QUASICONFORMAL MAPS WITH BILIPSCHITZ OR IDENTITY BOUNDARY VALUES IN BANACH SPACES

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Abstract. Suppose that E and E' denote real Banach spaces with dimension at least 2 and that $D \subsetneq E$ and $D' \subsetneq E'$ are uniform domains with homogeneously dense boundaries. We consider the class of all φ-FQC (freely φ-quasiconformal) maps of D onto D' with bilipschitz boundary values. We show that the maps of this class are η-quasisymmetric. As an application, we show that if D is bounded, then maps of this class satisfy a two sided Hölder condition. Moreover, replacing the class φ-FQC by the smaller class of M-QH maps, we show that M-QH maps with bilipschitz boundary values are bilipschitz. Finally, we show that if f is a φ-FQC map which maps D onto itself with identity boundary values, then there is a constant C, depending only on the function φ, such that for all $x \in D$, the quasihyperbolic distance satisfies $k_D(x, f(x)) \leq C$.

1. Introduction and main results

Many results of classical function theory have their counterparts in the context of quasiconformal maps in the Euclidean n-dimensional space $\mathbf{R}^n$. Väisälä [21, 22, 24] has developed a theory of quasiconformality in the Banach space case which differs from the finite dimensional theory in many respects because tools such as conformal invariants and measures of sets are no longer available. These classical tools are replaced by fundamental objects from metric space geometry such as curves, their lengths, and approximately length minimizing curves. Väisälä used these notions in the setup of several metric space structures on the same underlying Banach space and developed effective methods based on these basic notions. In addition to the norm metric he considered two hyperbolic type metric structures, the quasihyperbolic metric and the distance ratio metric. The quasihyperbolic metric k_D of a domain D has a key role as quasiconformality is defined in terms of it in the Banach space case. Only recently some basic properties of quasihyperbolic metric have been studied: the convexity of quasihyperbolic balls was studied by Klén [6, 7], Rasila and Talponen [8, 14], Väisälä [25]. Rasila and Talponen also proved the smoothness of quasihyperbolic geodesics in [15] applying now probabilistic methods.

Given domains D, D' in Banach spaces E and E', respectively, our basic problem is to study the class of homeomorphisms $f \in QC_\varphi^L(D, D')$, where

$$QC_\varphi^L(D, D') = \{f \colon \overline{D} \to \overline{D}' \text{ homeo} \mid f|_D \text{ is a } \varphi\text{-FQC map and } f|_{\partial D} \text{ is } L\text{-bilipschitz}\}.$$

For the definition of φ-FQC and L-bilipschitz maps see Section 2. The class $QC_\varphi^L(D, D')$ is very wide and many particular cases of interest are obtained by choosing D, D', φ, L in a suitable way as we will see below.

Our first result deals with the case when both D and D' are uniform domains. In this case we prove that the class $QC_\varphi^L(D, D')$ consists of quasisymmetric maps. More precisely, we prove the following theorem.

Theorem 1.1. *Let $D \subsetneq E$, $D' \subsetneq E'$ be c-uniform domains. If $f \in QC_\varphi^L(D, D')$, then f is η-QS in $\overline{D}$ with η depending on c, L and φ only.*

Applying this result to the case of a bounded domain D we obtain the second result. Recall that in the case of $\mathbf{R}^n$ results of this type have been proved by Näkki and Palka [13].

Theorem 1.2. *Let $D \subsetneq E$, $D' \subsetneq E'$ be c-uniform domains. If $f \in QC_\varphi^L(D, D')$ and D is bounded, then for all $x, y \in D$,*

$$\frac{|x - y|^{1/\alpha}}{C} \le |f(x) - f(y)| \le C|x - y|^\alpha,$$

where $C \ge 1$ and $\alpha \in (0, 1)$ depend on c, L, φ and $\mathrm{diam}(D)$.

Our third result concerns the case when both D and D' are uniform domains and $\varphi(t) = Mt$ for some fixed $M \ge 1$. We also require a density condition of the boundary of a domain. This (r_1, r_2)-HD condition will be defined in Section 2.

Theorem 1.3. *Let $D \subsetneq E$, $D' \subsetneq E'$ be c-uniform domains and the boundary of D be (r_1, r_2)-HD. If $f \in QC_\varphi^L(D, D')$ with $\varphi(t) = Mt$, then f is M'-bilipschitz in $\overline{D}$, where M' depends only on c, r_1, r_2, L and M.*

Our fourth result deals with the case when $D = D'$, $L = 1$ and, moreover, the boundary mapping $f|_{\partial D} \colon \partial D \to \partial D$ is the identity. This problem has been studied very recently in [10, 11, 27]. Originally, the problem was motivated by Teichmüller's work on plane quasiconformal maps [9, 16] and then extended to the higher dimensional case by several authors: [1], [11], [10, 27]. Our result is as follows.

Theorem 1.4. *Let $D \subsetneq E$ be a c-uniform domain with (r_1, r_2)-HD boundary. If f is a φ-FQC map which maps D onto itself with identity boundary values, then for all $x \in D$,*

$$k_D(x, f(x)) \le C,$$

where C is a constant depending on r_1, r_2, c and φ only.

For the case $n = 2$, when D is the unit disk, the sharp bound is due to Teichmüller [9, 16]. For the case of unit ball in $\mathbf{R}^n$, $n \ge 2$, nearly sharp results appear in [10, 27]. In both of these cases one uses the hyperbolic metric in place of the quasihyperbolic metric.

We do not know whether there are sharp results for the Banach spaces, too. For instance, it is an open problem whether Theorem 1.4 could be refined for the case $D = \mathbf{B}$, the unit ball, to the effect that $C \to 0$ when φ approaches the identity map.

The organization of this paper is as follows. In Section 3, we will prove Theorems 1.1, 1.2, 1.3 and 1.4. In Section 2, some preliminaries are stated.

2. Preliminaries

2.1. Notation. We adopt mostly the standard notation and terminology from Väisälä [21, 24]. We always use E and E' to denote real Banach spaces with dimension at least 2. The norm of a vector z in E is written as $|z|$, and for every pair of points z_1, z_2 in E, the distance between them is denoted by $|z_1 - z_2|$, the closed line segment with endpoints z_1 and z_2 by $[z_1, z_2]$. Moreover, we use $\mathbf{B}(x, r)$ to denote the ball with center $x \in E$ and radius r (> 0), and its boundary and closure are denoted by $\mathbf{S}(x, r)$ and $\overline{\mathbf{B}}(x, r)$, respectively. In particular, we use $\mathbf{B}$ to denote the unit ball $\mathbf{B}(0, 1)$. The one-point extension of E is the Hausdorff space $\dot{E} = E \cup \{\infty\}$, where the neighborhoods of ∞ are the complements of closed bounded sets of E. The boundary ∂A and the closure $\overline{A}$ of a set $A \subset E$ are taken in $\dot{E}$.

2.2. Quasihyperbolic distance and uniform domains. The *quasihyperbolic length* of a rectifiable arc or a path α in the norm metric in D is the number (cf. [2, 3, 21]):

$$\ell_k(\alpha) = \int_\alpha \frac{|dz|}{d_D(z)},$$

where $d_D(z)$ denotes the distance from z to the boundary ∂D of D. For each pair of points z_1, z_2 in D, the *quasihyperbolic distance* $k_D(z_1, z_2)$ between z_1 and z_2 is defined in the usual way:

$$k_D(z_1, z_2) = \inf \ell_k(\alpha),$$

where the infimum is taken over all rectifiable arcs α joining z_1 to z_2 in D. For each pair of points z_1, z_2 in D, the *distance ratio metric* $j_D(z_1, z_2)$ between z_1 and z_2 is defined by

$$j_D(z_1, z_2) = \log\left(1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}}\right).$$

For all z_1, z_2 in D, we have (cf. [21])

$$(2.3) \qquad \begin{aligned} k_D(z_1, z_2) &\geq \inf\left\{\log\left(1 + \frac{\ell(\alpha)}{\min\{d_D(z_1), d_D(z_2)\}}\right)\right\} \\ &\geq j_D(z_1, z_2) \geq \left|\log\frac{d_D(z_2)}{d_D(z_1)}\right|, \end{aligned}$$

where the infimum is taken over all rectifiable curves α in D connecting z_1 and z_2. Moreover, if $|z_1 - z_2| < d_D(z_1)$, we have [21, 26]

$$(2.4) \qquad k_D(z_1, z_2) \leq \log\left(1 + \frac{|z_1 - z_2|}{d_D(z_1) - |z_1 - z_2|}\right).$$

Gehring and Palka [3] introduced the quasihyperbolic metric of a domain in $\mathbf{R}^n$ and it has been recently used by many authors in the study of quasiconformal mappings and related questions [4, 8, 14] etc.

Definition 2.5. A domain D in E is called *c-uniform* in the norm metric provided there exists a constant c with the property that each pair of points z_1, z_2 in D can be joined by a rectifiable arc α in D satisfying (see [12, 20, 23])

(1) $\min\limits_{j=1,2} \ell(\alpha[z_j, z]) \le c\, d_D(z)$ for all $z \in \alpha$, and

(2) $\ell(\alpha) \le c\, |z_1 - z_2|$,

where $\ell(\alpha)$ denotes the length of α and $\alpha[z_j, z]$ the part of α between z_j and z. Moreover, α is said to be a *uniform arc*.

In [22], Väisälä characterized uniform domains as follows.

Lemma 2.6. [22, Theorem 6.16] *For a domain D, the following are quantitatively equivalent:*

(1) *D is a c-uniform domain;*
(2) *$k_D(z_1, z_2) \le c'\, j_D(z_1, z_2)$ for all $z_1, z_2 \in D$;*
(3) *$k_D(z_1, z_2) \le c_1'\, j_D(z_1, z_2) + d$ for all $z_1, z_2 \in D$.*

In the case of domains in $\mathbf{R}^n$, the equivalence of items (1) and (3) in Theorem D is due to Gehring and Osgood [2] and the equivalence of items (2) and (3) due to Vuorinen [26]. Many of the basic properties of this metric may be found in [2, 8, 14, 21, 22].

In [24], Väisälä proved the following examples for some special uniform domain.

Example 2.7. [24, Examples 10.4] (1) Each ball $B \subset E$ is 2-uniform;
(2) Every bounded convex domain $G \subset E$ is uniform;
(3) Half space $H \subset E$ is c-uniform for all $c > 2$.

2.8. Bilipschitz and FQC maps.

Definition 2.9. Suppose $G \subsetneq E$, $G' \subsetneq E'$, and $M \ge 1$. We say that a homeomorphism $f\colon G \to G'$ is *M-bilipschitz* if

$$|x - y|/M \le |f(x) - f(y)| \le M\, |x - y|$$

for all $x, y \in G$, and *M-QH* if

$$k_G(x, y)/M \le k_{G'}(f(x), f(y)) \le M\, k_G(x, y)$$

for all $x, y \in G$.

Clearly, if f is M-bilipschitz or M-QH, then also f^{-1} has the same property.

Definition 2.10. Let $G \subsetneq E$ and $G' \subsetneq E'$ be metric spaces, and let $\varphi\colon [0, \infty) \to [0, \infty)$ be a growth function, that is, a homeomorphism with $\varphi(t) \ge t$. We say that a homeomorphism $f\colon G \to G'$ is *φ-semisolid* if

$$k_{G'}(f(x), f(y)) \le \varphi(k_G(x, y))$$

for all $x, y \in G$, and *φ-solid* if both f and f^{-1} satisfy this condition.

We say that f is *fully φ-semisolid* (resp. *fully φ-solid*) if f is φ-semisolid (resp. φ-solid) on every subdomain of G. In particular, when $G = E$, the corresponding subdomains are taken to be proper ones. Fully φ-solid maps are also called *freely φ-quasiconformal maps*, or briefly *φ-FQC maps*.

Clearly, if f is freely φ-quasiconformal, then so is f^{-1}.

If $E = \mathbf{R}^n = E'$, then f is FQC if and only if f is quasiconformal (cf. [21]). See [18, 26] for definitions and properties of K-quasiconformal maps, or briefly K-QC maps.

2.11. Quasisymmetric and quasimöbius maps. Let X be a metric space and $\dot{X} = X \cup \{\infty\}$. By a triple in X we mean an ordered sequence $T = (x, a, b)$ of

three distinct points in X. The ratio of T is the number

$$\rho(T) = \frac{|a - x|}{|b - x|}.$$

If $f: X \to Y$ is an injective map, the image of a triple $T = (x, a, b)$ is the triple $fT = (fx, fa, fb)$.

Definition 2.12. Let X and Y be two metric spaces, and let $\eta: [0, \infty) \to [0, \infty)$ be a homeomorphism. An embedding $f: X \to Y$ is said to be η-*quasisymmetric*, or briefly η-QS, if $\rho(f(T)) \leq \eta(\rho(T))$ for each triple T in X.

It is known that an embedding $f: X \to Y$ is η-QS if and only if $\rho(T) \leq t$ implies that $\rho(f(T)) \leq \eta(t)$ for each triple T in X and $t \geq 0$ (cf. [17]).

A quadruple in X is an ordered sequence $Q = (a, b, c, d)$ of four distinct points in X. The cross ratio of Q is defined to be the number

$$\tau(Q) = |a, b, c, d| = \frac{|a - b|}{|a - c|} \cdot \frac{|c - d|}{|b - d|}.$$

Observe that the definition is extended in the well known manner to the case where one of the points is ∞. For example,

$$|a, b, c, \infty| = \frac{|a - b|}{|a - c|}.$$

If $X_0 \subset \dot{X}$ and if $f: X_0 \to \dot{Y}$ is an injective map, the image of a quadruple Q in X_0 is the quadruple $fQ = (fa, fb, fc, fd)$.

Definition 2.13. Let X and Y be two metric spaces and let $\eta: [0, \infty) \to [0, \infty)$ be a homeomorphism. An embedding $f: X \to Y$ is said to be η-*quasimöbius* (cf. [19]), or briefly η-QM, if the inequality $\tau(f(Q)) \leq \eta(\tau(Q))$ holds for each quadruple Q in X.

Observe that if $\infty \in \dot{X}$ and if $f: \dot{X} \to \dot{Y}$ is η-quasimöbius with $f(\infty) = \infty$, then f is η-quasisymmetric (see [24, 6.18]). Conversely, the following result holds.

Lemma 2.14. [19, Theorem 3.12] *Suppose that X and Y are bounded spaces, and that $f: X \to Y$ is θ-quasimöbius. If there are $\lambda > 0$ and $z_1, z_2, z_3 \in X$ such that*

$$|z_i - z_j| \geq \operatorname{diam}(X)/\lambda, \quad |f(z_i) - f(z_j)| \geq \operatorname{diam}(Y)/\lambda$$

for $i \neq j$. Then f is η-quasisymmetric with $\eta = \eta_{\theta, \lambda}$.

Concerning the relation between the class of uniform domains and quasimöbius maps, Väisälä proved the following result.

Lemma 2.15. [24, Theorems 11.8 and 11.15] *Suppose that $D \subsetneq E$ and $D' \subsetneq E'$, that D and D' are c-uniform domains, and that $f: D \to D'$ is a φ-FQC map. Then f extends to a homeomorphism $\overline{f}: \overline{D} \to \overline{D}'$ and $\overline{f}$ is θ_1-QM in $\overline{D}$.*

Finally we introduce the concept of homogeneous density from [17].

Definition 2.16. [17, Definition 3.8] *A space X is said to be* homogeneously dense, *abbreviated HD, if there are numbers r_1, r_2 such that $0 < r_1 \leq r_2 < 1$ and such that for each pair of points $a, b \in X$ there is $x \in X$ satisfying the condition*

$$r_1 |b - a| \leq |x - a| \leq r_2 |b - a|.$$

We also say that X is (r_1, r_2)-HD or simply r-HD, where $r = (r_1, r_2)$.

By the definition, obviously, a HD space has no isolated point. And for all $0 < r_1 \leq r_2 < 1$, every connected domain is (r_1, r_2)-HD, $[0, 1] \cup [2, 3]$ is $(\frac{1}{6}, \frac{1}{4})$-HD (see [17]). Particularly, a finite union of connected nondegenerate sets (i.e., the set is not a point) is (r_1, r_2)-HD with some constants $0 < r_1 \leq r_2 < 1$.

For a HD space, Tukia and Väisälä proved the following properties in [17].

Lemma 2.17. [17, Lemma 3.9] (1) *Let X be (r_1, r_2)-HD and let m be a positive integer. Then X is (r_1^m, r_2^m)-HD.*

(2) *Let X be r-HD and let $f\colon X \to Y$ be η-QS. Then fX is μ-HD, where μ depends only on η and r.*

Moreover, we prove the following property.

Lemma 2.18. *Let $D \subsetneq E$ be a domain with (r_1, r_2)-HD boundary and let $x \in D$. Then for all $x_0 \in \partial D$ with $|x - x_0| \leq 2d_D(x)$ there exists some point $x_1 \in \partial D$ such that*

$$(2.19) \qquad \frac{1}{2}d_D(x) \leq |x_0 - x_1| \leq \left(2 + \frac{17}{2r_1}\right) d_D(x).$$

Proof. By Lemma 2.17 we may assume that $0 < r_1 \leq r_2 < \frac{1}{3}$. For example, if $r_2 \geq \frac{1}{3}$, then there exists a positive integer m such that $r_2^m < \frac{1}{3}$. In fact we can choose $m - 1$ to be the integer part of $\log_{r_2} \frac{1}{3}$, and by Lemma 2.17 the (r_1, r_2)-HD property of ∂D implies that ∂D is (r_1^m, r_2^m)-HD with $r_2^m < \frac{1}{3}$.

For a given $x \in D$, let $x_0 \in \partial D$ be such that $|x - x_0| \leq 2d_D(x)$. We divide the proof into three cases.

Case I: $\partial D \subset \overline{\mathbf{B}}\left(x, \frac{5}{2}d_D(x)\right)$. Obviously, D is bounded. Let $x_1 \in \partial D$ be such that $|x_0 - x_1| \geq \frac{1}{3}\operatorname{diam}(D)$. Then

$$\frac{2}{3}d_D(x) \leq |x_0 - x_1| \leq 5d_D(x),$$

which shows that x_1 is the desired point and satisfies (2.19).

Case II: $\partial D \cap \left(\mathbf{B}\left(x, \frac{1}{r_1}d_D(x)\right) \setminus \overline{\mathbf{B}}\left(x, \frac{5}{2}d_D(x)\right)\right) \neq \emptyset$, where $0 < r_1 < \frac{1}{3}$. Let $x_2 \in \partial D \cap \mathbf{B}\left(x, \frac{1}{r_1}d_D(x)\right) \setminus \overline{\mathbf{B}}\left(x, \frac{5}{2}d_D(x)\right)$. Then

$$|x_0 - x_2| \geq |x_2 - x| - |x - x_0| \geq \frac{1}{2}d_D(x)$$

and

$$|x_0 - x_2| \leq |x_0 - x| + |x - x_2| \leq \left(\frac{1}{r_1} + 2\right)d_D(x).$$

Obviously, x_2 is the needed point.

Case III: $\partial D \cap \left(\mathbf{B}\left(x, \frac{1}{r_1}d_D(x)\right) \setminus \overline{\mathbf{B}}\left(x, \frac{5}{2}d_D(x)\right)\right) = \emptyset$, where $0 < r_1 < \frac{1}{3}$. Let $\omega = \partial D \cap \left(E \setminus \mathbf{B}\left(x, \frac{1}{r_1}d_D(x)\right)\right)$ and d_1 denote the distance from ω to $\mathbf{B}\left(x, \frac{1}{r_1}d_D(x)\right)$, i.e., $d_1 = d\left(\omega, \mathbf{B}\left(x, \frac{1}{r_1}d_D(x)\right)\right)$. If $d_1 = 0$, let $x_3 \in \omega$ be such that $d\left(x_3, \mathbf{B}\left(x, \frac{1}{r_1}d_D(x)\right)\right) \leq \frac{1}{2}d_D(x)$. Hence

$$\left(\frac{1}{r_1} - 2\right)d_D(x) \leq |x_0 - x_3| \leq |x_0 - x| + |x - x_3| \leq \left(\frac{1}{r_1} + \frac{5}{2}\right)d_D(x).$$

So x_3 is the desired point.

If $d_1 > 0$, let $x_4 \in \omega$ be such that

$$(2.20) \qquad d\left(x_4, \mathbf{B}\left(x, \frac{1}{r_1}d_D(x)\right)\right) \leq \frac{3}{2}d_1.$$

We claim that the point x_4 satisfies (2.19). To see this, we first prove

$$(2.21) \qquad d_1 < \frac{5}{r_1}d_D(x).$$

Suppose on the contrary that

$$(2.22) \qquad d_1 \geq \frac{5}{r_1}d_D(x).$$

Since ∂D is (r_1, r_2)-HD, we get that there must exist some point $u \in \partial D$ such that

$$r_1|x_0 - x_4| \leq |u - x_0| \leq r_2|x_0 - x_4|.$$

Hence, first, we have

$$|u - x_0| \geq r_1|x_0 - x_4| \geq r_1(|x_4 - x| - |x - x_0|)$$
$$\geq r_1\left(\frac{6}{r_1} - 2\right)d_D(x) = (6 - 2r_1)d_D(x) > 5d_D(x),$$

which shows that

$$u \in \partial D \cap \left(E \setminus \overline{\mathbf{B}}\left(x, \frac{5}{2}d_D(x)\right)\right).$$

And then, by (2.20), (2.22) and the assumption on r_1 and r_2, namely $0 < r_1 \leq r_2 < \frac{1}{3}$, we have

$$|u - x_0| \leq r_2|x_0 - x_4| \leq r_2(|x_0 - x| + |x - x_4|) \leq r_2\left(2 + \frac{1}{r_1}\right)d_D(x) + \frac{3r_2}{2}d_1 < d_1,$$

which shows that $u \in \partial D \cap \mathbf{B}\left(x, \frac{1}{r_1}d_D(x)\right)$. Therefore, we have

$$u \in \partial D \cap \left(\mathbf{B}\left(x, \frac{1}{r_1}d_D(x)\right) \setminus \overline{\mathbf{B}}\left(x, \frac{5}{2}d_D(x)\right)\right).$$

This is a contradiction. Hence (2.21) holds.

By (2.20), (2.21) and the choice of x_0, we have

$$|x_4 - x_0| \geq |x_4 - x| - |x - x_0| \geq \left(\frac{1}{r_1} - 2\right)d_D(x)$$

and

$$|x_4 - x_0| \leq |x_4 - x| + |x - x_0| \leq \left(2 + \frac{1}{r_1}\right)d_D(x) + \frac{3}{2}d_1 \leq \left(2 + \frac{17}{2r_1}\right)d_D(x).$$

Hence the point x_4 has the required properties, and so the proof of the lemma is complete. $\qquad\square$

The discussions in the case III also follows from [5, Lemma 11.7].

3. Proofs of Theorems 1.1, 1.2, 1.3 and 1.4

For convenience, in the following, we always assume that $x, y, z, \ldots$ denote points in D and $x', y', z', \ldots$ the images in D' of $x, y, z, \ldots$ under f, respectively. We start with some known results that are necessary for the following proofs.

Lemma 3.1. [21, Lemma 2.5] *Suppose that* $x, y \in D \neq E$ *and that either* $|x - y| \leq \frac{1}{2} d_D(x)$ *or* $k_D(x, y) \leq 1$. *Then*

$$\frac{1}{2} \frac{|x - y|}{d_D(x)} \leq k_D(x, y) \leq 2 \frac{|x - y|}{d_D(x)}.$$

Lemma 3.2. [21, Lemma 2.6] *Suppose that* X *is connected, that* $f : X \to Y$ *is* η-*quasisymmetric, and that* $A \subset X$ *is bounded. Then* $f|_A$ *satisfies a two-sided Hölder condition*

$$|x - y|^{1/\alpha}/M \leq |fx - fy| \leq M|x - y|^\alpha \quad \text{for } x, y \in A,$$

where $\alpha = \alpha(\eta) \leq 1$ *and* $M = M(\eta, d(A), d(fA)) \geq 1$.

3.3. The proof of Theorem 1.1. We divide the proof of Theorem 1.1 into two cases.

Case I. D is bounded. We first prove that D' is also bounded. Suppose on the contrary that D' is unbounded. Then let $w'_1, w'_2 \in \partial D'$ such that $|w'_1 - w'_2| \geq 4L \operatorname{diam}(D)$. Since $f : \partial D \to \partial D'$ is L-bilipschitz, we know that

$$\operatorname{diam}(D) \geq \frac{1}{2}|w_1 - w_2| \geq \frac{1}{2L}|w'_1 - w'_2| \geq 2\operatorname{diam}(D),$$

which is a contradiction. Hence, D' is bounded.

Since $f : \partial D \to \partial D'$ is L-bilipschitz, then

$$(3.4) \qquad \frac{1}{4L} \operatorname{diam}(D) \leq \operatorname{diam}(D') \leq 4L \operatorname{diam}(D).$$

Let $z_1, z_2 \in \partial D$ be such that $|z_1 - z_2| \geq \frac{1}{2} \operatorname{diam}(D)$ and let $z'_3 \in \partial D'$ be such that

$$\min\{|z'_1 - z'_3|, |z'_2 - z'_3|\} \geq \frac{1}{6} \operatorname{diam}(D').$$

Then by (3.4), we have

$$|z'_1 - z'_2| \geq \frac{1}{L}|z_1 - z_2| \geq \frac{1}{2L} \operatorname{diam}(D) \geq \frac{1}{8L^2} \operatorname{diam}(D')$$

and

$$\min\{|z_1 - z_3|, |z_2 - z_3|\} \geq \frac{1}{L} \min\{|z'_1 - z'_3|, |z'_2 - z'_3|\} \geq \frac{1}{24L^2} \operatorname{diam}(D),$$

which, in combination with Lemma 2.14 and Lemma 2.15, shows that f is η-QS in $\overline{D}$ with η depending on c, L and φ.

Case II. D is unbounded. Then obviously, $\infty \in \partial D$. Since $f : \partial D \to \partial D'$ is L-bilipschitz, we have $\infty \in \partial D'$ and $f(\infty) = \infty$. Hence by the assumptions in Theorem 1.1 and Lemma 2.15, f is η-QS in $\overline{D}$ with η depending on c, L and φ. $\square$

3.5. The proof of Theorem 1.2 The proof of Theorem 1.2 easily follows from Theorem 1.1 and Lemma 3.2. $\square$

In the remaining part of this paper, we always assume that D and D' are c-uniform subdomains in E and E', respectively, that the boundary of D is (r_1, r_2)-homogeneously dense, that $f\colon D \to D'$ is a φ-FQC map, and that f extends to a homeomorphism $\overline{f}\colon \overline{D} \to \overline{D'}$ such that $\overline{f}\colon \partial D \to \partial D'$ is L-bilipschitz.

We first show that the following lemma holds.

Lemma 3.6. *There is a constant $M_1 = M_1(c, L, \varphi, r_1, r_2)$ such that for given $x \in D$ the following hold:*

(1) *For $x_0 \in \partial D$ with $|x - x_0| \le 2d_D(x)$, we have*

$$|x_0' - x'| \le M_1 d_D(x).$$

(2) *For all $x_1 \in \partial D$, we have*

$$(3.7) \qquad \frac{1}{2(2L + M_1)}|x_1 - x| \le |x_1' - x'| \le 2(2L + M_1)|x_1 - x|.$$

Proof. We first prove (1). For a fixed $x \in D$, let $x_0 \in \partial D$ be such that $|x - x_0| \le 2d_D(x)$. Let x_2 be the intersection point of $\mathbf{S}(x, \frac{1}{2}d_D(x))$ with $[x_0, x]$. Then by (2.4) we have

$$k_D(x_2, x) \le \log\left(1 + \frac{|x - x_2|}{d_D(x) - |x - x_2|}\right) = \log 2,$$

which implies that

$$\log\frac{|x_2' - x'|}{|x_2' - x_0'|} \le k_{D'}(x_2', x') \le \varphi(k_D(x_2, x)) = \varphi(\log 2).$$

Hence

$$(3.8) \qquad |x_2' - x'| \le e^{\varphi(\log 2)}|x_2' - x_0'|,$$

and so

$$(3.9) \qquad |x' - x_0'| \le |x' - x_2'| + |x_2' - x_0'| \le (e^{\varphi(\log 2)} + 1)|x_2' - x_0'|.$$

Since ∂D is (r_1, r_2)-HD, we see from Lemma 2.18 that there must exist some point $x_3 \in \partial D$ such that

$$(3.10) \qquad \frac{1}{2}d_D(x) \le |x_3 - x_0| \le \left(2 + \frac{17}{2r_1}\right)d_D(x).$$

Hence

$$(3.11) \qquad |x - x_3| \le |x - x_0| + |x_0 - x_3| \le \left(4 + \frac{17}{2r_1}\right)d_D(x)$$

and

$$(3.12) \qquad \frac{1}{2L}d_D(x) \le \frac{1}{L}|x_3 - x_0| \le |x_3' - x_0'| \le L|x_3 - x_0| \le L\left(2 + \frac{17}{2r_1}\right)d_D(x).$$

By Lemma 2.15 we see that f^{-1} is θ-quasimöbius in $\overline{D}$, where $\theta = \theta(c, \varphi)$. It follows from (3.8), (3.10), (3.11) and (3.12) that

$$\frac{1}{6\left(4 + \frac{17}{2r_1}\right)} \le \frac{|x_3 - x_0|}{|x_2 - x_0|} \cdot \frac{|x_2 - x|}{|x - x_3|} \le \theta\left(\frac{|x_3' - x_0'|}{|x_2' - x_0'|} \cdot \frac{|x_2' - x'|}{|x' - x_3'|}\right)$$

$$\le \theta\left(\frac{L\left(2 + \frac{17}{2r_1}\right)e^{\varphi(\log 2)}d_D(x)}{|x' - x_3'|}\right),$$

which, together with (3.9), shows that

$$|x' - x_0'| \le |x' - x_3'| + |x_3' - x_0'| \le \left(\lambda L \left(2 + \frac{17}{2r_1}\right) e^{\varphi(\log 2)} + L \left(2 + \frac{17}{2r_1}\right)\right) d_D(x)$$

$$= L \left(2 + \frac{17}{2r_1}\right) (\lambda e^{\varphi(\log 2)} + 1) d_D(x),$$

where $\lambda = 1/\theta^{-1}\left(\frac{1}{6\left(4 + \frac{17}{2r_1}\right)}\right)$. Thus the proof of (1) is complete by taking $M_1 = L\left(2 + \frac{17}{2r_1}\right)(\lambda e^{\varphi(\log 2)} + 1)$.

Now we are going to prove (2). We first observe that $f\colon \partial D \to \partial D'$ is η-QS with $\eta(t) = L^2 t$. Hence Lemma 2.17 shows that $\partial D'$ is (λ_1, λ_2)-HD with λ_1, λ_2 depending only on L, r_1 and r_2. Since f^{-1} is also a φ-FQC map, it is easily seen that we only need to prove the right hand side of (3.7). For $x \in D$, we let $y_1 \in \partial D$ be such that

$$(3.13) \qquad\qquad |x - y_1| \le 2d_D(x).$$

Then it follows from Lemma 3.6 (1) that

$$(3.14) \qquad\qquad |x' - y_1'| \le M_1 d_D(x) \le M_1 |x - y_1|.$$

For $x_1 \in \partial D$, on one hand, if $|y_1 - x_1| \le 2|x - y_1|$, then by (3.13),

$$|x' - x_1'| \le |x' - y_1'| + |y_1' - x_1'| \le M_1 |x - y_1| + L|y_1 - x_1|$$
$$\le (2L + M_1)|x - y_1| \le 2(2L + M_1)d_D(x) \le 2(2L + M_1)|x - x_1|.$$

On the other hand, if $|y_1 - x_1| > 2|x - y_1|$, then we have

$$|x - x_1| > |y_1 - x_1| - |x - y_1| > \frac{1}{2}|y_1 - x_1|,$$

which, together with (3.14), shows that

$$|x' - x_1'| \le |x' - y_1'| + |y_1' - x_1'| \le M_1 |x - y_1| + L|y_1 - x_1|$$
$$\le 2M_1 d_D(x) + 2L|x - x_1| \le 2(L + M_1)|x - x_1|.$$

Hence the proof of (3.7) is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.15. The proof of Theorem 1.3. Supposing that $f \in QC_\varphi^L(D, D')$ is M-QH, we show that f is M'-bilipschitz from $\overline{D}$ to $\overline{D}'$. Lemma 2.17 yields that $\partial D'$ is (λ_1, λ_2)-HD with λ_1, λ_2 depending only on L, r_1 and r_2. Then by Lemma 3.6 and the fact that "f^{-1} is also M-QH and a M-QH map is a φ-FQC map with $\varphi(t) = Mt$" we know that it suffices to show that for all $z_1, z_2 \in D$, the following holds:

$$(3.16) \qquad\qquad |z_1' - z_2'| \le M'|z_1 - z_2|.$$

Fix $z_1, z_2 \in D$. Without loss of generality, we may assume that

$$\max\{d_D(z_1), d_D(z_2)\} = d_D(z_1).$$

Consider first the case $|z_1 - z_2| \le \frac{1}{2M} d_D(z_1)$. Then by Lemma 3.1,

$$k_{D'}(z_1', z_2') \le M k_D(z_1, z_2) \le 2M \frac{|z_1 - z_2|}{d_D(z_1)} \le 1,$$

which shows that

$$\frac{1}{2} \frac{|z_1' - z_2'|}{d_{D'}(z_1')} \le k_D(z_1', z_2') \le M k_D(z_1, z_2) \le 2M \frac{|z_1 - z_2|}{d_D(z_1)}.$$

Let $w \in \partial D$ be a point such that $|z_1 - w| \leq 2d_D(z_1)$. Then by Lemma 3.6, we have

$$d_{D'}(z_1') \leq |z_1' - w'| \leq M_1 d_D(z_1),$$

which shows that

$$(3.17) \qquad |z_1' - z_2'| \leq 4M \frac{|z_1 - z_2|}{d_D(z_1)} d_{D'}(z_1') \leq 4MM_1|z_1 - z_2|.$$

Next we consider the case $|z_1 - z_2| > \frac{1}{2M} d_D(z_1)$. We let $z \in \partial D$ be such that $|z_1 - z| \leq 2d_D(z_1)$. If $|z_1 - z| \leq \frac{1}{2}|z_2 - z|$, then

$$|z_1 - z_2| \geq |z_2 - z| - |z_1 - z| \geq \frac{1}{2}|z_2 - z|,$$

and so Lemma 3.6 yields

$$(3.18) \qquad \begin{aligned} |z_1' - z_2'| &\leq |z_1' - z'| + |z_2' - z'| \leq M_1 d_D(z_1) + 2(2L + M_1)|z_2 - z| \\ &\leq 2(MM_1 + 4L + 2M_1)|z_1 - z_2|. \end{aligned}$$

On the other hand, if $|z_1 - z| \geq \frac{1}{2}|z_2 - z|$, then by Lemma 3.6 we have

$$(3.19) \qquad \begin{aligned} |z_1' - z_2'| &\leq |z_1' - z'| + |z_2' - z'| \leq M_1 d_D(z_1) + 2(2L + M_1)|z_2 - z| \\ &\leq M_1 d_D(z_1) + 4(2L + M_1)|z_1 - z| \leq 2M(9M_1 + 16L)|z_1 - z_2|. \end{aligned}$$

By taking $M' = 2M(9M_1 + 16L)$ we see from (3.17), (3.18) and (3.19) that (3.16) holds. Hence the proof of Theorem 1.3 is complete. $\qquad\square$

Remark 3.20. (1) In Theorem 1.3, the hypothesis "f is FQC" alone does not imply the conclusion "f is bilipschitz". As an example, we consider the radial power map $f_\alpha \colon \mathbf{B} \to \mathbf{B}$ with $f_\alpha(x) = |x|^{\alpha-1}x$ and $\alpha \geq 1$. By [21, 6.5] we see that f_α is a FQC map and $f_\alpha|_{\partial\mathbf{B}}$ is the identity on the boundary, but f_α is not bilipschitz (see [24, 6.8]).

(2) If the boundary of D is not HD, then "f being QH" does not always imply that "f is bilipschitz". We still consider the radial power map $f_\alpha \colon E \setminus \{0\} \to E \setminus \{0\}$ with $f_\alpha(x) = |x|^{\alpha-1}x$ and $\alpha \geq 1$. On one hand, the domain $E \setminus \{0\}$ has only two boundary components: $\{0\}$ and $\{\infty\}$, and so the boundary is not HD. On the other hand, f is α-QH (see [24, 5.21]) and it is the identity on the boundary. But it is not bilipschitz.

3.21. The proof of Theorem 1.4. Given $x \in D = D'$, let $z' \in \partial D'$ be such that $d_{D'}(x') \geq \frac{1}{2}|x' - z'|$. Then Lemma 3.6 yields

$$d_{D'}(x') \geq \frac{1}{4(2L + M_1)}|x - z| \geq \frac{1}{4(2L + M_1)} d_D(x).$$

Let $z_1 \in \partial D$ be such that $|x - z_1| \leq 2d_D(x)$. Then it follows from Lemma 3.6 that

$$|x - x'| \leq |x - z_1| + |x' - z_1| \leq (2 + M_1)d_D(x).$$

Hence by Lemma 2.6 we see that

$$k_D(x, x') \leq c' \log\left(1 + \frac{|x - x'|}{\min\{d_D(x), d_D(x')\}}\right) \leq c' \log\left(1 + 4(2 + M_1)(2L + M_1)\right). \quad \square$$

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