheta \in \mathcal{P} = (-\infty, \hat\lambda_1)$ and let $u_\vartheta \in S(\vartheta)$. Then

$$u_\lambda^* \leqslant u_\vartheta \quad \text{(see Proposition 10)} \implies u_\lambda^* \leqslant u_\vartheta^*.$$

In fact, by considering $(u_\lambda^*)^\delta = u_\lambda^* + \delta \in \operatorname{int} C_+$ $(\delta > 0)$ as in the proof of Proposition 13, via hypothesis $H(f)_1(v)$, we show that

$$u_\vartheta^* - u_\lambda^* \in \operatorname{int} C_+. \qquad \square$$

Summarizing the situation in the case of superlinear perturbations, we can state the following theorem.

Theorem 20. *If hypotheses $H(\beta)$ and $H(f)_1$ hold, then for every $\lambda \in (-\infty, \hat{\lambda}_1)$ problem (P_λ) has at least two positive solutions*

$$u_\lambda, v_\lambda \in \operatorname{int} C_+, \quad u_\lambda \leqslant v_\lambda, \quad u_\lambda \neq v_\lambda;$$

also it admits a smallest positive solution $u_\lambda^* \in \operatorname{int} C_+$ and the map $\lambda \to u_\lambda^*$ is strictly increasing, that is,

$$\lambda < \vartheta \in (-\infty, \hat{\lambda}_1) \implies u_\vartheta^* - u_\lambda^* \in \operatorname{int} C_+;$$

finally for $\lambda \geqslant \hat{\lambda}_1$ problem (P_λ) has no positive solution.

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