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ON L^∞-REGULARITY RESULT FOR SOME DEGENERATE NONLINEAR ELLIPTIC EQUATIONS

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Abstract. We prove an existence result of bounded solutions for some degenerate nonlinear elliptic equations whose prototype is

$$-\operatorname{div}\left(\frac{w(x)}{(1+|u|)^\theta}\nabla u\right) = f - \operatorname{div} g, \quad \text{in a bounded open } \Omega \subset \mathbf{R}^N,$$

in the setting of the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$. By means of the relative rearrangement, we prove an L^∞-estimate which we use to obtain an existence result.

1. Introduction

Let Ω be a bounded open subset of $\mathbf{R}^N$ with $N \geq 2$. We are interested in the study of the existence of bounded solutions for the following problem

$$\begin{cases} -\operatorname{div} a(x,u,\nabla u) = f - \operatorname{div} g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where a is a Carathéodory function having natural growth of order $p-1$, $p>1$, with respect to $|u|$ and $|\nabla u|$ and the data f and g satisfy suitable summability assumptions. We suppose that $A(u) = -\operatorname{div} a(x,u,\nabla u)$ is a nonlinear degenerate elliptic operator in the sense that there exist a real valued non-negative function w such that

$$a(x,u,\nabla u)\cdot\nabla u \geq \frac{w(x)}{(1+|u|)^{\theta(p-1)}}|\nabla u|^p, \tag{1.2}$$

with θ a real number such that $0 \leq \theta \leq 1$. In the case where A is a uniformly elliptic operator ($\theta = 0$ and $w = \text{const}$), the existence of bounded weak solutions was obtained in [6] where the crucial step of uniforme L^∞-estimate was obtained by means of the well-known Stampacchia's lemma and by a different method based

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on rearrangement techniques in [12] for a quasi-linear operator and then in [13] for nonlinear operator.

Regarding the weighted case with $\theta = 0$, Murthy and Stampacchia in [19] had initiated the first results in the linear case and since many results were given (see, for instance, [3, 4, 8, 14]). In the non-coercive case where w is a nonzero constant, the existence of bounded weak solutions can be found in [2] with $f \in L^m(\Omega)$, $m > \max(\frac{N}{p}, 1)$, and $g \equiv 0$ and in [5] with $g \in (L^r(\Omega))^N$, $r > \frac{N}{p-1}$, and $f \equiv 0$. In [21] the author proved the existence of bounded solutions for the Dirichlet problem associated to the equation $-\operatorname{div} a(x, u, \nabla u) + F(x, u, \nabla u) = -\operatorname{div} g$, where the function a grows like

$$|a(x, s, \xi)| \leq \beta(|s|)(a_0(x) + |\xi|^{p-1}), \tag{1.3}$$

with $\beta\colon [0, +\infty) \to [0, +\infty)$ an increasing function and $g \in (L^r(\Omega))^N$, $r > \frac{N}{p-1}$.

In the present paper, we consider the problem (1.1) under, among others, the assumptions (1.2) and (1.3). As far as the function w is involved, the natural setting in which such equations are considered is that of weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ defined below. In this framework, the author in [11] has dealt with (1.1) in the quasilinear case with $g = 0$, while in [9] one can find a related topic. The model example we have in mind is the following boundary value problem:

$$\begin{cases} -\operatorname{div}\left(w(x)\frac{(1+|u|)^m}{(1+|u|)^{\theta(p-1)}}|\nabla u|^{p-2}\nabla u\right) = f - \operatorname{div} g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $m > 0$. We emphasize that since no growth condition is required for the function β, it is not obvious that the function $a(x, u, \nabla u)$ belongs to $(L^{p'}(\Omega))^N$ that is to say the operator $-\operatorname{div} a(x, u, \nabla u)$ is meaningless in general even as a distribution. In fact such an operator is only defined on $W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$.

Despite this, there are two other difficulties in dealing with (1.1). The first one is due to the fact that by (1.2), the diffusion may disappear when the unknown has large values and leads, in general, to a non-coercive operator to which, unfortunately, standard Leray–Lions surjectivity result cannot be applied (see [16]) even if the datum belongs to the dual space $W^{-1,\frac{p}{p-1}}(\Omega, w^{-\frac{1}{p-1}})$. We overcome this problem by approximating the operator A with another one defined by "cutting", by means of truncatures, the nonlinearity $a(x, s, \xi)$ obtaining a coercive and pseudomonotone differential operator on $W_0^{1,p}(\Omega, w)$, to which we can apply an existence result in [16]. The second difficulty stems from the term $\operatorname{div} g$ which generates the gradient of the unknown function. This difficulty is overcome by choosing a suitable test function built from the unknown and by means of rearrangement techniques (see [15, 18, 22], without using the weighted version of the Sobolev imbedding [19, Theorem 3.1].

The paper is organized as follows. Some reminders on weighted Sobolev spaces that we use, some properties on rearrangement of functions and the main result are given in Section 2. In Section 3 we prove an a priori L^∞-estimate which allows us to prove our main result (see Section 4).

2. Some prerequisites and assumptions

2.1. Sobolev spaces with weight. We recall some facts about Sobolev spaces with weight that can be found in [10]. Let Ω be an open subset of $\mathbf{R}^N$, $N \geq 2$, and $1 \leq p < \infty$ a real number. Let $w = w(x)$ be a weighted function, that is a function which is measurable and positive a.e. in Ω. Define $L^p(\Omega, w) = \{u \text{ measurable}\colon uw^{\frac{1}{p}} \in L^p(\Omega)\}$. We shall denote by $W^{1,p}(\Omega, w)$ the function space which consists of all real functions $u \in L^p(\Omega, w)$ such that their weak derivatives $\frac{\partial u}{\partial x_i}$, $i = 1, \cdots, N$ (in the sense of distributions) satisfy $\frac{\partial u}{\partial x_i} \in L^p(\Omega, w)$, for all $i = 1, \cdots, N$. Endowed with the norm

$$\||u|\|_{p,w} = \left(\int_\Omega |u|^p w(x)\, dx + \int_\Omega |\nabla u|^p w(x)\, dx \right)^{\frac{1}{p}}, \tag{2.1}$$

$W^{1,p}(\Omega, w)$ is a Banach space. Further we suppose that

$$w \in L^1_{\text{loc}}(\Omega), \tag{2.2}$$

$$w^{-\frac{1}{p-1}} \in L^1(\Omega). \tag{2.3}$$

Due to condition (2.2), $\mathcal{C}_0^\infty(\Omega)$ is a subset of $W^{1,p}(\Omega, w)$. Therefore, we denote by $W_0^{1,p}(\Omega, w)$ the closure of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{p,w} = \left(\int_\Omega |\nabla u|^p w(x)\, dx \right)^{\frac{1}{p}}.$$

We remark that condition (2.3) implies that $W^{1,p}(\Omega, w)$ as well as $W_0^{1,p}(\Omega, w)$ are reflexive Banach spaces if $1 < p < \infty$. The elements of the dual space $W^{-1,p'}(\Omega, w^{1-p'})$ of $W_0^{1,p}(\Omega, w)$ can be represented as the sum

$$f_0 - \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

where $\frac{f_i}{w} \in L^{p'}(\Omega, w)$, $i = 0, 1, \cdots, n$.

Since we are dealing with compactness methods to get solutions of nonlinear elliptic equations, a compact imbedding is necessary. This leads us to suppose that the weight function w also satisfies

$$w^{-q} \in L^1(\Omega), \quad 1 + \frac{1}{q} < p \quad \text{and} \quad q > \frac{N}{p}. \tag{2.4}$$

Condition (2.4) ensures that the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega) \tag{2.5}$$

is compact.

2.2. A reminder on the rearrangements. Let Ω be an open bounded subset of $\mathbf{R}^N$, $N \geq 1$, and let $u\colon \Omega \to \mathbf{R}$ be a measurable function. We denote by $|E|$ the Lebesgue measure of a subset E of Ω and by Ω^* the interval $]0, |\Omega|[$. The distribution function $\mu_u(t)$ of u is defined as follows

$$\mu_u(t) = |\{x \in \Omega\colon |u(x)| > t\}|, \quad t \in \mathbf{R}.$$

The decreasing rearrangement u^* of u is defined as the generalized inverse function of $\mu_u(t)$, that is

$$u^*(s) = \sup\{t > 0\colon \mu_u(t) > s\}, \quad s \in \overline{\Omega^*}.$$

We observe that $u^*(0) = \|u\|_\infty$ and recall that u and u^* are equimeasurable, i.e.

$$\mu_u(t) = \mu_{u^*}(t), \quad t \in \mathbf{R}.$$

This implies that for any non-negative Borel function F it holds that

$$\int_\Omega F(u(x))\,dx = \int_0^{|\Omega|} F(u^*(s))\,ds,$$

and in particular

$$\|u^*\|_{L^p(\Omega^*)} = \|u\|_{L^p(\Omega)}, \quad 1 \le p \le \infty.$$

Given a measurable subset $E \subset \Omega$, the following Hardy–Littlewood inequality holds

$$\int_E |u(x)|\,dx \le \int_0^{|E|} u^*(s)\,ds. \tag{2.6}$$

At least, if $\psi\colon \mathbf{R} \to \mathbf{R}$ is a non-decreasing function, then

$$(\psi(u))^* = \psi(u^*). \tag{2.7}$$

These and other properties of decreasing rearrangement can be found, for instance, in [15, 22].

Now we recall the notion of relative rearrangement. Given a function $v \in L^1(\Omega)$, we define the function $w\colon \overline{\Omega^*} \to \mathbf{R}$ by

$$w(s) = \begin{cases} \displaystyle\int_{u>u^*(s)} v(x)\,dx & \text{if } |\{u = u^*(s)\}| = 0, \\ \displaystyle\int_{u>u^*(s)} v(x)\,dx + \int_0^{s-|\{u>u^*(s)\}|} \left(v_{|P_u(s)}\right)^*(\sigma)\,d\sigma & \text{if } |\{u = u^*(s)\}| > 0. \end{cases}$$

Here, $v_{|P_u(s)}$ denotes the restriction of v to the set

$$P_u(s) = \{u = u^*(s)\} := \{x \in \Omega\colon u(x) = u^*(s)\}$$

and $\left(v_{|P_u(s)}\right)^*$ is its decreasing rearrangement. Next, we state the following lemma.

Lemma 2.1. *Let $v \in L^p(\Omega)$ for some $1 \le p \le +\infty$ and let u be a measurable function from Ω into $\mathbf{R}$. Then*

(i) *$w \in W^{1,p}(\Omega^*)$ with $\Omega^* = (0, |\Omega|)$,*
(ii) *$\|\frac{dw}{ds}\|_{L^p(\Omega^*)} \le \|v\|_{L^p(\Omega)}$.*

The proof of the above Lemma can be found in [18] if $p = +\infty$ and in [17] if $1 \le p \le +\infty$. The following definition was introduced in [18].

Definition 2.1. The function $\frac{dw}{ds}$ is called the relative rearrangement of v with respect to u and it is denoted by v_{*u}.

This function has many properties (see, for instance, [18, 22]), including the following property that will be useful later and which is proved in [21].

Lemma 2.2. *Let Ω be a bounded open set of $\mathbf{R}^N$. Let $u \in W_0^{1,1}(\Omega)$, $u \ge 0$, and let $v \in L^1(\Omega)$. Then for almost every $t \in (0, \operatorname{ess\,sup} u)$, it holds*

$$\frac{d}{dt}\int_{u>t} v(x)\,dx = v_{*u}(\mu_u(t)) \times \mu_u'(t).$$

In addition, if f is a positive locally integrable function in Ω^*, then for all $s, s' \in \overline{\Omega^*}$ with $s \le s'$

$$\int_{u^*(s')}^{u^*(s)} f(\mu_u(\theta))(-\mu_u'(\theta))\,d\theta \le \int_s^{s'} f(\sigma)\,d\sigma. \tag{2.8}$$

2.3. Assumptions. Let Ω be an open bounded subset of $\mathbf{R}^N$, $N \geq 2$, and $p > 1$ a real number. Let w be a non-negative real valued measurable function defined on Ω which satisfies (2.2), (2.3) and (2.4). Throughout the paper we will be interested in the problem

$$\begin{cases} A(u) = f - \operatorname{div} g \text{ in } \Omega, \\ u \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega), \end{cases} \tag{2.9}$$

and we assume that $a\colon \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ is a Carathéodory function satisfying the following assumptions for a.e. $x \in \Omega$, for every $s \in \mathbf{R}$ and for every $\xi, \mu \in \mathbf{R}^N$

$$|a(x,s,\xi)| \leq w^{\frac{1}{p}}(x)\beta(|s|)(b(x) + w^{1-\frac{1}{p}}(x)|\xi|^{p-1}), \tag{2.10}$$

$$a(x,s,\xi) \cdot \xi \geq w(x) h^{p-1}(s)|\xi|^p, \tag{2.11}$$

$$(a(x,s,\xi) - a(x,s,\mu)) \cdot (\xi - \mu) > 0, \text{ whenever } \xi \neq \mu, \tag{2.12}$$

where $\beta\colon [0,+\infty) \to (0,+\infty)$ is a continuous function and $h\colon \mathbf{R} \to (0,+\infty)$ is a continuous, decreasing, strictly positive and such that its primitive

$$H(t) = \int_0^t h(\tau)\, d\tau,$$

is unbounded. As regards the source terms we assume that

$$f \in L^m(\Omega), \quad \frac{1}{m} < \frac{p}{N} - \frac{1}{q}, \quad m > 1 \tag{2.13}$$

and

$$g w^{-\frac{1}{p}} \in (L^{mp'}(\Omega))^N. \tag{2.14}$$

We use the following definition of solution:

Definition 2.2. A function $u \in W_0^{1,1}(\Omega)$ is said to be a weak solution of (2.9) if $a(\cdot, u, \nabla u) \in (L^{p'}(\Omega, w^{-p'/p}))^N$ and

$$\int_\Omega a(x,u,\nabla u) \cdot \nabla \phi\, dx = \int_\Omega f\phi\, dx + \int_\Omega g \cdot \nabla\phi\, dx, \quad \forall \phi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega). \tag{2.15}$$

Remark 2.1. We cannot expect that (2.15) holds for all $\phi \in W_0^{1,p}(\Omega, w)$ since, in view of the assumption on exponents (2.13) and (2.5) the first integral in the right-hand side is meaningless in general.

We use the following weighted version of the Stampacchia composition result (see [20]):

Lemma 2.3. (see [1]) *Assume that (2.5) holds. Let $F\colon \mathbf{R} \to \mathbf{R}$ be a uniformly Lipschitz function such that $F(0) = 0$. Then, F maps $W_0^{1,p}(\Omega, w)$ into itself. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u)\dfrac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega\colon u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega\colon u(x) \in D\}. \end{cases}$$

In the sequel, we apply this Lemma to the following truncation functions defined on $\mathbf{R}$ by $T_k(s) = \max\{-k, \min\{k, s\}\}$ and $G_k(s) = s - T_k(s)$, $k > 0$.

3. L^∞-a priori estimate

In this section we prove the following

Theorem 3.1. *Let us assume that (2.2), (2.3), (2.4), (2.12), (2.10), (2.11), (2.13) and (2.14) hold true. Then any weak solution u of (2.9) which satisfies (2.15) is such that*

$$H(\|u\|_\infty) \le M := M_1 + M_2 \tag{3.1}$$

where

$$M_1 = \frac{2^{p'}p'}{C_N^{p'}}\|f\|_{L^m(\Omega)}^{p'/p}\|w^{-q}\|_{L^1(\Omega)}^{\frac{1}{q(p-1)}}\left[\frac{N\rho}{q}\times\frac{q(p-1)-1}{p\rho-N}\right]^{\frac{q(p-1)-1}{q(p-1)}}|\Omega|^{\frac{p\rho-N}{N\rho}},$$

$$M_2 = \frac{p'C_p^{1/p}}{C_N}\|gw^{-\frac{1}{p}}\|_{(L^{mp'}(\Omega))^N}^{\frac{1}{p-1}}\|w^{-q}\|_{L^1(\Omega)}^{1/pq}\left(\frac{N(p\rho-1)}{p\rho-N}\right)^{1-1/p\rho}|\Omega|^{\frac{p\rho-N}{Np\rho}},$$

ρ is such that $1/\rho = 1/m + 1/q$, $C_p = 2/p'(2/p)^{1/(p-1)}$ and $C_N = Nw_N^{1/N}$ where w_N denotes the volume of the unit ball in $\mathbf{R}^N$.

Proof. Let $\epsilon > 0$ and $t > 0$. Observe that the function $\phi = \frac{1}{\epsilon}T_\epsilon(G_t(H(u)))$ belongs to $W_0^{1,p}(\Omega, w)\cap L^\infty(\Omega)$ and so it is an admissible test function in (2.9). This choice yields to

$$\frac{1}{\epsilon}\int_{A_{t,\epsilon}} a(x,u,\nabla u)\cdot\nabla u\, h(u)\,dx = \int_{A_t} f\phi\,dx + \frac{1}{\epsilon}\int_{A_{t,\epsilon}} g\cdot\nabla v\,dx$$

where we set for simplicity $v = H(u)$

$$A_{t,\epsilon} = \{x\in\Omega\colon t < |v(x)| \le t+\epsilon\} \quad\text{and}\quad A_t = \{x\in\Omega\colon |v(x)| > t\}.$$

Using (2.11) and Young's inequality we get

$$\frac{1}{\epsilon}\int_{A_{t,\epsilon}} w(x)|\nabla v|^p\,dx \le 2\int_{A_t}|f|\,dx + \frac{C_p}{\epsilon}\int_{A_{t,\epsilon}}|g|^{p'}w^{-\frac{p'}{p}}\,dx$$

where $C_p = \frac{2}{p'}\left(\frac{2}{p}\right)^{\frac{p'}{p}}$. Then, letting ϵ tend to 0^+, we obtain

$$-\frac{d}{dt}\int_{A_t} w(x)|\nabla v|^p\,dx \le 2\int_{A_t}|f|\,dx + C_p\left(-\frac{d}{dt}\int_{A_t}G(x)\,dx\right),$$

with $G(x) = |g(x)|^{p'}w^{-\frac{p'}{p}}(x)$. Since $v\in W_0^{1,p}(\Omega,w)\subset W_0^{1,1}(\Omega)$ and G belongs at least to $L^1(\Omega)$, one has

$$\frac{d}{dt}\int_{A_t}G(x)\,dx = G_{*|v|}(\mu(t))\times(\mu'(t)),\quad\text{for a.e. } t>0,$$

where $\mu(t) = |A_t|$ and $G_{*|v|}$ is the relative rearrangement of G with respect to $|v|$. Hence, by Hardy's inequality we get

$$-\frac{d}{dt}\int_{A_t} w(x)|\nabla v|^p\,dx \le 2\int_0^{\mu(t)} f^*(\sigma)\,d\sigma + C_pG_{*|v|}(\mu(t))\times(-\mu'(t)). \tag{3.2}$$

On the other hand and thanks to Hölder's inequality, we can easily check that

$$-\frac{d}{dt}\int_{A_t}|\nabla v|\,dx \le \left(-\frac{d}{dt}\int_{A_t} w(x)|\nabla v|^p\,dx\right)^{\frac{1}{p}}\left(-\frac{d}{dt}\int_{A_t} w(x)^{-\frac{1}{p-1}}\,dx\right)^{1-\frac{1}{p}}. \tag{3.3}$$

Since $w(x)^{-\frac{1}{p-1}} \in L^1(\Omega)$, we write

$$-\frac{d}{dt}\int_{A_t} w(x)^{-\frac{1}{p-1}}\,dx = \left(w(x)^{-\frac{1}{p-1}}\right)_{*|v|}(\mu(t)) \times (-\mu'(t)), \tag{3.4}$$

for almost every $t > 0$, where $\left(w(x)^{-\frac{1}{p-1}}\right)_{*|v|}$ is the relative rearrangement of $w(x)^{-\frac{1}{p-1}}$ with respect to $|v|$. As a consequence of the Fleming–Rishel formula, one has

$$-\frac{d}{dt}\int_{A_t} |\nabla v| dx \geq C_N \mu(t)^{1-\frac{1}{N}}. \tag{3.5}$$

for almost every $t > 0$. Therefore, combining (3.2), (3.3), (3.4) and (3.5) we obtain

$$\begin{aligned} 1 \leq{}& \frac{2\left(w^{-\frac{1}{p-1}}\right)^{\frac{1}{p'}}_{*|v|}(\mu(t)) \times (-\mu'(t))^{\frac{1}{p'}}}{C_N \mu(t)^{1-\frac{1}{N}}} \left(\int_0^{\mu(t)} f^*(\sigma)\,d\sigma\right)^{\frac{1}{p}} \\ &+ \frac{C_p^{\frac{1}{p}}\left(w^{-\frac{1}{p-1}}\right)^{\frac{1}{p'}}_{*|v|}(\mu(t)) \times \left(G_{*|v|}\right)^{\frac{1}{p}}(\mu(t)) \times (-\mu'(t))}{C_N \mu(t)^{1-\frac{1}{N}}}. \end{aligned}$$

Let $\epsilon > 0$ and $\tau \in]0, |\Omega|[$. Integration both sides of the last inequality between $v^*(\tau+\epsilon)$ and $v^*(\tau)$, then using the definition of decreasing rearrangement, inequality (2.8) and passing to the limit as ϵ goes to 0^+, we obtain

$$\begin{aligned} -v^{*\prime}(\tau) \leq{}& \frac{2\left(w^{-\frac{1}{p-1}}\right)^{\frac{1}{p'}}_{*|v|}(\tau) \times (-v^{*\prime}(\tau))^{\frac{1}{p}}}{C_N \tau^{1-\frac{1}{N}}} \left(\int_0^{\tau} f^*(\sigma)\,d\sigma\right)^{\frac{1}{p}} \\ &+ \frac{C_p^{\frac{1}{p}}\left(w^{-\frac{1}{p-1}}\right)^{\frac{1}{p'}}_{*|v|}(\tau) \times \left(G_{*|v|}\right)^{\frac{1}{p}}(\tau)}{C_N \tau^{1-\frac{1}{N}}}. \end{aligned}$$

Thus, Young's inequality enables us to get

$$\begin{aligned} -v^{*\prime}(\tau) \leq{}& \frac{2^{p'}p'\left(w^{-\frac{1}{p-1}}\right)_{*|v|}(\tau)}{C_N^{p'} \tau^{p'(1-\frac{1}{N})}} \left(\int_0^{\tau} f^*(\sigma)\,d\sigma\right)^{\frac{p'}{p}} \\ &+ \frac{p' C_p^{\frac{1}{p}}\left(w^{-\frac{1}{p-1}}\right)^{\frac{1}{p'}}_{*|v|}(\tau) \times \left(G_{*|v|}\right)^{\frac{1}{p}}(\tau)}{C_N \tau^{1-\frac{1}{N}}}. \end{aligned}$$

Therefore, integrating the previous inequality between 0 and $|\Omega|$ taking into account (2.7), we obtain

$$\begin{aligned} H(\|u\|_\infty) = \|v\|_\infty \leq{}& \int_0^{|\Omega|} \frac{2^{p'}p'\left(w^{-\frac{1}{p-1}}\right)_{*|v|}(t)}{C_N^{p'} t^{p'(1-\frac{1}{N})}} \left(\int_0^{t} f^*(\sigma)\,d\sigma\right)^{\frac{p'}{p}} dt \\ &+ \int_0^{|\Omega|} \frac{p' C_p^{\frac{1}{p}}\left(w^{-\frac{1}{p-1}}\right)^{\frac{1}{p'}}_{*|v|}(t) \times \left(G_{*|v|}\right)^{\frac{1}{p}}(t)}{C_N t^{1-\frac{1}{N}}}\,dt \\ ={}& I_1 + I_2. \end{aligned}$$

In order to estimate I_1, we remark that $\|f^*\|_{L^m((0,|\Omega|))} = \|f\|_{L^m(\Omega)}$ and we use Hölder's inequality obtaining

$$\int_0^t f^*(\sigma)\, d\sigma \leq \|f\|_{L^m(\Omega)} \times t^{1-\frac{1}{m}},$$

which we use to get

$$I_1 \leq \frac{2^{p'}p'}{C_N^{p'}}\|f\|_{L^m(\Omega)}^{p'/p}\int_0^{|\Omega|}\left(w^{-\frac{1}{p-1}}\right)_{*|v|}(t)\times t^{\frac{p'}{p}(1-\frac{1}{m})-p'(1-\frac{1}{N})}\, dt.$$

Since by (2.4) one has $q(p-1) > 1$, we use again Hölder's inequality to obtain

$$I_1 \leq \frac{2^{p'}p'}{C_N^{p'}}\|f\|_{L^m(\Omega)}^{p'/p}\|w^{-q}\|_{L^1(\Omega)}^{\frac{1}{q(p-1)}}\left(\int_0^{|\Omega|} t^{\left(\frac{p'}{p}(1-\frac{1}{m})-p'(1-\frac{1}{N})\right)\frac{q(p-1)}{q(p-1)-1}}\, dt\right)^{1-\frac{1}{q(p-1)}}.$$

The assumptions on exponents (2.4) and (2.13) allow us to get

$$I_1 \leq \frac{2^{p'}p'}{C_N^{p'}}\|g\|_{L^m(\Omega)}^{p'/p}\|w^{-q}\|_{L^1(\Omega)}^{\frac{1}{q(p-1)}}\left(\frac{N\rho}{q}\times\frac{q(p-1)-1}{p\rho-N}\right)^{1-\frac{1}{q(p-1)}}|\Omega|^{\frac{p\rho-N}{N\rho(p-1)}}.$$

We now turn to estimate I_2. Since $p\rho > N > 1$, we can consider its Hölder conjugate exponent $\alpha = \frac{p\rho}{p\rho-1}$. It's easy to check that α satisfies the identity

$$\frac{1}{qp}+\frac{1}{mp}+\frac{1}{\alpha} = 1,$$

so that by Hölder's inequality we obtain

$$I_2 \leq \frac{p'C_p^{\frac{1}{p}}}{C_N}\|w^{-q}\|_{L^1(\Omega)}^{\frac{1}{qp}}\left(\int_\Omega |g|^{mp'}w^{-\frac{m}{p-1}}\, dx\right)^{\frac{1}{mp}}\left(\int_0^{|\Omega|} t^{\alpha(\frac{1}{N}-1)}\, dt\right)^{\frac{1}{\alpha}}.$$

A straightforward calculation gives

$$I_2 \leq \frac{p'C_p^{\frac{1}{p}}}{C_N}\|w^{-q}\|_{L^1(\Omega)}^{\frac{1}{qp}}\|g\|_{(L^{mp'}(\Omega,w^{-\frac{1}{p}}))^N}^{\frac{1}{p-1}}\left(\frac{N(p\rho-1)}{p\rho-N}\right)^{1-\frac{1}{p\rho}}|\Omega|^{\frac{p\rho-N}{Np\rho}}. \qquad \square$$

4. Application to an existence result

The aim of this section is to prove the following existence result:

Theorem 4.1. *Suppose that the assumptions (2.2), (2.3), (2.4), (2.10), (2.11), (2.12), (2.13) and (2.14) hold true. Then there exists at least one weak solution $u \in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)$ of problem (2.9). Moreover, we also have $H(u) \in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega)$.*

Remark 4.1. (1) The conclusion of the previous result does not depend on the function h and is similar to the one obtained for operators A with w or h or both w and h are nonzero positive constants. This seems to be natural, since if one looks for bounded solutions, the degeneracy of the operator A (generated by unbounded functions) "disappears".

(2) The conclusion of Theorem 4.1 remains valid if we try to change the growth assumption (2.10) by the following usual one

$$|a(x,s,\xi)| \leq w^{\frac{1}{p}}(x)(b(x)+|s|^{p-1}+w^{1-\frac{1}{p}}(x)|\xi|^{p-1}), \tag{4.1}$$

where $b \in L^{p'}(\Omega)$, $p' := \frac{p}{p-1}$

Remark 4.2. Observe that if $q = +\infty$, the degeneration of the operator A in problem 2.9 is produced only when the unknown function has large values. Hence, Theorem 4.1 extends the results contained in [2, 5, 20, 21].

Our result involves partially a related topic in [6, 12, 13] when $q = +\infty$ and h is a nonzero constant and in [3] when only h is a nonzero constant.

4.1. Approximate problem. Let $n \in \mathbf{N}$. We consider the sequence of approximate problems

$$\begin{cases} A_n(u) = f_n - \operatorname{div} g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where $f_n = T_n(f)$ and $A_n(u) = -\operatorname{div} a(x, T_n(u), \nabla u)$. The operator A_n enjoys the properties in Lemma A.1 (see the Appendix). Since the source term $f_n - \operatorname{div} g$ belongs to the dual space $W^{-1,p'}(\Omega, w^{1-p'})$, in view of [16, Theorem 2.7] (p. 180), there exists at least a function $u_n \in W_0^{1,p}(\Omega, w)$ solution to problem (2.9) in the sense

$$\int_\Omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla\phi \, dx = \int_\Omega f_n \phi \, dx + \int_\Omega g \cdot \nabla\phi \, dx, \quad \forall \phi \in W_0^{1,p}(\Omega, w). \tag{4.3}$$

Using Stampacchia's method [20], one can prove that $u_n \in L^\infty(\Omega)$ for fixed n. Thus, by virtue of Theorem 3.1 we get

$$\|u_n\|_\infty \le C_\infty := H^{-1}(M), \tag{4.4}$$

where H^{-1} stands for the inverse function of H. Taking $\phi = u_n$ as test function in the formulation (4.3) and then using (2.11) and Young's inequality, we arrive at

$$\begin{aligned} &\frac{h^{p-1}(C_\infty)}{2} \int_\Omega |\nabla u_n|^p w(x) \, dx \\ &\le C_\infty \|f\|_{L^1(\Omega)} + \frac{p-1}{p} \left(\frac{2}{p h^{p-1}(C_\infty)} \right)^{\frac{p'}{p}} \int_\Omega |g|^{p'} w^{-\frac{p'}{p}}(x) \, dx. \end{aligned}$$

Observe that thanks to (2.14), $|g| w^{-\frac{1}{p}}$ belongs at least to $L^{p'}(\Omega)$. Thus, we conclude that u_n is uniformly bounded in $W_0^{1,p}(\Omega, w)$. Therefore, there exists a function $u \in W_0^{1,p}(\Omega, w)$ such that for a subsequence still denoted by u_n, we have

$$u_n \rightharpoonup u \ \text{ weakly in } W_0^{1,p}(\Omega, w), \tag{4.5}$$

by the compact imbedding (2.5) the sequence $\{u_n\}_n$ converges strongly to u in $L^p(\Omega)$, so that we can deduce that

$$u_n \to u \ \text{ a.e. in } \Omega \tag{4.6}$$

and by (4.4) we get

$$u_n \rightharpoonup u \ \text{ in } L^\infty(\Omega) \text{ for } \sigma^*(L^\infty, L^1). \tag{4.7}$$

4.2. Almost everywhere convergence of the gradients. We now shall prove that the weak convergence (4.5) is strong, that is

$$u_n \to u \ \text{ strongly in } W_0^{1,p}(\Omega, w). \tag{4.8}$$

Using $\phi = u_n - u$, for $n > C_\infty$, as test function in the formulation (4.3), we get

$$\int_\Omega a(x, u_n, \nabla u_n) \cdot (\nabla u_n - \nabla u) \, dx = \int_\Omega f_n (u_n - u) \, dx + \int_\Omega g \cdot (\nabla u_n - \nabla u) \, dx. \tag{4.9}$$

Since $|g| \in L^{p'}(\Omega, w^{-\frac{p'}{p}}) = (L^p(\Omega, w))'$, by (4.5) we have

$$\lim_{n\to\infty} \int_\Omega g \cdot (\nabla u_n - \nabla u)\, dx = 0.$$

For the first term in the right-hand side of (4.9), we can write

$$\begin{aligned}\left|\int_\Omega f_n(u_n - u)\, dx\right| &\le \int_\Omega |(f_n - f)(u_n - u)|\, dx + \left|\int_\Omega f(u_n - u)\, dx\right| \\ &\le 4C_\infty \|f\|_m |\{|f| > n\}|^{1-\frac{1}{m}} + \left|\int_\Omega f(u_n - u)\, dx\right|\end{aligned}$$

In view of (4.7) and (2.13), we obtain

$$\lim_{n\to\infty} \int_\Omega f_n(u_n - u)\, dx = 0.$$

Thus, one has

$$\lim_{n\to\infty} \int_\Omega a(x, u_n, \nabla u_n) \cdot (\nabla u_n - \nabla u)\, dx = 0. \tag{4.10}$$

On the other hand, we write

$$\begin{aligned}&\int_\Omega (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)) \cdot (\nabla u_n - \nabla u)\, dx \\ &= \int_\Omega a(x, u_n, \nabla u_n) \cdot (\nabla u_n - \nabla u)\, dx - \int_\Omega a(x, u_n, \nabla u) \cdot (\nabla u_n - \nabla u)\, dx.\end{aligned}$$

By virtue of (4.6) and Vitali's theorem, we obtain

$$a(x, u_n, \nabla u) \to a(x, u, \nabla u) \ \text{ strongly in } (L^{p'}(\Omega, w^{-\frac{p'}{p}}))^N.$$

It follows from (4.5) that

$$\lim_{n\to\infty} \int_\Omega a(x, u_n, \nabla u) \cdot (\nabla u_n - \nabla u)\, dx = 0,$$

which with (4.10) allow us to get

$$\lim_{n\to\infty} \int_\Omega (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)) \cdot (\nabla u_n - \nabla u)\, dx = 0.$$

Hence, arguing as in [7, Lemma 5], we get the strong convergence of u_n in $W_0^{1,p}(\Omega, w)$ which in turn implies, for a subsequence still denoted by u_n,

$$\nabla u_n \to \nabla u \ \text{ a.e. in } \Omega. \tag{4.11}$$

4.3. Passage to the limit. Let $n > C_\infty$. By virtue of (2.10), the sequence $\{a(x, u_n, \nabla u_n) w^{-\frac{1}{p}}\}_n$ is uniformly bounded in $(L^{p'}(\Omega))^N$ and by (4.6) and (4.11) we have

$$a(x, u_n, \nabla u_n) w^{-\frac{1}{p}} \to a(x, u, \nabla u) w^{-\frac{1}{p}} \ \text{ a.e. in } \Omega.$$

So that by [16, Lemma 1.3], one has

$$a(x, u_n, \nabla u_n) w^{-\frac{1}{p}} \rightharpoonup a(x, u, \nabla u) w^{-\frac{1}{p}} \ \text{ weakly in } (L^{p'}(\Omega))^N.$$

Let $\phi \in (L^p(\Omega, w))^N$. We have $\phi w^{\frac{1}{p}} \in (L^p(\Omega))^N$. Thus, we get

$$\lim_{n\to\infty} \int_\Omega a(x, u_n, \nabla u_n) \cdot \phi \, dx = \int_\Omega a(x, u, \nabla u) \cdot \phi \, dx$$

which means that

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L^{p'}(\Omega, w^{-\frac{p'}{p}}))^N. \tag{4.12}$$

Let now $v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$. Since $f_n \to f$ strongly in $L^1(\Omega)$ and in view of (4.12), we can pass to the limit in (4.3) to get

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla v \, dx = \int_\Omega f v \, dx + \int_\Omega g \cdot \nabla v \, dx. \tag{4.13}$$

Let us now use $H(u_n) \in W_0^{1,p}(\Omega, w)$ as test function in (4.3). Using (2.11), (4.4) and Young's inequality, we arrive at

$$\int_\Omega |\nabla H(u_n)|^p w(x) \, dx \le 2M \|f\|_{L^1(\Omega)} + \frac{2}{p'} \left(\frac{2}{p}\right)^{\frac{1}{p-1}} \int_\Omega |g|^{p'} w^{-\frac{p'}{p}}(x) \, dx.$$

Since $|g| w^{-\frac{1}{p}}$ belongs at least to $L^{p'}(\Omega)$, we have that $\{H(u_n)\}_n$ is uniformly bounded in $W_0^{1,p}(\Omega, w)$ and then in view of (4.5) and (4.6), we obtain $H(u) \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$.

Remark 4.3. The weak convergence (4.12) is a hidden step in the proof of (4.8) as it was done in [7]. Moreover, the weak convergence will become stronger by using (4.8) and Vitali's theorem.

Exemple 4.1. We put ourselves in the situation $p = N = 2$. Let $\Omega = \{(x, y) \in \mathbf{R}^2 \colon x^2 + y^2 < 1\}$. The weight function $w(x, y) = (x^2 + y^2)^{\frac{1}{4}}$ is such that $w \in L^1(\Omega)$, $w^{-1} \in L^1(\Omega)$ and $w^{-3} \in L^1(\Omega)$. The functions $f(x, y) = (x^2 + y^2)^{-\frac{3}{8}} \cos(x^2 + y^2)^{-1}$ belongs to $L^2(\Omega)$ and $g(x, y) = ((x^2 + y^2)^{\frac{1}{8}} \cos xy, (x^2 + y^2)^{-\frac{1}{72}} \sin xy)$ is such that $g w^{-\frac{1}{2}} \in (L^4(\Omega))^2$. Therefore, by virtue of Theorem 3.1 the problem

$$-\operatorname{div}\left(\frac{(x^2 + y^2)^{\frac{1}{4}}}{\sqrt{1 + |u|}} \nabla u\right) = (x^2 + y^2)^{-\frac{3}{8}} \cos(x^2 + y^2)^{-1} - \operatorname{div} g \text{ in } \Omega, \;\; u(x, y) = 0 \text{ on } \partial\Omega$$

has at least a solution $u \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$.

Appendix

We need to use the following concept of operators (see [16]).

Definition A.1. Let V be a reflexive Banach space. An operator T mapping V into its dual V' is said to be pseudomonotone, if for any sequence $\{u_j\}_j$ in V with $u_j \rightharpoonup u$ weakly in V and $\limsup \langle T u_j, u_j - u\rangle \le 0$, it follows that

$$\liminf \langle T u_j, u_j - v\rangle \ge \langle T u, u - v\rangle, \quad \forall v \in V.$$

Lemma A.1. *The operator $A_n(u) = -\operatorname{div} a(x, T_n(u), \nabla u)$ maps $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^{1-p'})$. Moreover, A_n is bounded, coercive and pseudomonotone.*

Proof. Let $u \in W_0^{1,p}(\Omega, w)$. By the growth assumption (2.10), we get

$$\int_\Omega |a(x, T_n(u), \nabla u)|^{p'} w^{1-p'}(x) \, dx \le 2^{p'-1} \beta_n^{p'} \left(\int_\Omega b^{p'}(x) \, dx + \int_\Omega |\nabla u|^p w(x) \, dx\right),$$

where $\beta_n = \max_{|t|\le n} \beta(t)$. This yields $|a(x, T_n(u), \nabla u)| \in L^{p'}(\Omega, w^{1-p'})$. Thus, for each $n \in \mathbf{N}$ we have

$$A_n(u) \in W^{-1,p'}(\Omega, w^{1-p'}).$$

Let u, $v \in W_0^{1,p}(\Omega, w)$. Thanks to the growth assumption (2.10) and Hölder's inequality, we obtain

$$|\langle A_n(u), v\rangle| \le 2^{\frac{1}{p}} \beta_n \left(\int_\Omega b^{p'}(x)\, dx + \int_\Omega |\nabla u|^p w(x)\, dx \right)^{\frac{1}{p'}} \|v\|_{W_0^{1,p}(\Omega,w)}.$$

Hence,

$$\|A_n(u)\|_{W^{-1,p'}(\Omega,w^{1-p'})} \le C_n \left(1 + \|u\|^{p-1}_{W_0^{1,p}(\Omega,w)}\right),$$

where C_n is a constant depending on n, b, Ω and p. So that A_n is bounded from $W_0^{1,p}(\Omega, w)$ to $W^{-1,p'}(\Omega, w^{1-p'})$. Let $u \in W_0^{1,p}(\Omega, w)$. Using (2.11), one has

$$\langle A_n(u), u\rangle \ge h^{p-1}(n) \|u\|^p_{W_0^{1,p}(\Omega,w)}.$$

This implies that the operator A_n is coercive.

To show that A_n is pseudomonotone, let us consider a sequence $\{u_j\}_j$ in $W_0^{1,p}(\Omega, w)$ such that

$$u_j \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega, w)$$

and

$$\limsup \langle A_n u_j, u_j - u\rangle \le 0. \tag{A-1}$$

We shall prove that

$$\liminf \langle A_n u_j, u_j - v\rangle \ge \langle A_n u, u - v\rangle, \quad \forall v \in W_0^{1,p}(\Omega, w).$$

We have $\nabla u_j \rightharpoonup \nabla u$ in $(L^p(\Omega, w))^N$, that is $\nabla u_j w^{\frac{1}{p}} \rightharpoonup \nabla u w^{\frac{1}{p}}$ in $(L^p(\Omega))^N$. Being A_n bounded, we get

$$A_n u_j \rightharpoonup \chi^n \quad \text{weakly in } W^{-1,p'}(\Omega, w^{1-p'}). \tag{A-2}$$

Due to the growth assumption (2.10), we derive that

$$a(x, T_n(u_j), \nabla u_j) \rightharpoonup \psi^n \quad \text{weakly in } (L^{p'}(\Omega, w^{1-p'}))^N. \tag{A-3}$$

Actually by (A-2) and (A-3) one has

$$\chi^n = -\operatorname{div} \psi^n. \tag{A-4}$$

Therefore, it follows from (A-1) that

$$\limsup \langle A_n u_j, u_j\rangle \le \langle \chi^n, u\rangle. \tag{A-5}$$

Let $\phi \in (L^p(\Omega))^N$. We can write the monotonicity hypothesis (2.12)

$$(a(x, T_n(u_j), \nabla u_j) - a(x, T_n(u_j), \phi)) \cdot (\nabla u_j - \phi) > 0.$$

Thus

$$\begin{aligned} &\int_\Omega a(x, T_n(u_j), \nabla u_j) \cdot \nabla u_j\, dx \\ &> \int_\Omega a(x, T_n(u_j), \nabla u_j) \cdot \phi\, dx + \int_\Omega a(x, T_n(u_j), \phi) \cdot (\nabla u_j - \phi)\, dx. \end{aligned} \tag{A-6}$$

Since by (2.5) the imbedding $W_0^{1,p}(\Omega, w) \hookrightarrow L^p(\Omega)$ is compact, there exist a subsequence still denoted by $\{u_j\}$ such that $u_j \to u$ a.e. in Ω, and a function in $L^p(\Omega)$ that dominates $|u_j|$ a.e. in Ω. So that by Lebesgue's dominated convergence theorem, one has

$$a(x, T_n(u_j), \phi) \to a(x, T_n(u), \phi) \ \text{ strongly in } (L^{p'}(\Omega, w^{1-p'}))^N$$

and so

$$\lim_{j\to\infty} \int_\Omega a(x, T_n(u_j), \phi) \cdot (\nabla u_j - \phi)\, dx = \int_\Omega a(x, T_n(u), \phi) \cdot (\nabla u - \phi)\, dx. \tag{A-7}$$

Consequently, using (A-3), (A-4), (A-5), (A-7) and letting $j \to \infty$ in (A-6), we get

$$\int_\Omega (\psi^n - a(x, T_n(u), \phi)) \cdot (\nabla u - \phi)\, dx \geq 0.$$

Let $\theta \in (L^p(\Omega, w))^N$. Setting $\phi = \nabla u + t\theta$ we obtain

$$\int_\Omega [a(x, T_n(u), \nabla u + t\theta) - \psi^n] \cdot \theta\, dx \geq 0.$$

Letting $t \to 0$ and using Lebesgue's dominated convergence theorem, one has

$$\int_\Omega [a(x, T_n(u), \nabla u) - \psi^n] \cdot \theta\, dx \geq 0$$

which implies that

$$\psi^n = a(x, T_n(u), \nabla u) \ \text{ a.e. in } \Omega.$$

Hence, we have $\chi^n = A_n u$ and so

$$A_n u_j \rightharpoonup A_n u \ \text{ in } W^{-1,p'}(\Omega, w^{1-p'}). \tag{A-8}$$

By (A-5) we have

$$\limsup \langle A_n u_j, u_j \rangle \leq \langle A_n u, u \rangle.$$

Going back to inequality (A-6) written with $\phi = \nabla u$ and letting $j \to \infty$, we obtain

$$\liminf \int_\Omega a(x, T_n(u_j), \nabla u_j) \cdot \nabla u_j\, dx \geq \int_\Omega a(x, T_n(u), \nabla u) \cdot \nabla u\, dx.$$

It follows that

$$\lim_{j\to\infty} \langle A_n u_j, u_j \rangle = \langle A_n u, u \rangle,$$

which with (A-8) prove that A_n is pseudomonotone. □

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