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ON MINKOWSKI DIMENSION OF JORDAN CURVES

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Abstract. This paper has its origin in a question raised by McMullen [McM08]: Under what general circumstances does a smooth family of conformal maps $\phi_t\colon \mathbf{D} \to \overline{\mathbf{C}}$ with $\phi_0 = \mathrm{id}$ satisfy

$$\text{(a)}\qquad \frac{d^2}{dt^2}\mathrm{H.dim}\,(\phi_t(\partial\mathbf{D}))\Big|_{t=0} = \lim_{r\to 1}\frac{1}{4\pi|\log(1-r)|}\int_{|z|=r}|\dot{\phi}_0'(z)|^2|dz|?$$

McMullen has shown that (a) is true for some families (ϕ_t) arising from some dynamical systems. In order to answer this question, we consider a general analytic 1-parameter family (ϕ_t), $t \in U$, a neighborhood of 0, conformal maps with $\phi_0 = \mathrm{id}$ and $\phi_t(0) = 0$, $\forall t \in U$ defined as $\phi_t(z) = \int_0^z e^{tb(u)}\,du$, $b \in \mathcal{B}$, where $\mathcal{B}$ is the Bloch space. By using a probability argument, we first describe a relatively large class of functions in $\mathcal{B}$ for which $(\phi_t)_{t\in U}$ satisfies (a), where Hausdorff dimension is replaced by Minkowski dimension. This class is defined in terms of the square function of the associated dyadic martingale of $\mathrm{Re}(b)$. The second principal result of this paper is a counter-example which is reminiscent of Kahane and Piranian's construction of non-Smirnov domain. We have constructed a singular Bloch function b such that if we consider the associated family (ϕ_t) as above, then $\phi_t(\partial\mathbf{D})$ is rectifiable for $t < 0$. Using the properties of this Bloch function b, we prove that there exists $c > 0$ such that $\mathrm{M.dim}\,(\phi_t(\partial\mathbf{D})) \geq 1 + ct^2$ ($t > 0$ small), thus contradicting (a), where the Hausdorff dimension replaced by the Minkowski dimension.

1. Introduction

Let $\Omega \subsetneq \mathbf{C}$ be a simply connected domain containing 0: by the Riemann Mapping Theorem, there is a unique conformal map f from the unit disk $\mathbf{D} = \{|z| < 1\}$ onto Ω such that $f(0) = 0, f'(0) > 0$. In this paper, we are interested in domains with fractal boundary and more precisely in the (Hausdorff) dimension of these boundaries. Well-known examples of fractal curves which have deserved a lot of investigations and attentions are the Julia sets and the limit sets of quasifuchsian groups because of their dynamical properties. For instance, let us consider the family of quadratic polynomials

$$P_t(z) = z^2 + t, \quad t \in \mathbf{C},$$

in the neighborhood of $t = 0$. There is a smooth family of conformal map ϕ_t from $\overline{\mathbf{C}}\setminus\overline{\mathbf{D}}$ onto the basin of infinity of the polynomial $P_t(z)$ (the component containing ∞ of its Fatou set) with $\phi_0(z) = z$ and conjugating P_0 to P_t on their basins of infinity. We thus have

$$\text{(1)}\qquad \phi_t(P_0(z)) = P_t(\phi_t(z)), \quad z \in \overline{\mathbf{C}} \setminus \overline{\mathbf{D}}.$$

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Each ϕ_t extends to a quasiconformal map on the sphere $\overline{\mathbf{C}}$. Taking the derivative of the equation (1) with respect to t, we obtain the equation

$$\dot{\phi}_t(z^2) = 2\phi_t(z)\dot{\phi}_t(z) + 1, \tag{2}$$

where $\dot{\phi}_t = \frac{\partial \phi}{\partial t}$. Let $V(z)$ denote the holomorphic vector field of $V(z) = \frac{\partial \phi_t}{\partial t}\Big|_{t=0}$. Letting $t = 0$ in the equation (2), we get that the holomorphic vector field V satisfies the functional equation

$$V(z^2) = 2zV(z) + 1. \tag{3}$$

If we replace z by z^2 in the preceding equation, we obtain that

$$V(z^4) = 2z^2V(z^2) + 1. \tag{4}$$

Injecting $V(z^2)$ in (3) into (4), one gets

$$V(z) = -\left(\frac{1}{2z} + \frac{1}{2z2z^2}\right) + \frac{V(z^4)}{2z2z^2}.$$

By induction, we obtain

$$V(z) = -\sum_{k=1}^{n-1} \frac{1}{2z2z^2 \dots 2z^{2^k}} + \frac{V(z^{2^n})}{2z2z^2 \dots 2z^{2^{n-1}}},$$

where the term $\frac{V(z^{2^n})}{2^{n+1}z^{2^n-1}}$ tends to 0 as n tends to ∞. Therefore, one can write $V(z)$ in the form of infinite sum

$$V(z) = -z\sum_{k=0}^{\infty} \frac{1}{2^{k+1}z^{2^{k+1}}}. \tag{5}$$

Using thermodynamic formalism, Ruelle [Rul82] (see also [Zin96] and [McM08]) proved that

$$\frac{d^2}{dt^2}\mathrm{H.dim}(J(P_t))\bigg|_{t=0} = \lim_{r\to 1}\frac{1}{4\pi}\frac{1}{\log\frac{1}{1-r}}\int_{|z|=r}|V'(z)|^2|dz|. \tag{6}$$

Using then the explicit formula (5) of V, he could prove that

$$\mathrm{H.dim}(J(P_t)) = 1 + \frac{|t|^2}{4\log 2} + o(|t|^2). \tag{7}$$

for this particular family. The definition of Hausdorff dimension can be found in Apprendix A.

Passing to the disc instead of its complement, let us consider a general analytic one-parameter family (ϕ_t), $t \in U$, a neighborhood of $t = 0$, of conformal maps with $\phi_0 = \mathrm{id}$ and $\phi_t(0) = 0$, $\forall t \in U$. Then

$$\phi_t(z) = \int_0^z e^{\log \phi_t'(u)}\,du$$

and

$$\frac{\partial}{\partial t}\phi_t(z) = \int_0^z \frac{\partial}{\partial t}\left(\log \phi_t'(u)\right) e^{\log \phi_t'(u)}\,du.$$

From which follows that

$$V(z) = \frac{\partial}{\partial t}\phi_t(z)\bigg|_{t=0} = \int_0^z \frac{\partial}{\partial t}\left(\log \phi_t'(u)\right)\bigg|_{t=0} du,$$

and $b(z) = V'(z) = \frac{\partial}{\partial t}(\log \phi_t'(z))\big|_{t=0}$ belongs to the Bloch space $\mathcal{B}$, which is defined as follows:

$$\mathcal{B} = \left\{ b \text{ holomorphic in } \mathbf{D};\ \sup_{\mathbf{D}}(1-|z|^2)|b'(z)| < \infty \right\}.$$

For t small ($|t| \le \frac{1}{\|b\|_{\mathcal{B}}}$), it follows from Becker univalence criterion (see [Pom73], p. 172) that ϕ_t is a univalent map in the unit disc $\mathbf{D}$. It then follows from λ-lemma (see [IT91], p. 118) that ϕ_t has a quasiconformal extension to the plane if t is small enough ($t \le \min\{\frac{1}{3}, \frac{1}{\|b\|_{\mathcal{B}}}\}$). In particular, $\Gamma_t = \phi_t(\partial\mathbf{D})$ is well-defined.

In [McM08], McMullen asked under which condition on the family of (ϕ_t) it is true that

$$\frac{d^2}{dt^2}\mathrm{H.dim}(\Gamma_t)\bigg|_{t=0} = \lim_{r\to 1} \frac{1}{4\pi|\log(1-r)|}\int_{|z|=r}|b(z)|^2|dz|. \tag{8}$$

In other words, the question addresses the problem of how much formula (6) owes to dynamical properties.

Conversely, starting from a function $b \in \mathcal{B}$, it is known that if we put

$$\phi_t(z) = \int_0^z e^{tb(u)}\,du, \quad b \in \mathcal{B}, \tag{9}$$

then (ϕ_t) is an analytic family. There exists a neighborhood U of 0 such that if $t \in U$ then ϕ_t is a conformal map with quasiconformal extension. Denote by Γ_t the image of the unit circle by ϕ_t.

The aim of the present work is two-fold: We shall first describe a "large" family of function $b \in \mathcal{B}$ for which if ϕ_t is defined by (9) (t being real), then (8) is true with Hausdorff dimension replaced by Minkowski dimension (see Appendix A). This class will be defined in term of the square function of the associated of dyadic martingale of $\mathrm{Re}(b)$. Details and proper statement will be given in Section 2. Here we have written "large" since it is actually questionnable whether it is, indeed, large. This class generalizes martingales with constant square functions (see below) which form definitively a small class: we do not know how much more it may be generalized.

The second result of this paper is a counter-example: the starting point is the construction by Kahane and Piranian of a so-called "non Smirnov" rectifiable domain. These authors have constructed a Bloch function b such that if we consider the associated family (ϕ_t) as in (9) then $\phi_t(\partial\mathbf{D})$ is rectifiable for $t < 0$. This function is very singular in the sense that

$$b(z) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta),$$

where μ is singular with respect to Lebesgue measure on the circle. We use this feature to prove that there exists $c > 0$ such that

$$\mathrm{M.dim}(\Gamma_t) \ge 1 + ct^2, \quad t > 0 \text{ small},$$

which contradicts $\lim_{t\to 0}\frac{\mathrm{M.dim}(\Gamma_t)-1}{t^2} = 0$ by (8) with Hausdorff dimension replaced by Minkowski dimension. Even if we believe that Minkowski dimension is equal to Hausdorff dimension in this case we cannot prove it: the question whether there is a counterexample with Hausdorff dimension instead of Minkowski's thus remains open.

2. Martingale condition

Before giving statement of the first result of this paper, we recall some preliminaries on Bloch function and the notion of dyadic martingale.

2.1. Preliminaries on Bloch function.

Proposition 1. *If $b \in \mathcal{B}$ and $b(0) = 0$, then*

$$\frac{1}{2\pi}\int_{\mathbf{T}} |b(r\xi)|^{2n} |d\xi| \leq n! \|b\|_{\mathcal{B}}^{2n} \left(\log \frac{1}{1-r^2}\right)^n$$

for $0 < r < 1$ and $n = 0, 1, \ldots$.

Proof. See [Pom92], p. 186. □

This proposition implies that if $b \in \mathcal{B}$, $b(0) = 0$,

$$\limsup_{r \to 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 \, d\theta}{2\pi \log(\frac{1}{1-r})} \leq \|b\|_{\mathcal{B}}^2 < +\infty. \tag{10}$$

This proposition can be generalized as follows.

Corollary 1. *If $b \in \mathcal{B}$ and $b(0) = 0$, then there exists a constant C such that*

$$\int_{\mathbf{T}} |b(r\xi)|^p |d\xi| \leq C \left(\log \frac{1}{1-r}\right)^{p/2}$$

for $0 < r < 1$ and $p > 0$.

Proof. For $p > 0$, there exists a positive integer n such that $0 < \frac{p}{2n} < 1$. Applying the Hölder's inequality for $\alpha = \frac{p}{2n} < 1$, we deduce that

$$\int_{\mathbf{T}} |b(r\xi)|^{2n} |d\xi| \geq \left(\int_{\mathbf{T}} |b(r\xi)|^{2n\alpha} |d\xi|\right)^{1/\alpha} (2\pi)^{1/\beta},$$

where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then Proposition 1 implies that

$$\begin{aligned}\left(\int_{\mathbf{T}} |b(r\xi)|^p |d\xi|\right) = \left(\int_{\mathbf{T}} |b(r\xi)|^{2n\alpha} |d\xi|\right) &\leq (2\pi)^{-\alpha/\beta} \left(\int_{\mathbf{T}} |b(r\xi)|^{2n} |d\xi|\right)^{\alpha} \\ &\leq C \left(\log \frac{1}{1-r}\right)^{p/2},\end{aligned}$$

where $C = (2\pi)^{-\alpha/\beta}(n! \|b\|_{\mathcal{B}}^{2n})^{\alpha}$. □

For the construction of the dyadic martinagle of a Bloch function, it is necessary to introduce Zygmund functions. These are the functions h on the unit circle $\mathbf{T}$ and satisfying

$$\sup_{|z|=1} |h(e^{i\theta}z) - 2h(z) + h(e^{-i\theta}z)| \leq C\theta, \quad \text{for } \theta > 0.$$

Theorem 1. (Zygmund) *Let b be analytic on the disk $\mathbf{D}$ and let h be an antiderivative of b. Then b belongs to Bloch space $\mathcal{B}$ if and only if h is continuous on the closed disk $\overline{\mathbf{D}}$ and $h \mid \partial\mathbf{D}$ is a Zygmund function.*

Proof. See [Dur70], p. 76. □

Let b be a Bloch function. Then $h\colon z \mapsto \int_0^z b(u)\, du$ of b is a Zygmund function. Let $I = (e^{i\theta_1}, e^{i\theta_2})$ be a subarc of $\partial\mathbf{D}$. We can define b_I, the mean value of b on the arc $I \subset \partial\mathbf{D}$, as the limit $\lim_{r\to 1}(b_r)_I$, where $b_r(z) = b(rz)$, $z \in \mathbf{D}$. Integration by parts shows that

$$b_I = \lim_{r\to 1} \frac{1}{|I|}\int_I b(re^{i\theta})\,d\theta = \frac{-ie^{-i\theta_2}h(e^{i\theta_2}) + ie^{-i\theta_1}h(e^{i\theta_1})}{|I|} + \frac{1}{|I|}\int_I e^{-i\theta}h(e^{i\theta})\,d\theta$$

and by the property of continuity up to the boundary of the primitive function $h(z)$, the limit exists. Hence the definition of mean value of Bloch function is well-defined. We recall now the notion of dyadic martingale of a Bloch function.

2.2. Dyadic martingale. On the probability space $(\partial\mathbf{D}, |.|)$ ($|d\xi| = d\theta/2\pi$, $\xi = e^{i\theta} \in \partial\mathbf{D}$), we consider the increasing sequence of σ-algebras $\{\mathcal{F}_n, n \geq 0\}$ generated by the partitions of the unit circle by the intervals bounded by the (2^n)th roots of the unity.

Let b be a Bloch function, $b(0) = 0$. We define $S = (S_n, \mathcal{F}_n)$ by setting $S_n|I = b_I$ on each dyadic interval I of rank n. In other words, $S_n = \mathbf{E}(b|\mathcal{F}_n)$. Then

$$\forall \xi \in \partial\mathbf{D}, \quad S_n(\xi) = \sum_{I\in\mathcal{F}_n} b_I \chi_I(\xi).$$

This sequence is a martingale in the sense that $\mathbf{E}(S_{n+1}|\mathcal{F}_n) = S_n$. And it has the property:

$$\forall n, \forall \xi \in \partial\mathbf{D}, \quad \left|S_n(\xi) - b((1-2^{-n})\xi)\right| \leq C\|b\|_{\mathcal{B}}, \tag{11}$$

where C is an absolute constant (see [Mak90]). We consider the increasing sequence $\langle S\rangle_n^2 = \sum_{j=1}^n \mathbf{E}((\Delta S_j)^2|\mathcal{F}_{j-1})$, where $\Delta S_j = S_j - S_{j-1}$. In the dyadic case, ΔS_j^2 is $\mathcal{F}_{j-1}$ measurable, so that $\langle S\rangle_n^2 = \sum_{j=1}^n (\Delta S_j)^2$. We call $\langle S\rangle_\infty^2 = \sum_{j\geq 1}(\Delta S_j)^2$ the square function.

Our first result, Theorem 2 will follow from the computing of the integral means $\int_{|z|=r} e^{t\mathrm{Re}(b(z))}|dz|$, $b \in \mathcal{B}$ in which there is only the real part of a Bloch function b that appears. Therefore, we use the dyadic martingale which arises from the real part of the Bloch function. Let us state this result.

2.3. Statement of Theorem 2. Let b be a Bloch function and b_n be the dyadic martingale of $\mathrm{Re}(b)$. Let us assume the following condition for its square function $\langle S\rangle_n^2$:

$$\forall \theta \in [0, 2\pi], \quad \left|\langle S\rangle_n^2(\theta) - \frac{1}{2\pi}\int_0^{2\pi} \langle S\rangle_n^2(\theta)\,d\theta\right| \leq n\delta(n), \tag{$*$}$$

where $\delta(n)$ is a positive function which depends only on n and which tends to zero as n tends to ∞. Let us also write $d(t) = \mathrm{M.dim}(\Gamma_t)$.

Theorem 2. *If b belongs to $\mathcal{B}$ and satisfies the condition $(*)$, then the Minkowski dimension of Γ_t has the following development at zero:*

$$M.dim(\Gamma_t) = 1 + \limsup_{r\to 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2\,d\theta}{4\pi \log\frac{1}{1-r}}\frac{t^2}{2} + o(t^2). \tag{12}$$

By (10), $\limsup_{r\to 1} \dfrac{\int_0^{2\pi} |b(re^{i\theta})|^2\,d\theta}{\log\frac{1}{1-r}}$ exists.

Put $\Omega_t = \phi_t(\mathbf{D})$. The proof of Theorem 2 uses probability arguments and may be divided in two steps. In the first one, we shall deduce Minkowski dimension $d(t)$ of

Γ_t from the relation between Minkowski dimension $d(t)$ and the spectrum of integral means $\beta(d(t), \phi'_t)$ (the definition is in the following).

2.4. The first step of the proof. Given any real number p and $\phi'_t = \exp(tb(z))$ ($z \in \mathbf{D}$; $b \in \mathcal{B}$, $t \in \mathbf{R}$), the limes superior

$$\beta(p, \phi'_t) = \limsup_{r \to 1} \frac{\log \left(\int_0^{2\pi} (\exp\{t \operatorname{Re}(b(re^{i\theta}))\})^p \, d\theta \right)}{|\log(1-r)|}$$
$$= \limsup_{r \to 1} \frac{\log \left(\int_0^{2\pi} \exp\{tp \operatorname{Re}(b(re^{i\theta}))\} \, d\theta \right)}{|\log(1-r)|}$$

is referred to as the spectrum of integral means of ϕ'_t. Remark that in this general setting of $\beta(p, \phi'_t)$, t is some real number, so that ϕ_t need not to be conformal. In particular, let us consider the family of conformal maps $\phi_t(z) = \int_0^z e^{tb(u)} \, du$ ($z \in \mathbf{D}$; t real and small). We note that for t small $\Omega_t = \phi_t(\mathbf{D})$ is a quasidisk and its boundary $\Gamma_t = \partial\Omega_t$ is a quasicircles (boundaries of quasidisks). Denote ϕ_1 by ϕ. Then $\beta(p, \phi'_t) = \beta(tp, \phi')$.

Proposition 2. *If f maps $\mathbf{D}$ conformally onto a quasidisk Ω, then*

$$\text{M.dim}\partial\Omega = p,$$

where p is the unique solution of $\beta(p, f') = p - 1$.

Proof. See [Pom92], p. 241. □

As a consequence of Proposition 2, we deduce the

Proposition 3. *Let b be a Bloch function. If the spectrum of integral means of $\phi'(z) = \exp b(z)$ ($z \in \mathbf{D}$) has the development at $p = 0$,*

$$\beta(p, \phi') = ap^2 + o(p^2),$$

then the Minkowski dimension of Γ_t has the development at $t = 0$,

$$d(t) = 1 + at^2 + o(t^2).$$

Proof. We observe that $d(t) \to 1$, as $t \to 0$. Put $x(t) = d(t) - 1$. The Proposition 2 implies that

$$\beta(d(t), \phi'_t) = d(t) - 1.$$

Since $\beta(d(t), \phi'_t) = \beta(td(t), \phi')$, we get

$$\beta(t(1 + x(t)), \phi') = x(t). \tag{13}$$

By the assumption, we have $\beta(t(1+x(t)), \phi') = at^2(1+x(t))^2 + o(t^2(1+x(t))^2)$. Since $x(t) \to 0$ as $t \to 0$, $t^2(1 + x(t))^2 = t^2 + o(t^2)$. This implies that

$$\beta(t(1 + x(t)), \phi') = at^2 + o(t^2). \tag{14}$$

From (13) and (14), we obtain that $x(t) = at^2 + o(t^2)$. The result follows. □

Next we proceed with the second step of this proof.

2.5. The second step of the proof. As it follows from Proposition 3, it is sufficient to show that for any Bloch function $b(z)$ satisfying the condition $(*)$, the spectrum of integral means of $\phi'(z) = \exp b(z)$ expressed as $\beta(p, \phi') = ap^2 + o(p^2)$, where p small. This will be demonstrated in the following theorem.

Theorem 3. *If b belongs to $\mathcal{B}$ and satisfies the condition $(*)$, then the spectrum of the integral means of function $\phi'(z) = \exp b(z)$ has the following development at $p = 0$:*

$$\beta(p, \phi') = \frac{1}{4} \limsup_{r \to 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 \, d\theta}{2\pi \log(\frac{1}{1-r})} p^2 + \mathcal{O}(p^4).$$

Proof of Theorem 3. Let us give some remarks and the strategy for the proof of this theorem. First, we note that if $\gamma = \mathrm{Re}(b(0)) \neq 0$ then put $\hat{b}(z) = b(z) - b(0)$ and we have

$$\begin{aligned}
\beta(p, \phi') &= \limsup_{r \to 1} \frac{\log\left(\int_0^{2\pi} e^{p\gamma + p\mathrm{Re}(\hat{b}(re^{i\theta}))} \, d\theta\right)}{\log \frac{1}{1-r}} \\
&= \limsup_{r \to 1} \left\{ \frac{\log\left(\int_0^{2\pi} e^{p\mathrm{Re}(\hat{b}(re^{i\theta}))} \, d\theta\right)}{\log \frac{1}{1-r}} + \frac{p\gamma}{\log \frac{1}{1-r}} \right\} \\
&= \limsup_{r \to 1} \frac{\log\left(\int_0^{2\pi} e^{p\mathrm{Re}(\hat{b}(re^{i\theta}))} \, d\theta\right)}{\log \frac{1}{1-r}}.
\end{aligned}$$

This tells us that we do not loose generality if we assume that $b(0) = 0$. Moreover, we observe that for each $r \in (0, 1)$, there exists n such that $1/2^{n+1} \leq 1 - r \leq 1/2^n$ and from (11) $\big(|b_n(e^{i\theta}) - \mathrm{Re}(b(re^{i\theta}))| \leq C\|b\|_{\mathcal{B}}, \ (r = 1 - 2^{-n})\big)$, we deduce that

$$\beta(p, \phi') = \limsup_{r \to 1} \frac{\log\left(\int_0^{2\pi} e^{p\mathrm{Re}(b(re^{i\theta}))} \, d\theta\right)}{\log(\frac{1}{1-r})} = \limsup_{n \to \infty} \frac{\log\left(\int_0^{2\pi} e^{pb_n(e^{i\theta})} \, d\theta\right)}{n \log 2},$$

where b_n is the dyadic martinagle of $\mathrm{Re}(b)$ (the real part of the Bloch funtion b); of course, n depends on r and for simplicity's sake throughout this paper let us denote such $n(r)$ by n. Then, Theorem 3 will follow from the estimation of the integral $\int_0^{2\pi} e^{pb_n(e^{i\theta})} \, d\theta$. The main idea of this estimation is to make use of the exponential transformation of dyadic martingale b_n (the dyadic martingale of $\mathrm{Re}(b)$) which is defined as a sequence

$$\begin{cases} Z_0 = \exp pb_0; \\ Z_n = \frac{\exp pb_n}{\prod_{k=0}^{n-1} \cosh(p\Delta b_k)}, \quad n \geq 1. \end{cases}$$

Checking the condition $\mathbf{E}(Z_n|\mathcal{F}_{n-1}) = Z_{n-1}$, we see that $Z = (Z_n, \mathcal{F}_n)$ is a positive martingale. The integral $\int_0^{2\pi} e^{pb_n(e^{i\theta})} \, d\theta$ will be derived from the following equality which follows from the martingale's property that

$$\forall n \in \mathbf{N}, \quad \mathbf{E}(Z_n) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\exp pb_n(e^{i\theta})}{\prod_{k=0}^{n-1} \cosh(p\Delta b_k(e^{i\theta}))} \, d\theta = \mathbf{E}(Z_0) = 1.$$

In other words,

$$\frac{1}{2\pi} \int_0^{2\pi} e^{pb_n(e^{i\theta}) - \log(\prod_{k=0}^{n-1} \cosh(p\Delta b_k(e^{i\theta})))} \, d\theta = 1. \tag{15}$$

The rest part of the estimation of $\int_0^{2\pi} e^{pb_n(e^{i\theta})}\,d\theta$ is quite simple. We just apply the following inequalities and the condition $(*)$ to (15),

$$\left|\log\left(\prod_{k=0}^{n-1}\cosh(p\Delta b_k)\right)-\frac{p^2}{2}\langle S\rangle_n^2\right|\le\frac{p^4}{12}\sum_{k=0}^{n-1}(\Delta b_k)^4\le C'p^4\|b\|_{\mathcal{B}}^2\langle S\rangle_n^2, \tag{16}$$

where C' is an absolute constant and $\langle S\rangle_n^2$ is the square function of b_n. The first inequality of (16) follows from the estimation that

$$\left|\log(\cosh(x))-\frac{x^2}{2}\right|\le\frac{x^4}{12},\quad x\in\mathbf{R}.$$

Indeed, put $g(x)=\log(\cosh(x))-\frac{x^2}{2}-\frac{x^4}{12}$. We see that $g''(x)=-(\tanh x)^2-x^2\le 0$, $\forall x\in\mathbf{R}$. Hence, $g'(x)=\int_0^x g''(u)\,du\le 0$, $\forall x\in\mathbf{R}$. Therefore, $g(x)=\int_0^x g'(u)\,du\le 0$, $\forall x>0$ and as $g(x)$ is a even function, $g(x)\le 0$, $\forall x\in\mathbf{R}$. Similarly, put $h(x)=\log(\cosh(x))-\frac{x^2}{2}+\frac{x^4}{12}$. We observe that $h''(x)=-(\tanh x)^2+x^2\ge 0$, $\forall x\in\mathbf{R}$ because $|\tanh x|\le|x|$, $\forall x\in\mathbf{R}$. Analogously, we obtain that $\forall x\in\mathbf{R}$, $h(x)\ge 0$.

Besides, (11) ($\forall\xi\in\mathbf{T}$, $|\Delta b_n(\xi)|\le C\|b\|_{\mathcal{B}}$) implies that

$$\sum_{k=0}^{n-1}(\Delta b_k)^4=\sum_{k=0}^{n-1}(\Delta b_k)^2(\Delta b_k)^2\le C^2\|b\|_{\mathcal{B}}^2\sum_{k=0}^{n-1}(\Delta b_k)^2=C^2\|b\|_{\mathcal{B}}^2\langle S\rangle_n^2.$$

Then the second one of (16) follows.

Finally, we shall apply the following lemma to conclude that for p small

$$\beta(p,\phi')=\limsup_{n\to\infty}\frac{\log\left(\int_0^{2\pi}e^{pb_n(e^{i\theta})}\,d\theta\right)}{n\log 2}=\frac{p^2}{4}\limsup_{r\to 1}\frac{\int_0^2|b(re^{i\theta})|^2\,d\theta}{2\pi\log\frac{1}{1-r}}+\mathcal{O}(p^4).\quad\square$$

Lemma 1. *Let b be a Bloch function and $\langle S\rangle_n^2$ be the square function of the dyadic martingale b_n of* $\mathrm{Re}(b)$. *Then*

$$\limsup_{n\to\infty}\frac{\int_0^{2\pi}\langle S\rangle_n^2(\theta)\,d\theta}{n\log 2}=\limsup_{r\to 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2\,d\theta}{2\log\frac{1}{1-r}}\le\pi\|b\|_{\mathcal{B}}^2.$$

Proof. Denote $\tilde{b}=\mathrm{Re}(b)$ and recall b_n be a dyadic martingale of $\tilde{b}$. We have:

$$\|b_n\|_2^2=\int_0^{2\pi}b_n^2(\theta)\,d\theta=\int_0^{2\pi}\sum_{k=0}^{n-1}(\Delta b_k(\theta))^2\,d\theta=\int_0^{2\pi}\langle S\rangle_n^2(\theta)\,d\theta.$$

The second equality follows from Proposition 5.4.5 [Gra08], and the third one follows from the definition of the square function of the dyadic martingale b_n. Moreover, the fact that $|b_n(\theta)-\tilde{b}(re^{i\theta})|\le C\|b\|_{\mathcal{B}}$ if $r=1-2^{-n}$ (see (11)) implies that:

$$\left|\|b_n\|_2-\|\tilde{b}(re^{i\theta})\|_2\right|\le\|b_n(e^{i\theta})-\tilde{b}(re^{i\theta})\|_2\le\sqrt{2\pi}(C\|b\|_{\mathcal{B}}).$$

Therefore, if we divide both sides by $(n\log 2)^{1/2}$ of the above inequalities, and take the limit as $n\to\infty$, then we obtain:

$$\lim_{n\to\infty}\left\{\left(\frac{\int_0^{2\pi}(b_n)^2\,d\theta}{n\log 2}\right)^{1/2}-\left(\frac{\int_0^{2\pi}(\tilde{b}((1-2^{-n})e^{i\theta}))^2\,d\theta}{n\log 2}\right)^{1/2}\right\}=0. \tag{17}$$

By Proposition 1,

$$\frac{\int_0^{2\pi}(\tilde{b}((1-2^{-n})e^{i\theta}))^2\,d\theta}{n\log 2}$$

is bounded, then by (17)

$$\frac{\int_0^{2\pi}(b_n)^2\,d\theta}{n\log 2}$$

is also bounded. Furthermore, since the function x^2 is continuous uniformly on some compact set of $[0,+\infty)$, (17) implies that

$$\lim_{n\to\infty}\frac{\int_0^{2\pi}(b_n)^2\,d\theta}{n\log 2}-\frac{\int_0^{2\pi}(\tilde{b}((1-2^{-n})e^{i\theta}))^2\,d\theta}{n\log 2}=0.$$

Thus,

$$\limsup_{n\to\infty}\frac{\int_0^{2\pi}(b_n)^2\,d\theta}{n\log 2}=\limsup_{n\to\infty}\frac{\int_0^{2\pi}(\tilde{b}((1-2^{-n})e^{i\theta}))^2\,d\theta}{n\log 2}.$$

Then,

$$\limsup_{n\to\infty}\frac{\int_0^{2\pi}\langle S\rangle_n^2(\theta)\,d\theta}{n\log 2}=\limsup_{r\to 1}\frac{\int_0^{2\pi}(\tilde{b}(re^{i\theta}))^2\,d\theta}{\log(\frac{1}{1-r})}\quad (r=1-2^{-n}).$$

In addition, as b is holomorphic in the unit disk $\mathbf{D}$ and by Proposition 1, we have:

$$\limsup_{r\to 1}\frac{\int_0^{2\pi}(\mathrm{Re}(b(re^{i\theta})))^2\,d\theta}{\log(\frac{1}{1-r})}=\limsup_{r\to 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2d\theta}{2\log(\frac{1}{1-r})}\le\pi\|b\|_{\mathcal{B}}^2.$$

The lemma is proven. □

The proof of Theorem 3 remains the main step: that is to estimate the integral $\int_0^{2\pi}e^{pb_n(e^{i\theta})}\,d\theta$.

The main step of the proof. Put

$$\epsilon_n(\theta)=\begin{cases}\frac{\log(\prod_{k=0}^{n-1}\cosh(p\Delta b_k(e^{i\theta})))-\frac{p^2}{2}\langle S\rangle_n^2(\theta)}{\frac{p^2}{2}\langle S\rangle_n^2(\theta)}, & \text{if } \langle S\rangle_n^2(\theta)\neq 0,\\ 0, & \text{otherwise,}\end{cases}$$

$\theta\in[0,2\pi]$. This says that

$$\log\left(\prod_{k=0}^{n-1}\cosh(p\Delta b_k(e^{i\theta}))\right)=\frac{p^2}{2}\langle S\rangle_n^2(\theta)\,(1+\epsilon_n(\theta))\,, \tag{18}$$

where $|\epsilon_n(\theta)|\le C'p^2\|b\|_{\mathcal{B}}^2$ by (16).

Put $I_n=\frac{1}{2\pi}\int_0^{2\pi}\langle S\rangle_n^2(\theta)\,d\theta$. By (18), (15) is equivalent to

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{pb_n(e^{i\theta})-\frac{p^2}{2}\langle S\rangle_n^2(\theta)(1+\epsilon_n(\theta))\right\}d\theta=1.$$

By subtraction and adding the term $\frac{p^2}{2}I_n(1+\epsilon_n(\theta))$, we can rewrite the preceding equality as follows

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{pb_n(e^{i\theta})-\frac{p^2}{2}I_n(1+\epsilon_n(\theta))-\frac{p^2}{2}(\langle S\rangle_n^2(\theta)-I_n)(1+\epsilon_n(\theta))\right\}d\theta=1.$$

Remark that I_n is a number, so we can take the term $\exp\{\frac{p^2}{2}I_n\}$ out of the above integral, then the equality turns out to be

$$\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{pb_n(e^{i\theta})-\frac{\epsilon_n(\theta)p^2}{2}I_n-\frac{p^2}{2}(\langle S\rangle_n^2(\theta)-I_n)(1+\epsilon_n(\theta))\right\}d\theta=\exp\left(\frac{p^2}{2}I_n\right).$$

Put

$$I=\frac{1}{2\pi}\int_0^{2\pi}\exp\left\{pb_n(e^{i\theta})-\frac{\epsilon_n(\theta)p^2}{2}I_n-\frac{p^2}{2}(\langle S\rangle_n^2(\theta)-I_n)(1+\epsilon_n(\theta))\right\}d\theta.$$

Next, we shall estimate the integral I. Combining the condition $(*)$ $|\langle S\rangle_n^2(\theta)-I_n|\le n\delta(n)$ with the fact that $|\epsilon_n|\le C'p^2\|b\|_{\mathcal{B}}^2$, we have: $|(1+\epsilon_n(\theta))(\langle S\rangle_n^2(\theta)-I_n)|\le(1+C'p^2\|b\|_{\mathcal{B}}^2)n\delta(n)$. Then, this implies that:

$$\exp\left\{-C'p^4\|b\|_{\mathcal{B}}^2I_n-\frac{n\delta(n)}{2}p^2(1+C'p^2\|b\|_{\mathcal{B}}^2)\right\}\frac{1}{2\pi}\int_0^{2\pi}e^{pb_n(e^{i\theta})}\,d\theta\le I$$

and

$$I\le\exp\left\{C'p^4\|b\|_{\mathcal{B}}^2I_n+\frac{n\delta(n)}{2}p^2(1+C'p^2\|b\|_{\mathcal{B}}^2)\right\}\frac{1}{2\pi}\int_0^{2\pi}e^{pb_n(e^{i\theta})}\,d\theta.$$

Replacing I by $\exp(\frac{p^2}{2}I_n)$ and then taking logarithm of sides of the above inequalities, we deduce that

$$\log\left(\int_0^{2\pi}e^{pb_n(e^{i\theta})}\,d\theta\right)-\frac{n\delta(n)}{2}p^2(1+C'p^2\|b\|_{\mathcal{B}})-C'p^4\|b\|_{\mathcal{B}}^2I_n-\log(2\pi)\le\frac{p^2}{2}I_n$$

and

$$\frac{p^2}{2}I_n\le\log\left(\int_0^{2\pi}e^{pb_n(e^{i\theta})}\,d\theta\right)+\frac{n\delta(n)}{2}p^2(1+C'p^2\|b\|_{\mathcal{B}})+C'p^4\|b\|_{\mathcal{B}}^2I_n-\log(2\pi).$$

Next, if we divide both sides of the inequalities by $n\log 2$, we then obtain the inequalities

$$\begin{aligned}\frac{p^2}{2}\frac{I_n}{n\log 2}\ge&\frac{\log\left(\int_0^{2\pi}e^{pb_n(e^{i\theta})}\,d\theta\right)}{n\log 2}\\&-\left(p^2(1+C'p^2\|b\|_{\mathcal{B}}^2)\frac{n\delta(n)}{2n\log 2}+C'p^4\|b\|_{\mathcal{B}}^2\frac{I_n}{n\log 2}+\frac{\log(2\pi)}{n\log 2}\right)\end{aligned}$$

and

$$\begin{aligned}\frac{p^2}{2}\frac{I_n}{n\log 2}\le&\frac{\log\left(\int_0^{2\pi}e^{pb_n(e^{i\theta})}\,d\theta\right)}{n\log 2}\\&+\left(p^2(1+C'p^2\|b\|_{\mathcal{B}}^2)\frac{n\delta(n)}{2n\log 2}+C'p^4\|b\|_{\mathcal{B}}^2\frac{I_n}{n\log 2}-\frac{\log(2\pi)}{n\log 2}\right).\end{aligned}$$

Taking the limes superior as n tends to ∞ of these inequalities, then we get

$$\begin{aligned}\left(\frac{p^2}{2}-C'p^4\|b\|_{\mathcal{B}}^2\right)\limsup_{n\to\infty}\frac{I_n}{n\log 2}&\le\limsup_{n\to\infty}\frac{\log\left(\int_0^{2\pi}e^{pb_n(e^{i\theta})}\,d\theta\right)}{n\log 2}\\&\le\left(\frac{p^2}{2}+C'p^4\|b\|_{\mathcal{B}}^2\right)\limsup_{n\to\infty}\frac{I_n}{n\log 2}.\end{aligned}$$

Finally, we obtain the estimation

$$\left|\limsup_{n\to\infty}\frac{\log\left(\int_0^{2\pi}e^{pb_n(e^{i\theta})}\,d\theta\right)}{n\log 2}-\frac{p^2}{2}\limsup_{n\to\infty}\frac{I_n}{n\log 2}\right|\le C'p^4\|b\|_{\mathcal{B}}^2\limsup_{n\to\infty}\frac{I_n}{n\log 2},\tag{19}$$

where

$$\limsup_{n\to\infty}\frac{I_n}{n\log 2}=\limsup_{n\to\infty}\frac{\int_0^{2\pi}\langle S\rangle_n^2(\theta)\,d\theta}{2\pi n\log 2}=\limsup_{r\to 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2\,d\theta}{4\pi\log(\frac{1}{1-r})}\le\frac{\|b\|_{\mathcal{B}}^2}{2}<+\infty$$

by Lemma 1. Thus, the estimation (19) gives us the desired formula for the spectrum of integral means

$$\beta(p,\phi')=\limsup_{r\to 1}\frac{\log\left(\int_0^{2\pi}e^{pb(e^{i\theta})}\,d\theta\right)}{\log(\frac{1}{1-r})}=\frac{p^2}{4}\limsup_{r\to 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2\,d\theta}{2\pi\log(\frac{1}{1-r})}+\mathcal{O}(p^4),$$

as p tends to zero. This finishes the proof of Theorem 3. □

From Theorem 3 and Proposition 3, we conclude Theorem 2. For the sake of completeness of this section, we give a non-trivial example for Bloch function which satisfies condition $(*)$

2.6. An example with constant square function. First, we define the independent Bernoullian random variables ε_n on $\partial\mathbf{D}$ by the formula

$$\varepsilon_n(e^{2\pi ix})=\begin{cases}-1, & x_n=0 \text{ or } 3,\\ 1, & x_n=1 \text{ or } 2,\end{cases}\qquad (n=1,2,\ldots)$$

where x_n denotes the 4-adic nth digit of $x\in[0,1]$.

Proposition 4. *For any bounded sequence of a real numbers $\{a_n\}$, the 4-adic martingale $S_n=\sum_{k=1}^n a_k\varepsilon_k$ is a dyadic martingale (if considered as dyadic).*

Proof. See [Mak90]. □

Let $\{a_k\}$ be a bounded sequence of real numbers. Then $\limsup_{n\to\infty}\frac{\sum_{k=1}^n a_k^2}{n}=\alpha<+\infty$. By Proposition 4, there exists a Bloch function b whose real part $\mathrm{Re}(b)$ generates the dyadic martingale S_n.

Let $\phi_t(z)=\int_0^z e^{tb(u)}\,du$: these are conformal mappings for t close to 0. Denote $\Omega_t=\phi_t(\mathbf{D})$. Then the Minkowski dimension of $\Gamma_t=\partial\Omega_t$ has the following development at 0:

$$\mathrm{M.dim}(\Gamma_t)=1+\limsup_{r\to 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2\,d\theta}{4\pi\log\frac{1}{1-r}}\frac{t^2}{2}+o(t^2)=1+\frac{\alpha}{2\log 2}t^2+o(t^2).\tag{20}$$

Indeed, as $\Delta S_k=a_k\varepsilon_k$, $\langle S\rangle_n^2=\sum_{k=1}^n a_k^2$ is a constant square function. Thus, certainly the square function $\langle S\rangle_n^2$ satisfies the condition $(*)$. Besides, we have

$$\limsup_{r\to 1}\frac{\int_0^{2\pi}|b(re^{i\theta})|^2\,d\theta}{2\pi\log(\frac{1}{1-r})}=2\limsup_{n\to\infty}\frac{\int_0^{2\pi}\langle S\rangle_n^2(\theta)\,d\theta}{2\pi n\log 2}=2\limsup_{n\to\infty}\frac{\langle S\rangle_n^2}{n\log 2}=\frac{2\alpha}{\log 2},$$

$r=1-2^n$. Then, (20) follows from Theorem 2.

3. Counter-example

In this section we show that McMullen's property does not hold for all $b \in \mathcal{B}$. The counter-example we construct is reminiscent of Kahane's construction of a non-Smirnov domain.

3.1. Kahane measure and its Herglotz transform.

3.1.1. Kahane measure. First of all, let us recall the construction of Kahane measure. Denote by ω_0 the interval $[0,1]$ and by ω_j one of intervals of form 4-adic $[p4^{-j},(p+1)4^{-j}]$ contained in ω_0. We construct simultaneously a sequence of measure μ_j and their supports E_j as follows:

μ_0 is the Lebesgue measure on interval ω_0;
μ_j is proportional to the Lebesgue measure on each ω_j.

We denote by $D_j(\omega_j)$ its density on a given interval ω_j and its support E_j is the union of intervals ω_j where $D_j(\omega_j) \neq 0$. In order to obtain μ_{j+1} from μ_j, we divide each interval $\omega = \omega_j$ of rank j contained in E_j into four equal subintervals $\omega^1, \omega^2, \omega^3, \omega^4$ of rank $j+1$ and put

$$D_{j+1}(\omega^1) = D_{j+1}(\omega^4) = D_j(\omega) - 1,$$
$$D_{j+1}(\omega^2) = D_{j+1}(\omega^3) = D_j(\omega) + 1.$$

Put $\mu = \lim_{j\to\infty} \mu_j$ and $E = \bigcap_{j=0}^{\infty} E_j$. This measure μ is referred to as Kahane measure.

The set E can be rebuilt in another way as follows. Recall the independent Bernoullian random variables ε_k on $\partial\mathbf{D}$ (defined in 2.6): put $\Sigma_j(e^{2\pi ix}) = \sum_{k=1}^{j} \varepsilon_k(e^{2\pi ix})$ and let N be the first integer number j such that $1 + \sum_{k=1}^{j} \varepsilon_k(e^{2\pi ix}) = 0$ ($x \in [0,1]$). By the definition of D_k, we have:

$$\forall x \in [0,1], \;\; D_0(x) = 1; \;\; D_k(x) = (D_{k-1}(x) + \varepsilon_k(e^{2\pi ix}))1_{E_{k-1}}(x).$$

Therefore,

$$\begin{aligned} D_k(x) = \Big(\Big(\Big(\big(1 + \varepsilon_1(e^{2\pi ix})\big)\, 1_{E_0}(x) + \varepsilon_2(e^{2\pi ix})\Big)\, 1_{E_1}(x) + \ldots + \Big)\, 1_{E_{k-2}}(x) \\ + \varepsilon_k(e^{2\pi ix})\Big)1_{E_{k-1}}(x), \quad x \in [0,1]. \end{aligned}$$

Since $E_0 \supset E_1 \supset \ldots \supset E_{k-1}$, $1_{E_0} \ldots 1_{E_{k-1}} = 1_{E_{k-1}}$, we have

$$D_k(x) = (1 + \Sigma_k(e^{2\pi ix}))1_{E_{k-1}}(x).$$

This implies that the support of D_k: $E_k = E_{k-1} \cap \{1 + \Sigma_k > 0\}$. Then,

$$E_k = \{1 + \Sigma_1 > 0, \ldots, 1 + \Sigma_k > 0\}, \quad k = 1, 2, \ldots.$$

Moreover, for $x \in [0,1]$

$$\begin{aligned} D_k(x) &= (1 + \Sigma_k(e^{2\pi ix}))1_{E_{k-1}}(x) \\ &= (1 + \Sigma_k(e^{2\pi ix}))1_{E_k}(x) + (1 + \Sigma_k(e^{2\pi ix}))1_{E_{k-1}\setminus E_k}(x) \\ &= (1 + \Sigma_k(e^{2\pi ix}))1_{E_k}(x), \end{aligned}$$

because on the set $E_{k-1} \setminus E_k$ we have $1 + \Sigma_k(x) = 0$.

In his paper [Kah69], Kahane showed that the set $E = \bigcap_{k=0}^{\infty} E_k$ (support of the measure μ) has a null Lebesgue measure. Therefore this measure is totally singular.

3.1.2. Herglotz transform of Kahane measure. Let $b(z)$ be Herglotz transform of Kahane measure μ: that is

$$b(z) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta).$$

Kahane proved that $b \in \mathcal{B}$. Put $\Lambda_j(e^{2\pi ix}) = 1 + \Sigma_j(e^{2\pi ix})$ and

$$S_j(e^{2\pi ix}) = \Lambda_{j\wedge N}(e^{2\pi ix}) = \begin{cases} 1 + \Sigma_j(e^{2\pi ix}), & \text{if } x \in \{N > j\} = E_j \\ 0, & \text{otherwise,} \end{cases}$$

$x \in [0,1]$). Similarly to the example of the square constant function in 2.6 above, Λ_j is a dyadic martingale (if consider as dyadic). By the construction of Kahane measure μ, $\{N = j\} = \bigcup \omega_j = E_{j-1} \setminus E_j \in \mathcal{F}_j$, where ω_j is an interval 4-adic of rank $j-1$ i.e dyadic of rank j. Therefore N is a stopping time with respect to the σ-algebra $\{\mathcal{F}_j, j \geq 0\}$ (defined above). Thus, $S_j = \Lambda_{N\wedge j}$ is a dyadic martingale as well. Moreover, we have the following result.

Lemma 2. *S_j is the dyadic martingale of the Bloch function* $\mathrm{Re}(b)$.

Proof. Indeed, we recall $h(\theta)$ the cumulative distribution function of the Kahane measure μ, i.e., $h(\varphi) = \mu(\{\frac{\varphi}{2\pi} > 0\}$, $\varphi \in [0, 2\pi]$ and $h(0) = 0$. We observe that for $z \in \mathbf{D}$

$$b(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} h'(\varphi)\, d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \left(1 + 2\sum_{n=1}^{\infty} e^{-in\varphi} z^n\right) h'(\varphi)\, d\varphi. \tag{21}$$

By the Schwartz integral formula and $\mathrm{Im}\, b(0) = 0$, we have

$$\begin{aligned} b(z) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, \mathrm{Re}\, b(e^{i\varphi})\, d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(1 + 2\sum_{n=1}^{\infty} e^{-in\varphi} z^n\right) \mathrm{Re}\, b(e^{i\varphi})\, d\varphi. \end{aligned} \tag{22}$$

From (21),(22) we obtain $\int_0^{2\pi} e^{-in\varphi}(\mathrm{Re}\, b(e^{i\varphi}) - h'(\varphi))\, d\varphi = 0$, $n = 0, 1, 2, \ldots$. Since the sequence $\{e^{in\theta}\}$ $(n = 0, 1, 2, \ldots)$ constitutes a basis for $L^2([0, 2\pi])$,

$$\mathrm{Re}\, b(e^{i\varphi}) - h'(\varphi) = 0 \quad \text{in } L^2([0, 2\pi]).$$

Thus, $\mathrm{Re}\, b(e^{i\varphi}) = h'(\varphi)$ a.e. in $[0, 2\pi]$. We observe that for each subarc 4-adic $\omega_j = [\frac{\varphi_0}{2\pi}, \frac{\varphi_0}{2\pi} + \frac{\varphi}{2\pi}]$ of rank j of the interval $[0, 1]$,

$$\begin{aligned} \frac{\mu_j(\omega_j)}{|\omega_j|} &= \frac{1}{|\omega_j|}\int_{\omega_j} D_j(x)\, dx = \frac{1}{|\omega_j|}\int_{\omega_j} (1 + \Sigma_j(e^{2\pi ix}))1_{E_j}\, dx \\ &= \Lambda_{j\wedge N}(e^{2\pi ix})1_{\omega_j}(x) = S_j(e^{2\pi ix})|_{\omega_j}, \end{aligned}$$

while

$$\frac{\mu_j(\omega_j)}{|\omega_j|} = \frac{h(\varphi + \varphi_0) - h(\varphi_0)}{|\varphi|}.$$

Therefore,

$$S_j(e^{2\pi ix})|_{\omega_j} = \frac{h(\varphi + \varphi_0) - h(\varphi_0)}{|\varphi|} = \frac{1}{|\omega_j|}\int_{\omega_j} \mathrm{Re}(b(e^{2\pi ix}))\, dx = (\mathrm{Re}(b))_{\omega_j}.$$

It means that S_j is the dyadic martingale of $\mathrm{Re}(b)$. □

Now, let us state concretely the second result of this paper.

3.2. Statement of Theorem 4. Let μ be Kahane measure and $b(z)$ its Herglotz transform. We recall that Γ_t is the image of the unit circle $\mathbf{T}$ by the conformal map $\phi_t(z)$ which is defined as $\phi_t'(z) = e^{tb(z)}$, t small enough.

If a family of conformal maps $\phi_t(z) = \int_0^z e^{tb(u)}\,du$, $z \in \mathbf{D}$; $b \in \mathcal{B}$, satisfies (8) with Hausdorff dimension replaced by Minkowski dimension, then

$$\text{M.dim}(\Gamma_t) = 1 + \limsup_{r\to 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2\,d\theta}{4\pi \log\frac{1}{1-r}} \frac{t^2}{2} + o(t^2). \tag{23}$$

Theorem 4. *The behaviour of the curve Γ_t differs with the sign of t: In the case of negative t, the singular property of the Kahane measure μ (the density function of the probability measure μ is non negative and zero almost everywhere) makes $\phi_t' \in H^1$. This is equivalent to the rectifiability of Γ_t and then H.dim(Γ_t) = M.dim$(\Gamma_t) \equiv 1$. On the other hand, in the case of positive t, Γ_t is a fractal curve and its Minkowski dimension satisfies the following inequality:*

$$d(t) \geq 1 + \frac{t^2}{8\log 2}, \quad \forall t > 0 \text{ small enough},$$

as a consequence the family of conformal map $(\phi_t), t > 0$ gives a counter-example to (23).

Next, we shall give the proof of this theorem.

3.3. Proof of Theorem 4. First of all, we shall use the singularity of Kahane measure to prove that in the case of small negative t, H.dim(Γ_t) = M.dim$(\Gamma_t) \equiv 1$.

3.3.1. Negative *t*. We recall now two theorems on $H^p (p > 0)$ functions and then we shall show how they imply the first part of Theorem 4. Let us introduce some notions. Given a function $f(z) \not\equiv 0$ of class $H^p(p > 0)$. Let (a_n) (may be finite, or even empty) be the sequence zeroes of the function f. A function of the form

$$B(z) = z^m \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

is called a Blaschke product. A singular inner function is a function of the form

$$S(z) = \exp\left\{ -\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta) \right\},$$

where $\mu(\theta)$ be a bounded non-decreasing singular function ($\mu'(\theta) = 0$ a.e). And an outer function of class H^p is a function of form

$$F(z) = e^{i\gamma} \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|f(e^{i\theta})|\, d\theta \right\},$$

where γ is a real number, $|f(e^{i\theta})| \in L^p([0, 2\pi])$.

Theorem 5. (Canonical factorization theorem) *Every function $f(z) \not\equiv 0$ of class $H^p(p > 0)$ has a unique factorization of the form $f(z) = B(z)S(z)F(z)$, where $B(z)$ is a Blaschke product, $S(z)$ is a singular inner function and $F(z)$ is an outer function of class H^p. Conversely, every such product $B(z)S(z)F(z)$ belongs to H^p.*

Proof. See [Dur70], p. 193. □

Theorem 6. *Let $f(z)$ map the unit disc* $\mathbf{D}$ *conformally onto a Jordan domain* Ω. *Then the boundary* $\partial\Omega$ *is rectifiable if and only if* $f' \in H^1$.

Proof. See [Dur70], p. 42. □

Since t small enough and $b(z)$ is a Bloch function, by Becker univalence criterion ([Pom73], p. 172), the maps $\phi_t(z)$ map conformally the unit disk $\mathbf{D}$ onto a quasidisk Ω_t. And its derivative has the form

$$\phi_t'(z) = \exp\left\{ t \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta) \right\},$$

where $t < 0$ and μ is a positive singular measure, i.e., the density function $h'(\theta)$ of Kahane measure μ is non negative and zero almost everywhere on $[0, 2\pi]$ (mentioned above). Then, Theorem 5 yields $\phi_t' \in H^1$.

Since the fact that $\phi_t' \in H^1$ is equivalent to the rectifiability of the boundary Γ_t by Theorem 6, obviously

$$\text{H.dim}(\Gamma_t) = \text{M.dim}(\Gamma_t) \equiv 1. \tag{24}$$

The first part of Theorem 4 follows.

Now, we proceed to the main part of the proof of Theorem 4: the case of small positive t.

3.3.2. Positive *t*. In this part, we need to prove that for small positive t, $d(t) \geq 1 + \frac{t^2}{8\log 2}$. Analogously to the proof of Theorem 2, again by using Proposition 2, $d(t)$ can be deduced from the spectrum of integral means $\beta(p, \phi')$, where $\phi' = \exp b(z)$ and p small. Therefore, if one can show that $\beta(p, \phi')$ satisfies the following inequality

$$\beta(p, \phi') \geq \frac{p^2}{8\log 2}, \quad p > 0 \text{ small}, \tag{25}$$

then the inequality $d(t) \geq 1 + \frac{t^2}{8\log 2}$ easily follows. It follows from the fact: $|S_j(e^{i\theta}) - \text{Re}(b(re^{i\theta}))| \leq C\|b\|_{\mathcal{B}}$, $r = 1 - 2^{-j}$ (see (11)) that

$$\beta(p, \phi') = \limsup_{r\to 1} \frac{\int_0^{2\pi} e^{p\,\text{Re}\, b(re^{i\theta})}\, d\theta}{\log\frac{1}{1-r}} = \limsup_{j\to\infty} \frac{\int_0^{2\pi} e^{pS_j(e^{i\theta})}\, d\theta}{j\log 2}.$$

The above argument asks us to estimate the integral $\int_0^{2\pi} e^{pS_j(e^{i\theta})}\, d\theta$, $S_j = \Lambda_{j\wedge N}$, $p > 0$ small. The main issue in this estimation is that S_j is not a sum of independent random variables. However, we can go around this difficulty by using the stopping time of the random walk argument of the dyadic martingale S_j which will be introduced in the following.

3.3.3. Random walk argument. Let us describe this random walk on graph. On the lattice $\mathbf{Z}^+ \times \mathbf{Z}$, we consider that a particle moves in the direction parallel to two diagonals of the unit square. We denote the individual steps generically by $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ with the probability $p = \frac{1}{2}$ (defined above in 2.6) and the position of the particle by $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$. According to the assumption of the dyadic martingale S_j, the particle will stop as it reaches to the horizontal axis $y = 0$ on the lattice.

Denote the event {at epoch n the particle is at the position r} by $\{\Sigma_n = r\}$ and one can write the event $\{N > k\}$ by $\{1 + \Sigma_1 > 0, \ldots, 1 + \Sigma_k > 0\}$ and then by $\{\Sigma_1 \geq 0, \ldots, \Sigma_k \geq 0\}$. The following lemma is a basic result needed to prove the inequality (25).

Lemma 3. *For a random walk $\Sigma_n = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n$, where ε_k are Bernoulli independent random variables with the probability $p = \frac{1}{2}$, we have:*

$$P(\Sigma_1 \geq 0, \Sigma_2 \geq 0, \ldots, \Sigma_{2n} \geq 0) = P(\Sigma_{2n} = 0) = \frac{C_{2n}^n}{2^{2n}},$$

where $C_{2n}^n = \frac{(2n)!}{n!n!}$. Moreover, by Stirling's formula $P(N > 2n) \simeq \frac{1}{\sqrt{2n}}$.

Proof. See [Fel50], p. 76. □

In the proof of the inequality (25), we shall also use the remark that for each positive integer k, $\{N > 2k+1\} = \{N > 2k\}$. Indeed, $\{\Sigma_1 \geq 0, \ldots, \Sigma_{2k} \geq 0\} = \{\Sigma_1 \geq 0, \ldots, \Sigma_{2k} \geq 0, \Sigma_{2k+1} \geq 0\} \cap \{\Sigma_1 \geq 0, \ldots, \Sigma_{2k} \geq 0, \Sigma_{2k+1} < 0\}$. By the assumption of the stopping time, the particle will stop as it reaches to the axis $y = 0$, hence $\{\Sigma_1 \geq 0, \ldots, \Sigma_{2k} \geq 0, \Sigma_{2k+1} < 0\} = \emptyset$. Thus, $\{\Sigma_1 \geq 0, \ldots, \Sigma_{2k} \geq 0\} = \{\Sigma_1 \geq 0, \ldots, \Sigma_{2k} \geq 0, \Sigma_{2k+1} \geq 0\}$.

3.3.4. The main step of the proof. We shall estimate the integral $\int_{\mathbf{T}} e^{pS_j(e^{i\theta})}\, d\theta$. First we note that on the set $\{N \leq j\}$ $S_j(e^{i\theta}) = \Lambda_{j\wedge N}(e^{i\theta}) = 0$, then

$$\begin{aligned} \frac{1}{2\pi}\int_{\mathbf{T}} e^{pS_j(e^{i\theta})}\, d\theta &= \frac{1}{2\pi}\int_{\{N>j\}} e^{pS_j(e^{i\theta})}\, d\theta + \frac{1}{2\pi}\int_{\{N\leq j\}} e^{pS_j(e^{i\theta})}\, d\theta \\ &= \frac{1}{2\pi}\int_{\{N>j\}} e^{pS_j(e^{i\theta})}\, d\theta + P(\{N \leq j\}), \end{aligned} \tag{26}$$

where

$$\frac{1}{2\pi}\int_{\{N>j\}} e^{pS_j(e^{i\theta})}\, d\theta = \frac{1}{2\pi}\int_{\{N>j\}} e^{p(1+\Sigma_j(e^{i\theta}))}\, d\theta.$$

We observe that

$$\frac{1}{2\pi}\int_{\{N>j\}} e^{p(1+\Sigma_j(e^{i\theta}))}\, d\theta = \frac{1}{2\pi}\int_{\mathbf{T}} e^{p(1+\Sigma_j(e^{i\theta}))}\, d\theta - \frac{1}{2\pi}\int_{\{N\leq j\}} e^{p(1+\Sigma_j(e^{i\theta}))}\, d\theta.$$

As $\Sigma_j = \sum_{k=1}^{j} \varepsilon_k$, where ε_k with $k = 1, 2, \ldots$ are the independent random variables, we have

$$\frac{1}{2\pi}\int_{\mathbf{T}} e^{p(1+\Sigma_j(e^{i\theta}))}\, d\theta = e^p \prod_{k=1}^{j} \mathbf{E}(e^{p\varepsilon_k}) = e^p \prod_{k=1}^{j} \cosh p = e^p (\cosh p)^j.$$

Besides, the integral $\int_{\{N\leq j\}} e^{p(1+\Sigma_j(e^{i\theta}))}\, d\theta$ can be rewritten as:

$$\int_{\{N\leq j\}} e^{p(1+\Sigma_j(e^{i\theta}))}\, d\theta = \sum_{k=1}^{j} \int_{\{N=k\}} e^{p(1+\Sigma_j(e^{i\theta}))}\, d\theta.$$

The fact that $1 + \Sigma_k(e^{i\theta})$ is equal to zero on each set $\{N = k\}$ makes the value of the integral $\sum_{k=1}^{j} \int_{\{N=k\}} e^{p(1+\Sigma_j(e^{i\theta}))}\, d\theta$ unchanged if we divide the integrand $e^{p(1+\Sigma_j(e^{i\theta}))}$ by the term $e^{1+\Sigma_k(e^{i\theta})}$. Thus we have:

$$\begin{aligned} \sum_{k=1}^{j} \int_{\{N=k\}} e^{p(1+\Sigma_j(e^{i\theta}))}\, d\theta &= \sum_{k=1}^{j} \int_{\{N=k\}} e^{p(1+\Sigma_j(e^{i\theta})-1-\Sigma_k(e^{i\theta}))}\, d\theta \\ &= \sum_{k=1}^{j} \int_{\{N=k\}} e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))}\, d\theta. \end{aligned}$$

In addition, if we rewrite the integral

$$\int_{\{N=k\}} e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))}\, d\theta \quad \text{as} \quad \int_{\mathbf{T}} 1_{\{N=k\}} e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))}\, d\theta,$$

then by the independence of two random variables $1_{\{N=k\}}$ and $e^{p(\Sigma_j-\Sigma_k)}$ it follows that

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbf{T}} 1_{\{N=k\}} e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))}\, d\theta &= P(\{N=k\})\mathbf{E}(e^{p(\Sigma_j(e^{i\theta})-\Sigma_k(e^{i\theta}))})\\ &= P(\{N=k\})(\cosh p)^{j-k}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\frac{1}{2\pi}\int_{\{N>j\}} e^{pS_j(e^{i\theta})}\, d\theta &= e^p\cosh(p)^j\left(1-\sum_{k=1}^{j}\frac{P(\{N=k\})}{(\cosh p)^k e^p}\right)\\ &\geq e^p(\cosh p)^j\left(1-\sum_{k=1}^{j}P(\{N=k\})\right) \quad (p>0)\\ &= e^p(\cosh p)^j P(\{N>j\}) \quad (p>0). \qquad (27)\end{aligned}$$

The inequality above follows from the fact that for $p>0$ $(\cosh p)^k e^p \geq 1$, $k=1,2,\ldots,j$. From (26), (27) and Jensen's inequality, we deduce

$$\begin{aligned}\log\left(\int_{\mathbf{T}} e^{pS_j(e^{i\theta})}\, d\theta\right) &\geq \frac{1}{2}\log\left(2\int_{\{N>j\}} e^{pS_j(e^{i\theta})}\, d\theta\right)+\frac{1}{2}\log\left(4\pi P(\{N\leq j\})\right)\\ &\geq \frac{p}{2}+\log(4\pi)+\frac{1}{2}\log(\cosh(p)^j)+\frac{1}{2}\log(P(\{N>j\}))\\ &\quad+\frac{1}{2}\log(P(\{N\leq j\}))\end{aligned}$$

By Lemma 3: $\log(P(N>j))\simeq -\frac{\log j}{2}$ and $\log(P(N\leq j))\simeq -\frac{1}{\sqrt{j}}$ as $j\to\infty$, thus when we divide the above inequality by $j\log 2$ and take the limes superior as $j\to\infty$, we deduce that

$$\beta(p,\phi')\geq \limsup_{j\to\infty}\frac{\log(\cosh(p)^j)}{2j\log 2}=\frac{\log\cosh(p)}{2\log 2}, \quad p>0.$$

Moreover, the inequality $\log\cosh(x)\geq \frac{x^2}{2}-\frac{x^4}{12}$, $x>0$, (proved in 2.5) implies that $\log\cosh(x)\geq\frac{x^2}{4}$ for $x>0$ small enough, which implies (25): $\beta(p,\phi')\geq\frac{p^2}{8\log 2}$, $p>0$ small. As a consequence of (25), the spectrum of integral means $\beta(d(t),\phi_t')$ of the family of the conformal maps $\phi_t'(z)=\exp tb(z)$ satisfies the following inequality:

$$\beta(d(t),\phi_t')=\beta(td(t),\phi')\geq\frac{t^2 d(t)^2}{8\log 2}, \quad t>0 \text{ small},$$

where $d(t)=\mathrm{M.dim}(\Gamma_t)\geq 1$.

Finally, by Proposition 2: $d(t)=\beta(d(t),\phi_t')+1$, we deduce that

$$d(t)\geq 1+\frac{t^2}{8\log 2}, \quad t>0 \text{ small}. \qquad (28)$$

This means that (23) fails for the family of conformal map (ϕ_t), $t > 0$, because if this family holds for (23), then the fact that

$$\limsup_{r\to 1} \frac{\int_0^{2\pi} |b(re^{i\theta})|^2 \, d\theta}{\log \frac{1}{1-r}} = 2 \limsup_{r\to 1} \frac{\int_0^{2\pi} |\operatorname{Re} b(re^{i\theta})|^2 \, d\theta}{\log \frac{1}{1-r}} = 0 \tag{29}$$

which follows from the following results: Theorem 7 and then Corollary 2, would contradict (28). Theorem 4 is proven. □

Theorem 7. *Let $\langle S\rangle_j^2$ be the square function of the dyadic martingale S_j of $\operatorname{Re} b$ (b defined in 3.1.2) and p be a positive real number. Then there exist positive constants $M_1, M_2, K_1, K_2, T_1, T_2$ do not depend on j such that: if $p > 1$, then*

$$M_1 j^{(p-1)/2} \le \frac{1}{2\pi} \int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2} \, d\theta \le M_2 j^{(p-1)/2};$$

if $p = 1$, then

$$K_1 \log j \le \frac{1}{2\pi} \int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2} \, d\theta \le K_2 \log j;$$

if $p < 1$, then

$$T_1 \le \frac{1}{2\pi} \int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2} \, d\theta \le T_2.$$

Proof. First we shall show that

$$\frac{1}{2\pi} \int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2} \, d\theta = \sum_{k=1}^{j-1} ((k+1)^{p/2} - k^{p/2}) P(\{N > k\}) \tag{30}$$

and then we shall prove that there exist positive constant A_1, A_2 which do not depend on j such that

$$A_1 \sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}} \le \int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2} \, d\theta \le A_2 \sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}}, \tag{31}$$

where l is equal to $(j-2)/2$ if j is even number and equal to $(j-1)/2$ otherwise. The proof will follow from the estimation of the sum $\sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}}$. The equality (30) can be proved by using the stopping time's argument. By separating the unit circle into two sets $\{N > j\}$ and $\{N \le j\}$, we have

$$\frac{1}{2\pi} \int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2} \, d\theta = \frac{1}{2\pi} \int_{\{N>j\}} (\langle S\rangle_j^2(\theta))^{p/2} \, d\theta + \frac{1}{2\pi} \int_{\{N\le j\}} (\langle S\rangle_j^2(\theta))^{p/2} \, d\theta.$$

Observe that on the set $\{N > j\}$, $\langle S\rangle_j^2(\theta) = j$. Therefore

$$\frac{1}{2\pi} \int_{\{N>j\}} (\langle S\rangle_j^2(\theta))^{p/2} \, d\theta = j^{p/2} P(\{N > j\}).$$

Besides,

$$\int_{\{N\le j\}} (\langle S\rangle_j^2(\theta))^{p/2} \, d\theta = \sum_{k=1}^{j} \int_{\{N=k\}} (\langle S\rangle_j^2(\theta))^{p/2} \, d\theta.$$

Note that $\langle S\rangle_j^2(\theta) = k$ on $\{N = k\}$. Thus

$$\frac{1}{2\pi}\int_{\{N\le j\}}(\langle S\rangle_j^2(\theta))^{p/2}\,d\theta = \sum_{k=1}^{j}\frac{1}{2\pi}\int_{\{N=k\}}(\langle S\rangle_j^2(\theta))^{p/2}\,d\theta = \sum_{k=1}^{j}k^{p/2}P(\{N=k\}).$$

By using summation by parts, we have

$$\sum_{k=1}^{j}k^{p/2}P(\{N=k\}) = \sum_{k=1}^{j-1}((k+1)^{p/2}-k^{p/2})P(\{N>k\}) - j^{p/2}P(\{N>j\}).$$

This implies (30). Now, we estimate the left hand side of the equality (30). We observe that if $p \ge 2$, then

$$\frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))d\theta \ge \sum_{k=1}^{j-1}(p/2)k^{(p-2)/2}P(\{N>k\})$$

and

$$\frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))\,d\theta \le \sum_{k=1}^{j-1}(p/2)(k+1)^{(p-2)/2}P(\{N>k\}),$$

and since $(k+1)^{(p-2)/2} \le e^{(p-2)/2}k^{(p-2)/2}$, $k = 1, 2, \ldots$

$$\frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))\,d\theta \le e^{(p-2)/2}\sum_{k=1}^{j-1}(p/2)k^{(p-2)/2}P(\{N>k\}).$$

Thus,

$$\begin{aligned}\frac{p}{2}\sum_{k=1}^{j-1}k^{(p-2)/2}P(\{N>k\}) &\le \frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}\,d\theta\\ &\le \frac{pe^{(p-2)/2}}{2}\sum_{k=1}^{j-1}k^{(p-2)/2}P(\{N>k\}).\end{aligned}$$

If $p < 2$, by an analogous argument, we have the inverse inequality

$$\begin{aligned}\frac{p}{2}\sum_{k=1}^{j-1}k^{(p-2)/2}P(\{N>k\}) &\ge \frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}\,d\theta\\ &\ge \frac{pe^{(p-2)/2}}{2}\sum_{k=1}^{j-1}k^{(p-2)/2}P(\{N>k\}).\end{aligned}$$

In the following, we continue to estimate the above sums by using the remark in 3.3.3 that $P(\{N > 2n+1\}) = P(\{N > 2n\})$. In order to do so we shall consider two following cases: j is an even integer and j is an odd integer. In each one, we shall also consider two situations $p \ge 2$ and $p < 2$ respectively. In the case of $j = 2(l+1)$, if $p \ge 2$, then

$$\begin{aligned}p/2\left(1+2\sum_{n=1}^{l}(2n)^{(p-2)/2}P(\{N>2n\})\right) &\le \frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}\,d\theta\\ \le \frac{pe^{(p-2)/2}}{2}\left(1+(1+e^{(p-2)/2})\sum_{n=1}^{l}(2n)^{(p-2)/2}P(\{N>2n\})\right);\end{aligned}$$

if $p < 2$ then

$$p/2\left(1+2\sum_{n=1}^{l}(2n)^{(p-2)/2}P(\{N>2n\})\right)\geq\frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}\,d\theta$$
$$\geq\frac{pe^{(p-2)/2}}{2}\left(1+(1+e^{(p-2)/2})\sum_{n=1}^{l}(2n)^{(p-2)/2}P(\{N>2n\})\right).$$

In the case of $j = 2l + 1$, if $p \geq 2$, then

$$p/2\left(1+2\left(\sum_{n=1}^{l-1}(2n)^{(p-2)/2}P(\{N>2n\})\right)+(2l)^{(p-2)/2}P(\{N>2l\})\right)$$
$$\leq\frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}\,d\theta\leq\frac{pe^{(p-2)/2}}{2}\left(1+(1+e^{(p-2)/2})\sum_{n=1}^{l}(2n)^{(p-2)/2}P(\{N>2n\})\right);$$

if $p < 2$, then

$$p/2\left(1+2\sum_{n=1}^{l}(2n)^{(p-2)/2}P(\{N>2n\})\right)\geq\frac{1}{2\pi}\int_0^{2\pi}(\langle S\rangle_j^2(\theta))^{p/2}\,d\theta$$
$$\geq\frac{pe^{(p-2)/2}}{2}\left(1+(1+e^{(p-2)/2})\left(\sum_{n=1}^{l-1}(2n)^{(p-2)/2}P(\{N>2n\})\right)\right.$$
$$\left.+(2l)^{(p-2)/2}P(\{N>2l\})\right).$$

By Lemma 3, there exist absolute positive constants C_1, C_2 such that

$$C_1\frac{1}{\sqrt{2n}}\leq P(\{N>2n\})=P(\{\Sigma_{2n}=0\})\leq C_2\frac{1}{\sqrt{2n}}.$$

This implies that

$$C_1'\sum_{n=1}^{l}\frac{1}{(2n)^{(3-p)/2}}\leq\sum_{n=0}^{l}(2n)^{(p-2)/2}P(\{N>2n\})\leq C_2'\sum_{n=1}^{l}\frac{1}{(2n)^{(3-p)/2}},$$

where C_1' and C_2' are some absolute positive constants. This implies (31). The proof remains to estimate the sum $\sum_{n=1}^{l}\frac{1}{(2n)^{(3-p)/2}}$. Observe that if $p \geq 3$ the function $f(x) = \frac{1}{(2x)^{(3-p)/2}}$ is increasing on $[1, \infty)$, then

$$2^{(p-3)/2}+\int_1^l\frac{1}{(2x)^{(3-p)/2}}\,dx\leq\sum_{n=1}^{l+1}\frac{1}{(2n)^{(3-p)/2}}\leq\int_1^{l+1}\frac{1}{(2x)^{(3-p)/2}}\,dx,$$

where

$$\int_1^l\frac{1}{(2x)^{(3-p)/2}}\,dx=\frac{1}{p-1}\left((j-2)^{(p-1)/2}-2^{(p-1)/2}\right)$$

and

$$\int_1^{l+1}\frac{1}{(2x)^{(3-p)/2}}\,dx=\frac{1}{p-1}\left((j)^{(p-1)/2}-2^{(p-1)/2}\right).$$

If $p < 3$, the function $f(x) = \dfrac{1}{2n^{(3-p)/2}}$ is decreasing on $[1, \infty)$ and then if $\frac{3-p}{2} < 1$ ($\Leftrightarrow p > 1$), then

$$\int_1^{l+1} \frac{1}{(2x)^{(3-p)/2}}\,dx \le \sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}} \le \int_1^{l} \frac{1}{(2x)^{(3-p)/2}}\,dx + \frac{1}{2^{(3-p)/2}},$$

where

$$\int_1^{l} \frac{1}{(2x)^{(3-p)/2}}\,dx = \frac{1}{p-1}\left((j-2)^{(p-1)/2} - 2^{(p-1)/2}\right)$$

and

$$\int_1^{l+1} \frac{1}{(2x)^{(3-p)/2}}\,dx = \frac{1}{p-1}\left((j)^{(p-1)/2} - 2^{(p-1)/2}\right);$$

if $\frac{3-p}{2} = 1$ ($\Leftrightarrow p = 1$), then

$$\frac{1}{2}\int_1^{l+1} \frac{1}{x}\,dx \le \sum_{n=1}^{l} \frac{1}{2n} \le \frac{1}{2}\int_1^{l} \frac{1}{x}\,dx + \frac{1}{2},$$

where

$$\int_1^{l+1} \frac{1}{x}\,dx = \log(j) - \log 2 \quad \text{and} \quad \int_1^{l} \frac{1}{x}dx = \log(j-2) - \log 2;$$

if $\frac{3-p}{2} > 1$ ($\Leftrightarrow p < 1$), then $\sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}}$ converges as $j \to \infty$ because

$$\int_1^{l+1} \frac{1}{(2x)^{(3-p)/2}}\,dx \le \sum_{n=1}^{l} \frac{1}{(2n)^{(3-p)/2}} \le \int_1^{l} \frac{1}{(2x)^{(3-p)/2}}\,dx + \frac{1}{2^{(3-p)/2}},$$

where

$$\int_1^{l} \frac{1}{(2x)^{(3-p)/2}}\,dx = \frac{1}{p-1}\left((j-2)^{(p-1)/2} - 2^{(p-1)/2}\right)$$

and

$$\int_1^{l+1} \frac{1}{(2x)^{(3-p)/2}}\,dx = \frac{1}{p-1}\left((j)^{(p-1)/2} - 2^{(p-1)/2}\right)$$

converges as $j \to \infty$.

Now we can go to the conclusion that there exist positive constants M_1, M_2, K_1, K_2, T_1, T_2 which do not depend on j such that: if $p > 1$, then

$$M_1 j^{(p-1)/2} \le \frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2}\,d\theta \le M_2 j^{(p-1)/2};$$

if $p = 1$, then

$$K_1 \log j \le \frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2}\,d\theta \le K_2 \log j;$$

if $p < 1$, then

$$T_1 \le \frac{1}{2\pi}\int_0^{2\pi} (\langle S\rangle_j^2(\theta))^{p/2}\,d\theta \le T_2.$$

The theorem is proven. □

Corollary 2. *Let b be the Bloch function (defined in 3.1.2) and a real positive p. Then*

$$\limsup_{r\to 1} \frac{\int_0^{2\pi} |\operatorname{Re}(b(re^{i\theta}))|^p\, d\theta}{(\log\frac{1}{1-r})^{p/2}} = 0$$

Proof. The proof will be given as follows. First of all, we shall show that for $p > 0$,

$$\limsup_{j\to\infty} \frac{\int_0^{2\pi} |\operatorname{Re}(b((1-2^{-j})e^{i\theta}))|^p\, d\theta}{(j\log 2)^{p/2}} = \limsup_{j\to\infty} \frac{\int_0^{2\pi} |S_j(e^{i\theta})|^p\, d\theta}{(j\log 2)^{p/2}}. \tag{32}$$

Then we shall estimate

$$\frac{\int_0^{2\pi} |S_j(e^{i\theta})|^p\, d\theta}{(j\log 2)^{p/2}}$$

by using two following facts with respect to two cases: $1 < p < \infty$ and $0 < p \le 1$. Firstly for $1 < p < \infty$ (see [Bur66]) there exist absolute positive constants b_p and B_p such that

$$b_p \|\langle S\rangle_j^2\|_{(p/2)}^{1/2} \le \|S_j\|_p \le B_p \|\langle S\rangle_j^2\|_{(p/2)}^{1/2}$$

and secondly for $0 < p \le 1$ (see [Wan91]) there also exists a positive absolute constant ν_p such that $\|S_j\|_p \le \nu_p \|\langle S\rangle_j^2\|_{(p/2)}^{1/2}$, where the square function $\langle S\rangle_j^2(\theta) = \sum_{k=1}^{j} (\Delta S_k(\theta))^2$. Then the proof will follow by Theorem 7.

Now, let us prove (32). We start it with the case $1 < p < \infty$. The fact that $|S_j(e^{i\theta}) - \operatorname{Re}(b(re^{i\theta}))| \le C\|b\|_{\mathcal{B}}$ if $r = 1-2^{-j}$ (see (11)) implies that for $p \ge 1$,

$$\big|\|S_j(e^{i\theta})\|_p - \|\operatorname{Re} b(re^{i\theta})\|_p\big| \le \|S_j(e^{i\theta}) - \operatorname{Re}(b(re^{i\theta}))\|_p \le (2\pi)^{1/p}(C\|b\|_{\mathcal{B}}).$$

Therefore if we divide both sides by $(j\log 2)^{1/2}$ of the above inequalities and take the limit as j tends to ∞, then we obtain

$$\lim_{j\to\infty} \left(\frac{\int_0^{2\pi} |S_j(e^{i\theta})|^p\, d\theta}{(j\log 2)^{p/2}}\right)^{1/p} - \left(\frac{\int_0^{2\pi} |\operatorname{Re}(b((1-2^{-j})e^{i\theta}))|^p\, d\theta}{(j\log 2)^{p/2}}\right)^{1/p} = 0. \tag{33}$$

According to Corollary 1 the quantity

$$\frac{\int_0^{2\pi} |\operatorname{Re}(b((1-2^{-j})e^{i\theta}))|^p\, d\theta}{(j\log 2)^{p/2}}$$

is bounded and then by (33) the quantity

$$\frac{\int_0^{2\pi} |S_j(e^{i\theta})|^p\, d\theta}{(j\log 2)^{p/2}}$$

is also bounded. Moreover, as the function x^p is continuous uniformly on some compact set of $[0, +\infty)$, (33) implies that

$$\lim_{j\to\infty} \frac{\int_0^{2\pi} |S_j(e^{i\theta})|^p\, d\theta}{(j\log 2)^{p/2}} - \frac{\int_0^{2\pi} |\operatorname{Re}(b((1-2^{-j})e^{i\theta}))|^p\, d\theta}{(j\log 2)^{p/2}} = 0$$

Thus,

$$\limsup_{j\to\infty} \frac{\int_0^{2\pi} |S_j(e^{i\theta})|^p\, d\theta}{(j\log 2)^{p/2}} = \limsup_{j\to\infty} \frac{\int_0^{2\pi} |\operatorname{Re}(b((1-2^{-j})e^{i\theta}))|^p\, d\theta}{(j\log 2)^{p/2}}.$$

In the case of $0 < p \leq 1$, again the fact that $|S_j(e^{i\theta}) - \mathrm{Re}(b(re^{i\theta}))| \leq C\|b\|_{\mathcal{B}}$ if $r = 1 - 2^{-j}$ (see (11)) implies that

$$\left| \int_0^{2\pi} |S_j(e^{i\theta})|^p \, d\theta - \int_0^{2\pi} |\mathrm{Re}(b((1-2^{-j})e^{i\theta}))|^p \, d\theta \right|$$
$$\leq \int_0^{2\pi} |S_j(e^{i\theta}) - \mathrm{Re}(b((1-2^{-j})e^{i\theta}))|^p \, d\theta \leq 2\pi (C\|b\|_{\mathcal{B}})^p.$$

Analogously, if we devide the above inequalities by $(j \log 2)^{p/2}$ and take the limit as j tends to ∞, then we have

$$\lim_{j\to\infty} \frac{\int_0^{2\pi} |S_j(e^{i\theta})|^p \, d\theta}{(j\log 2)^{p/2}} - \frac{\int_0^{2\pi} |\mathrm{Re}(b((1-2^{-j})e^{i\theta}))|^p \, d\theta}{(j\log 2)^{p/2}} = 0$$

which implies that

$$\limsup_{j\to\infty} \frac{\int_0^{2\pi} |S_j(e^{i\theta})|^p \, d\theta}{(j\log 2)^{p/2}} = \limsup_{j\to\infty} \frac{\int_0^{2\pi} |\mathrm{Re}(b((1-2^{-j})e^{i\theta}))|^p \, d\theta}{(j\log 2)^{p/2}}.$$

Then (32) follows.

According to Theorem 7, if we divide the integral

$$\frac{1}{2\pi} \int_0^{2\pi} (\langle S \rangle_j^2(\theta))^{p/2} \, d\theta$$

by $(j \log 2)^{p/2}$ and let $j \to \infty$, then we have

$$\limsup_{j\to\infty} \frac{\int_0^{2\pi} (\langle S \rangle_j^2(\theta))^{p/2} \, d\theta}{(j\log 2)^{p/2}} = 0.$$

This finishes the proof. □

Appendix A. Hausdorff dimension and Minkowski dimension

Let $\alpha > 0$. The *α-dimensional Hausdorff measure* of a Borel set $E \subset \mathbf{C}$ is defined by

$$\Lambda_\alpha(E) = \lim_{\epsilon\to 0} \inf_{(B_k)} \sum_k (\mathrm{diam} B_k)^\alpha,$$

where the infimum is taken over the covers (B_k) of E with $\mathrm{diam} B_k \leq \epsilon$ for all k. The *Haudorff dimension* is defined by

$$\mathrm{H.dim} E = \inf\{\alpha \colon \Lambda_\alpha(E) = 0\}.$$

For reference, see [Pom92], Chapter 10, p. 222.

Let E be a bounded set in $\mathbf{C}$ and let $N(\varepsilon, E)$ denote the minimal numbers of disks of diameter ε that are needed to cover E. Up to bounded multiplies it is the same as the number of squares of grid of mesh size ε that intersect E. We defined the *Minkowski dimension* of E by

(34) $$\mathrm{M.dim} E = \limsup_{\varepsilon\to 0} \frac{\log N(\varepsilon, E)}{\log(1/\varepsilon)}.$$

Proposition 5. *If E is any bounded set in $\mathbf{C}$, then*

(35) $$\mathrm{H.dim} E \leq \liminf_{\varepsilon\to 0} \frac{\log N(\varepsilon, E)}{\log(1/\varepsilon)} \leq \mathrm{M.dim} E.$$

Proof. Let β be any number greater than the limes inferior in (35). Then there are $\varepsilon_n \to 0$ such that $N_n = N(\varepsilon_n, E) < \varepsilon_n^{-\beta}$. If E is covered by the discs $D_1, D_2, \dots, D_{N_n}$ of diameter ε_n then, for $\alpha > \beta$,

$$\sum_{k=1}^{N_n} (\operatorname{diam} D_k)^\alpha = N_n \varepsilon_n^\alpha < \varepsilon_n^{\alpha-\beta} \to 0 \quad \text{as} \quad n \to \infty$$

and thus H.dim$E \leq \alpha$ which implies (35). □

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