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UNIVERSAL TEICHMÜLLER SPACE AND $\mathcal{Q}_K$ SPACES

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Abstract. In this article, we study the universal Teichmüller space $T(1)$ and give relationship between $\mathcal{T}_K \subset T(1)$ and a more general function space $\mathcal{Q}_K$. Our results extend Astala and Zinsmeister's BMO-Teichmüller theory to the $\mathcal{Q}_K$-Teichmüller theory.

1. Introduction

By results of Ahlfors–Bers [1, 2], Gehring [9] and Astala–Gehring [3], the universal Teichmüller space, denoted by $T(1)$, can be defined as a set of all functions $\log f'$ in the unit disc $\mathbf{D}$, where f is conformal in $\mathbf{D}$ and has a quasiconformal extension to the complex plane $\mathbf{C}$. Denote by S the set of all mappings $\log f'(z)$, where f is conformal in $\mathbf{D}$. By the Koebe distortion theorem, S is a bounded subset of the Bloch space $\mathcal{B}$ which consists of all functions f analytic in $\mathbf{D}$ with

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbf{D}} (1-|z|^2)|f'(z)| < \infty.$$

We know that the universal Teichmüller space $T(1)$ is the interior of S in $\mathcal{B}$ and as a bridge between space of univalent functions and general Teichmüller spaces, it is the simplest Teichmüller space. More characterizations of $T(1)$, see [1] and [10].

The Green function in the unit disc with singularity at $a \in \mathbf{D}$ is given by $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation of $\mathbf{D}$. There are many ways to define BMOA, the analytic space of bounded mean oscillation; see [5] and [8]. For the purposes of this paper, a function f analytic in $\mathbf{D}$ is said to belong to BMOA if

$$\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 g(z,a)\, dA(z) < \infty,$$

where $dA(z) = dx\, dy$, $z = x + iy$.

A very useful tool in the study of function spaces is the Carleson measure. For a subarc I of the unit circle $\mathbf{T}$ with $\ell(I) < 1$, define the Carleson box by

$$S_G(I) = \begin{cases} \{r\zeta \in G \colon 1-\ell(I) < r < 1, \zeta \in I\}, & G = \mathbf{D}, \\ \{r\zeta \in G \colon 1 < r < 1+\ell(I), \zeta \in I\}, & G = \mathbf{C}\setminus\overline{\mathbf{D}}. \end{cases}$$

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For $0 < p < \infty$, we say that a positive Borel measure ν on G is a p-Carleson measure if

$$\sup_{I \subset \mathbf{T}} \frac{\nu(S_G(I))}{(\ell(I))^p} < \infty.$$

When $p = 1$ and $G = \mathbf{D}$, we get the (classical) Carleson measure.

Let f be a C^1 homeomorphism from one region to another. It is said to be quasiconformal if

$$D_f(z) = \frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|}$$

is bounded, where $\mu_f(z) = \frac{\overline{\partial} f}{\partial f}(z)$ is called the complex dilatation of f. Note that if f is quasiconformal then

$$\|\mu_f\|_\infty = \sup_{z \in \mathbf{C}} |\mu_f(z)| < 1.$$

Astala and Zinsmeister [4] introduced a new subset $\mathcal{T}$ of $T(1)$; that is, the set $\mathcal{T}$ consists of all functions $\log f' \in T(1)$ and

$$|\mu_f(z)|^2 (|z|^2 - 1)^{-1} \, dA(z)$$

is a Carleson measure on $\mathbf{C} \backslash \overline{\mathbf{D}}$. As important parts of their BMO-Teichmüller theory, Astala and Zinsmeister [4] gave the relations between BMOA and $\mathcal{T}$ as follows.

Theorem A. *$\mathcal{T}$ is a subset of BMOA.*

Theorem B. *$\mathcal{T}$ is open in BMOA with $\mathcal{T}_b = \{\log f' \in \mathcal{T};\ f(\mathbf{D})$ bounded$\}$ connected. Furthermore, $\mathcal{T}_b$ and $\mathcal{T}_\theta = \{\log f' \in \mathcal{T};\ f(e^{i\theta}) = \infty\}$, $\theta \in [0, 2\pi]$, are the connected components of $\mathcal{T}$.*

The main goal of this paper is to introduce subsets, denoted by $\mathcal{T}_K$ with weight K, of the universal Teichmüller space $T(1)$ and to give relationship between $\mathcal{T}_K$ and a more general function space $\mathcal{Q}_K$, which has attracted a lot of attention in recent years. Of course, for choosing a special function K, our results are just Theorems A and B above.

For a nonnegative and nondecreasing function K on $[0, \infty)$, the space $\mathcal{Q}_K$ consists of analytic functions in $\mathbf{D}$ for which

$$\|f\|_{\mathcal{Q}_K}^2 = \sup_{a \in \mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 K(g(z,a)) \, dA(z) < \infty.$$

If $K(t) = t^p$ for $0 \le p < \infty$, the space $\mathcal{Q}_K$ gives a $\mathcal{Q}_p$ space. We refer to [19] and [20] for the general theory of $\mathcal{Q}_p$ spaces. In particular, if $K(t) = t$, then $\mathcal{Q}_K$=BMOA. Note that $\mathcal{Q}_K$ spaces are always contained in the Bloch space. By [6], $\mathcal{Q}_K = \mathcal{B}$ if and only if

$$\int_0^1 K(\log(1/r))(1 - r^2)^{-2} r \, dr < \infty.$$

We know that $\mathcal{Q}_K$ is nontrivial, containing non-constant functions, if and only if

$$\sup_{t \in (0,1)} \int_0^1 \frac{(1-t)^2}{(1-tr^2)^3} K\left(\log \frac{1}{r}\right) r \, dr < \infty.$$

Throughout this paper we always assume that the condition above is satisfied, so that the space $\mathcal{Q}_K$ here is nontrivial. We also assume that $K(0) = 0$. Otherwise, $\mathcal{Q}_K$ coincides with the Dirichlet space [6]. For more results about the spaces $\mathcal{Q}_K$, see [6] and [7].

To define $\mathcal{T}_K$, we need the following K-Carleson measure. A positive Borel measure ν on $G = \mathbf{D}$ or $G = \mathbf{C} \setminus \overline{\mathbf{D}}$ is said to be a K-Carleson measure if

$$\sup_{I \subset \mathbf{T}} \int_{S_G(I)} K\left(\frac{|1-|z||}{\ell(I)}\right) d\nu(z) < \infty.$$

Clearly, if $K(t) = t^p$, then ν is a K-Carleson measure on G if and only if $|1-|z||^p d\nu(z)$ is a p-Carleson measure on G.

Define $\mathcal{T}_K$ the set of all functions $\log f' \in T(1)$ such that

$$|\mu_f(z)|^2(|z|^2-1)^{-2}\, dA(z)$$

is a K-Carleson measure on $\mathbf{C} \setminus \overline{\mathbf{D}}$. Our first observation is that $\mathcal{T}_K$ is not trivial. In fact, let $f(z) = e^z$. Then the Schwarzian derivative of f

$$S_f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = -\frac{1}{2}.$$

Since

$$\|S_f\|_{\mathbf{D}} = \sup_{z \in \mathbf{D}} |S_f(z)| \left(\frac{1}{1-|z|^2}\right)^{-2} = \frac{1}{2},$$

by Theorem II.5.1 in [10], f is univalent in $\mathbf{D}$ and can be extended to a quasiconformal mapping of the complex plane and the complex dilatation

$$\mu_f\left(\frac{1}{\bar{z}}\right) = -\frac{1}{2}\left(\frac{z}{\bar{z}}\right)^2 (1-|z|^2)^2 S_f(z) = \frac{1}{4}\left(\frac{z}{\bar{z}}\right)^2 (1-|z|^2)^2$$

for z in $\mathbf{D}$. This implies

$$\begin{aligned} &\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} |\mu_f(z)|^2(|z|^2-1)^{-2} K\left(\frac{|z|-1}{\ell(I)}\right) dA(z) \\ &\le C \int_{S_{\mathbf{D}}(I)} \left|\mu_f\left(\frac{1}{\bar{z}}\right)\right|^2 (1-|z|^2)^{-2} K\left(\frac{1-|z|}{\ell(I)}\right) dA(z) \le CK(1). \end{aligned}$$

Hence, $z = \log(e^z)' \in \mathcal{T}_K$.

Remark 1. If $K_1(t) = t$, then $\mathcal{T}_{K_1} = \mathcal{T}$.

Remark 2. Let $K_2(t) = t^p$ for $0 < p < \infty$. By the definition we have that $\mathcal{T}_{K_2} = \mathcal{T}_p$ coincides with the universal Teichmüller space $T(1)$ for $1 < p < \infty$. In fact, suppose that f is conformal on $\mathbf{D}$ and admits a quasiconformal extension to $\mathbf{C}$. Since $\|\mu_f\|_\infty < 1$, for any $I \subset \mathbf{T}$, we have

$$\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} |\mu_f(z)|^2(|z|^2-1)^{-2} K_2\left(\frac{|z|-1}{\ell(I)}\right) dA(z) \le \frac{(2+2\pi)^{p-1}}{2(p-1)}.$$

It is easy to see that $\mathcal{T}_p \subset \mathcal{T} = \mathcal{T}_1 \subset \mathcal{T}_q = T(1)$ for $0 < p < 1 < q < \infty$. For more general case, we give a sufficient and necessary condition for $\mathcal{T}_K = T(1)$ in Section 2.

To study $\mathcal{T}_K$ we consider the auxiliary function

$$\varphi_K(s) = \sup_{0 \le t \le 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty,$$

which plays a key role in the study of $\mathcal{Q}_K$ spaces; see [7], [17] and [18] for example. Our methods require two more constraints on K as follows:

$$\int_0^1 \frac{\varphi_K(s)}{s}\, ds < \infty \tag{1.1}$$

and

$$\int_1^\infty \frac{\varphi_K(s)}{s^{1+p}}\, ds < \infty, \quad 0 < p < 2. \tag{1.2}$$

The main results provided in this paper are the following Theorems 1.1 and 1.2, which not only generalize Theorems A and B, but also the classical theory related to the Bloch space and the universal Teichmüller space. In particular, our results are also new for $\mathcal{Q}_p$ spaces.

Theorem 1.1. *Let K satisfy (1.1) and (1.2). Then $\mathcal{T}_K$ is a subset of $\mathcal{Q}_K$ space.*

Theorem 1.2. *Let K satisfy (1.1) and (1.2). Then $\mathcal{T}_K$ is open in $\mathcal{Q}_K$. Furthermore, $\mathcal{T}_{K,b} = \{\log f' \in \mathcal{T}_K \colon f(\mathbf{D}) \text{ is bounded}\}$ and $\mathcal{T}_{K,\theta} = \{\log f' \in \mathcal{T}_K \colon f(e^{i\theta}) = \infty\}$, $\theta \in [0, 2\pi]$, are the connected components of $\mathcal{T}_K$.*

In this paper, the letter C denotes a positive constant whose value may change from one occurrence to another.

2. Basic properties of $\mathcal{T}_K$ spaces

Theorem 2.1. *Assume that $K(c) > 0$ for $0 < c < \infty$ and define $K_1(t) = \inf(K(t), K(c))$. Then $\mathcal{T}_K = \mathcal{T}_{K_1}$.*

Proof. Since $K_1 \le K$ and K_1 is nondecreasing, it is clear that $\mathcal{T}_K \subset \mathcal{T}_{K_1}$. It remains to prove that $\mathcal{T}_{K_1} \subset \mathcal{T}_K$.

Let $\log f' \in \mathcal{T}_{K_1}$. If $c \ge 1$, the result is clear. For $c < 1$ and $I \subset \mathbf{T}$,

$$\begin{aligned}
&\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} |\mu_f(z)|^2(|z|^2-1)^{-2} K\left(\frac{|z|-1}{\ell(I)}\right) dA(z)\\
&= \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)\cap\left\{z:\frac{|z|-1}{\ell(I)}<c\right\}} |\mu_f(z)|^2(|z|^2-1)^{-2} K\left(\frac{|z|-1}{\ell(I)}\right) dA(z)\\
&\quad + \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)\cap\left\{z:\frac{|z|-1}{\ell(I)}\ge c\right\}} |\mu_f(z)|^2(|z|^2-1)^{-2} K\left(\frac{|z|-1}{\ell(I)}\right) dA(z)\\
&= \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)\cap\left\{z:\frac{|z|-1}{\ell(I)}<c\right\}} |\mu_f(z)|^2(|z|^2-1)^{-2} K_1\left(\frac{|z|-1}{\ell(I)}\right) dA(z)\\
&\quad + \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)\cap\left\{z:\frac{|z|-1}{\ell(I)}\ge c\right\}} |\mu_f(z)|^2(|z|^2-1)^{-2} K\left(\frac{|z|-1}{\ell(I)}\right) dA(z)\\
&\le \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} |\mu_f(z)|^2(|z|^2-1)^{-2} K_1\left(\frac{|z|-1}{\ell(I)}\right) dA(z)\\
&\quad + \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)\cap\left\{z:\frac{|z|-1}{\ell(I)}\ge c\right\}} (c\ell(I))^{-2} K(1)\, dA(z)
\end{aligned}$$

$$\leq C + \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} |\mu_f(z)|^2(|z|^2-1)^{-2} K_1\left(\frac{|z|-1}{\ell(I)}\right) dA(z).$$

Hence, $\log f' \in \mathcal{T}_K$. This proves the theorem. □

The significance of Theorem 2.1 is that the space $\mathcal{T}_K$ only depends on the behavior of $K(t)$ for t close to 0. In particular, when studying $\mathcal{T}_K$ spaces, we can always assume that $K(t) = K(c)$ for $t \geq c$.

The following result gives a sufficient and necessary condition for $\mathcal{T}_K = T(1)$. This result also shows that $\mathcal{T} \neq T(1)$.

Theorem 2.2. *The following are equivalent:*

(i) $\mathcal{T}_K = T(1)$;
(ii) $\int_0^1 \frac{K(t)}{t^2}\, dt < \infty$.

Proof. Let us first assume that $\int_0^1 \frac{K(t)}{t^2}\, dt < \infty$. To show $\mathcal{T}_K = T(1)$, we need only to prove $T(1) \subset \mathcal{T}_K$. Indeed,

$$\begin{aligned}&\sup_{I\subset\mathbf{T}} \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} |\mu_f(z)|^2(|z|^2-1)^{-2} K\left(\frac{|z|-1}{\ell(I)}\right) dA(z)\\ &\leq C \sup_{I\subset\mathbf{T}} \int_{S_{\mathbf{D}}(I)} (1-|z|^2)^{-2} K\left(\frac{1-|z|}{\ell(I)}\right) dA(z).\end{aligned}$$

Let $a = (1-\ell(I))e^{i\theta}$ for the middle point $e^{i\theta}$ of I. Then

$$\frac{1-|a|^2}{|1-\bar{a}z|^2} \approx \frac{1}{\ell(I)}$$

for all $z \in S_{\mathbf{D}}(I)$. Thus

$$\begin{aligned}&\sup_{I\subset\mathbf{T}} \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} |\mu_f(z)|^2(|z|^2-1)^{-2} K\left(\frac{|z|-1}{\ell(I)}\right) dA(z)\\ &\leq C \sup_{a\in\mathbf{D}} \int_{\mathbf{D}} (1-|z|^2)^{-2} K(1-|\varphi_a(z)|^2)\, dA(z) = \pi C \int_0^1 \frac{K(t)}{t^2}\, dt < \infty.\end{aligned}$$

Hence $\log f' \in \mathcal{T}_K$ and $\mathcal{T}_K = T(1)$.

Conversely, we assume that $\mathcal{T}_K = T(1)$. Define a measurable function μ in $\mathbf{C}$ as follows:

$$|\mu(z)| = \begin{cases} \frac{1}{2}, & 1 < |z| < 10,\\ 0, & \text{others.}\end{cases}$$

By Existence Theorem in [10], there is a quasiconformal mapping f in $\mathbf{C}$ whose complex dilatation agrees with μ almost everywhere. In this way, f is conformal in $\mathbf{D}$ and admits a quasiconformal extension in $\mathbf{C}$. Hence $\log f' \in T(1)$ and

$$\sup_{I\subset\mathbf{T}} \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} |\mu(z)|^2(|z|^2-1)^{-2} K\left(\frac{|z|-1}{\ell(I)}\right) dA(z) < \infty.$$

Since

$$\int_0^{1/2} \frac{K(t)}{t^2}\, dt \leq 2\int_0^{1/2} \frac{K(t)}{t^2}(1-t)\, dt,$$

we have

$$\begin{aligned}
\int_0^{1/2} \frac{K(t)}{t^2}\,dt &\le 2\int_0^1 \frac{K(t)}{t^2}(1-t\ell(I))\,dt \\
&\le 2\int_{1-\ell(I)}^1 \frac{\ell(I)}{(1-|z|)^2} K\left(\frac{1-|z|}{\ell(I)}\right) d|z| \\
&\le \frac{C}{\pi}\sup_{I\subset \mathbf{T}} \int_{S_{\mathbf{D}}(I)} (1-|z|^2)^{-2} K\left(\frac{1-|z|}{\ell(I)}\right) dA(z) \\
&\le C\sup_{I\subset \mathbf{T}} \int_{S_{\mathbf{D}}(I)\cap\{z\in\mathbf{D}:|z|\le\frac12\}} (1-|z|^2)^{-2} K\left(\frac{1-|z|}{\ell(I)}\right) dA(z) \\
&\quad + C\sup_{I\subset \mathbf{T}} \int_{S_{\mathbf{D}}(I)\cap\{z\in\mathbf{D}:|z|>\frac12\}} (1-|z|^2)^{-2} K\left(\frac{1-|z|}{\ell(I)}\right) dA(z) \\
&\le C + C\sup_{\ell(I)\le\frac12} \int_{S_{\mathbf{D}}(I)} (1-|z|^2)^{-2} K\left(\frac{1-|z|}{\ell(I)}\right) dA(z) \\
&\le C + C\sup_{I\subset \mathbf{T}} \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} (|z|^2-1)^{-2} K\left(\frac{|z|-1}{\ell(I)}\right) dA(z) \\
&= C + 4C\sup_{I\subset \mathbf{T}} \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} |\mu(z)|^2(|z|^2-1)^{-2} K\left(\frac{|z|-1}{\ell(I)}\right) dA(z) \\
&< \infty.
\end{aligned}$$

The proof of Theorem 2.2 is complete. □

3. Proof of Theorem 1.1

By [15] if K satisfies condition (1.2), we may assume that there exists $c > 0$ such that $K(t)/t^{p-c}$ is non-increasing and $K(2t) \approx K(t)$ for $0 < t < \infty$. The following results will be used in the proof of Theorem 1.1.

Theorem C. [21] *Let K satisfy the condition (1.1). If f is conformal on $\mathbf{D}$, then the following are equivalent:*

(i) $\log f' \in \mathcal{Q}_K$;
(ii) $(1-|z|^2)^2|S_f(z)|^2\,dA(z)$ *is a K-Carleson measure on $\mathbf{D}$.*

Lemma D. [16] *Let K satisfy the conditions (1.1) and (1.2). Let $b+\alpha \ge 1+p$, $b \ge p$ and $\alpha > 0$. There exists $\beta \in (0,1)$ and constant C such that*

$$\int_{\mathbf{D}} \frac{K\left(\frac{1-|z|}{\ell(I)}\right)(1-|w|^2)^{b-1}}{(1-|z|)^{1-\alpha+\beta}\,|1-\bar{w}z|^{b+\alpha}}\,dA(z) \le C\frac{K\left(\frac{1-|w|}{\ell(I)}\right)}{(1-|w|)^{\beta}}$$

for all $w \in \mathbf{D}$ and arc I on $\mathbf{T}$.

Proof of Theorem 1.1. We prove the result by two steps.

Step 1. Suppose that f is defined in $\mathbf{C}\setminus\overline{\mathbf{D}}$ instead of $\mathbf{D}$ for technical purposes. Denote by the same notation f for its extension to $\mathbf{C}$. We will show that

$$\sup_I \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} (|z|^2-1)^2|S_f(z)|^2 K\left(\frac{|z|-1}{\ell(I)}\right) dA(z)$$
$$\le C\sup_I \int_{S_{\mathbf{D}}(I)} \frac{|\mu_f(\zeta)|^2}{(1-|\zeta|^2)^2} K\left(\frac{1-|\zeta|}{\ell(I)}\right) dA(\zeta).$$

Note that f is conformal in $\mathbf{C}\setminus\overline{\mathbf{D}}$. We normalize f such that

$$f(z) = z + \frac{b_1}{z} + \cdots$$

at infinity. By the proof of Theorem 1 in [4], we know that

$$(|z_0|^2-1)^2|S_f(z_0)|^2 \le C\int_{\mathbf{D}} \frac{|\mu_f(\zeta)|^2}{|\zeta - z_0|^4}\, dA(\zeta) \tag{3.1}$$

for any $z_0\in\mathbf{C}\setminus\overline{\mathbf{D}}$. To prove that $(|z|^2-1)^2|S_f(z)|^2\, dA(z)$ is a K-Carleson measure on $\mathbf{C}\setminus\overline{\mathbf{D}}$, by (3.1), we have to estimate

$$\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} \int_{\mathbf{D}} \frac{|\mu_f(\zeta)|^2}{|\zeta - z|^4}\, dA(\zeta) K\left(\frac{|z|-1}{\ell(I)}\right) dA(z).$$

We cut the integral above into two parts as follows:

$$P_1 = \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} \int_{S_{\mathbf{D}}(2I)} \frac{|\mu_f(\zeta)|^2}{|\zeta - z|^4}\, dA(\zeta) K\left(\frac{|z|-1}{\ell(I)}\right) dA(z)$$

and

$$P_2 = \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} \int_{\mathbf{D}\setminus S_{\mathbf{D}}(2I)} \frac{|\mu_f(\zeta)|^2}{|\zeta - z|^4}\, dA(\zeta) K\left(\frac{|z|-1}{\ell(I)}\right) dA(z).$$

Here $2I$ is the arc with the same center as I but with double length.

Note that if $z\in S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)$, then $1<|z|\le 1+2\pi$ and $w=\frac{1}{\bar z}\in S_{\mathbf{D}}(I)$. For the first part, we have

$$\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} |\zeta - z|^{-4} K\left(\frac{|z|-1}{\ell(I)}\right) dA(z) \le C\int_{S_{\mathbf{D}}(I)} |1-\bar w\zeta|^{-4}\, K\left(\frac{1-|w|}{\ell(I)}\right) dA(w)$$
$$\le C(1-|\zeta|)^{-2} K\left(\frac{1-|\zeta|}{\ell(I)}\right).$$

The last inequality above holds by taking $\alpha=\beta+1$ and $b=3-\beta$ in Lemma D. Therefore,

$$P_1 = \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} \int_{S_{\mathbf{D}}(2I)} \frac{|\mu_f(\zeta)|^2}{|\zeta - z|^4}\, dA(\zeta) K\left(\frac{|z|-1}{\ell(I)}\right) dA(z)$$
$$\le C\int_{S_{\mathbf{D}}(2I)} \frac{|\mu_f(\zeta)|^2}{(1-|\zeta|)^2} K\left(\frac{1-|\zeta|}{\ell(I)}\right) dA(\zeta)$$
$$\le C\sup_I \int_{S_{\mathbf{D}}(I)} \frac{|\mu_f(\zeta)|^2}{(1-|\zeta|^2)^2} K\left(\frac{1-|\zeta|}{\ell(I)}\right) dA(\zeta).$$

To handle the other part, denote by z_I the center of I. Set

$$S_n = S_{\mathbf{D}}(2^nI) = \{r\xi\in\mathbf{D}\colon 1-2^n\ell(I)<r<1,\ \xi\in 2^nI\},\quad n=1,2,\cdots.$$

Let n_I be the minimum such that $2^{n_I}\ell(I) \geq 1$. Then $S_n = \mathbf{D}$ when $n \geq n_I$. Write $z_1 = (1+\ell(I)/2)z_I$. If $z \in S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)$ and $\zeta \in S_n \setminus S_{n-1}$, $1 < n < n_I$, then

$$\frac{2}{\pi}2^{n-2}\ell(I) \leq |\zeta - z_I| \leq \frac{3}{2}2^n\ell(I).$$

Hence,

$$|\zeta - z_1| \leq |\zeta - z_I| + |z_I - z_1| \leq \frac{3}{2}2^n\ell(I) + \frac{\ell(I)}{2} \leq 3\cdot 2^n\ell(I)$$

and

$$|\zeta - z_1| \geq |\zeta - z_I| - |z_I - z_1| \geq \frac{2}{\pi}2^{n-2}\ell(I) - \frac{\ell(I)}{2} \geq \frac{4-\pi}{8\pi}2^n\ell(I).$$

Thus,

$$|z| - 1 < \ell(I) \leq 8\pi(4-\pi)^{-1}2^{-n}|\zeta - z_1|$$

and

$$1 - |\zeta| < 2^n\ell(I) \leq 8\pi(4-\pi)^{-1}|\zeta - z_1|.$$

Note that

$$\begin{aligned}|\zeta - z_1| &\leq |\zeta - z| + |z_I - z| + |z_I - z_1| \\ &\leq |\zeta - z| + \frac{3}{2}\ell(I) + \frac{1}{2}\ell(I) \leq |\zeta - z| + 2\pi|\zeta - z|.\end{aligned}$$

Since K satisfies (1.2), we can assume that $K(t)/t^{p-c}$ is non-increasing for some small $c > 0$. Thus

$$\begin{aligned}P_2 &= \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} \int_{\mathbf{D}\setminus S_{\mathbf{D}}(2I)} \frac{|\mu_f(\zeta)|^2}{|\zeta - z|^4}\, dA(\zeta) K\left(\frac{|z|-1}{\ell(I)}\right) dA(z) \\ &= \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} \sum_{n=2}^{n_I} \int_{S_n\setminus S_{n-1}} \frac{|\mu_f(\zeta)|^2}{|\zeta - z|^4}\, dA(\zeta) K\left(\frac{|z|-1}{\ell(I)}\right) dA(z) \\ &\leq C\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} dA(z) \sum_{n=2}^{n_I} \int_{S_n\setminus S_{n-1}} \frac{K\left(\frac{|\zeta - z_1|}{2^n\ell(I)}\right)|\mu_f(\zeta)|^2}{|\zeta - z_1|^4}\, dA(\zeta) \\ &\leq C(\ell(I))^2 \sum_{n=2}^{\infty} \int_{S_n\setminus S_{n-1}} \frac{K\left(\frac{1-|\zeta|}{2^n\ell(I)}\right)|\mu_f(\zeta)|^2}{(1-|\zeta|)^{p-c}|\zeta - z_1|^{4-p+c}}\, dA(\zeta) \\ &\leq C(\ell(I))^2 \sum_{n=2}^{\infty} \frac{1}{(2^n\ell(I))^2} \int_{S_n} \frac{K\left(\frac{1-|\zeta|}{2^n\ell(I)}\right)|\mu_f(\zeta)|^2}{(1-|\zeta|)^2}\, dA(\zeta) \\ &\leq C\sup_I \int_{S_{\mathbf{D}}(I)} \frac{|\mu_f(\zeta)|^2}{(1-|\zeta|^2)^2} K\left(\frac{1-|\zeta|}{\ell(I)}\right) dA(\zeta).\end{aligned}$$

Combining our estimates for P_1 and P_2, we obtain

$$\begin{aligned}&\sup_I \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} (|z|^2-1)^2|S_f(z)|^2 K\left(\frac{|z|-1}{\ell(I)}\right) dA(z) \\ &\leq C\sup_I \int_{S_{\mathbf{D}}(I)} \frac{|\mu_f(\zeta)|^2}{(1-|\zeta|^2)^2} K\left(\frac{1-|\zeta|}{\ell(I)}\right) dA(\zeta).\end{aligned}$$

Therefore, if $|\mu_f(z)|^2(1-|z|^2)^{-2}dA(z)$ is a K-Carleson measure on $\mathbf{D}$, then $(|z|^2 - 1)^2|S_f(z)|^2dA(z)$ is a K-Carleson measure on $\mathbf{C}\setminus\overline{\mathbf{D}}$.

Step 2. We will prove that if

$$|\mu_f(z)|^2(|z|^2-1)^{-2}\,dA(z)$$

is a K-Carleson measure on $\mathbf{C}\setminus\overline{\mathbf{D}}$, then

$$(1-|z|^2)^2|S_f(z)|^2\,dA(z)$$

is a K-Carleson measure on $\mathbf{D}$.

It is well known that, for all univalent functions f,

$$\sup_{z\in\mathbf{D}}(1-|z|^2)^2|S_f(z)|\le 6.$$

For $I\subset\mathbf{T}$, if $\ell(I)>\frac{1}{3}$, we have

$$\begin{aligned}
&\int_{S_{\mathbf{D}}(I)}(1-|z|^2)^2|S_f(z)|^2K\left(\frac{1-|z|}{\ell(I)}\right)dA(z)\\
&\le\int_{\{z\in S_{\mathbf{D}}(I):|z|\le\frac{3}{4}\}}(1-|z|^2)^2|S_f(z)|^2K\left(\frac{1-|z|}{\ell(I)}\right)dA(z)\\
&\quad+\int_{\{z\in S_{\mathbf{D}}(I):|z|>\frac{3}{4}\}}(1-|z|^2)^2|S_f(z)|^2K\left(\frac{1-|z|}{\ell(I)}\right)dA(z)\\
&\le 36\int_{\{z\in S_{\mathbf{D}}(I):|z|\le\frac{3}{4}\}}(1-|z|^2)^{-2}K\left(\frac{1-|z|}{\ell(I)}\right)dA(z)\\
&\quad+8\pi\sup_{J\subset\mathbf{T},\ell(J)\le\frac{1}{4}}\int_{S_{\mathbf{D}}(J)}(1-|z|^2)^2|S_f(z)|^2K\left(\frac{1-|z|}{\ell(J)}\right)dA(z)\\
&\le C+8\pi\sup_{J\subset\mathbf{T},\ell(J)\le\frac{1}{4}}\int_{S_{\mathbf{D}}(J)}(1-|z|^2)^2|S_f(z)|^2K\left(\frac{1-|z|}{\ell(J)}\right)dA(z).
\end{aligned}$$

Thus, it suffices to consider the case $\ell(I)\le\frac{1}{3}$. Let $z\in S_{\mathbf{D}}(I)$ and then $g(z)=\frac{1}{z}\in S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I')$ where $\ell(I')=2\ell(I)$. If $z\in S_{\mathbf{D}}(I')$, then $g(z)\in S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I'')$ where $\ell(I'')=6\ell(I)$ and I'' has the same middle point with I. Clearly, $S_{\mathbf{D}}(I)$ and $S_{\mathbf{D}}(I')$ do not contain the center of $\mathbf{D}$. By *Step 1*,

$$\begin{aligned}
&\int_{S_{\mathbf{D}}(I)}(1-|z|^2)^2|S_f(z)|^2K\left(\frac{1-|z|}{\ell(I)}\right)dA(z)\\
&\le\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I')}\left(1-\left|\frac{1}{w}\right|^2\right)^2\left|S_f\left(\frac{1}{w}\right)\right|^2K\left(\frac{1-\left|\frac{1}{w}\right|}{\ell(I')/2}\right)\frac{dA(w)}{|w|^4}\\
&=\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I')}(|w|^2-1)^2\left|S_f\left(\frac{1}{w}\right)\left(\left(\frac{1}{w}\right)'\right)^2\right|^2K\left(\frac{|w|-1}{\ell(I')/2}\left|\frac{1}{w}\right|\right)dA(w)\\
&\le C\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I')}(|w|^2-1)^2|S_{f\circ g}(w)|^2K\left(\frac{|w|-1}{\ell(I')}\right)dA(w)\\
&\le C\sup_{I'}\int_{S_{\mathbf{D}}(I')}\frac{|\mu_{f\circ g}(\zeta)|^2}{(1-|\zeta|^2)^2}K\left(\frac{1-|\zeta|}{\ell(I')}\right)dA(\zeta).
\end{aligned}$$

Note that

$$\begin{aligned}\int_{S_{\mathbf{D}}(I')} \frac{|\mu_{f\circ g}(\zeta)|^2}{(1-|\zeta|^2)^2} K\left(\frac{1-|\zeta|}{\ell(I')}\right) dA(\zeta) &= \int_{S_{\mathbf{D}}(I')} \frac{|\mu_f(1/\zeta)|^2}{(1-|\zeta|^2)^2} K\left(\frac{1-|\zeta|}{\ell(I')}\right) dA(\zeta)\\ &\le \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I'')} \frac{|\mu_f(z)|^2}{\left(1-\left|\frac{1}{z}\right|^2\right)^2} K\left(\frac{1-\left|\frac{1}{z}\right|}{\ell(I'')/2}\right) \frac{dA(z)}{|z|^4}\\ &\le C \sup_{I''} \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I'')} \frac{|\mu_f(z)|^2}{(|z|^2-1)^2} K\left(\frac{|z|-1}{\ell(I'')}\right) dA(z).\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup_I \int_{S_{\mathbf{D}}(I)} (1-|z|^2)^2 |S_f(z)|^2 K\left(\frac{1-|z|}{\ell(I)}\right) dA(z)\\ &\le C + C \sup_{I''} \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I'')} \frac{|\mu_f(z)|^2}{(|z|^2-1)^2} K\left(\frac{|z|-1}{\ell(I'')}\right) dA(z).\end{aligned}$$

We have proved that if $\log f' \in \mathcal{T}_K$, then

$$(1-|z|^2)^2 |S_f(z)|^2 \, dA(z)$$

is a K-Carleson measure on $\mathbf{D}$. By Theorem C, $\log f' \in \mathcal{Q}_K$. The proof of Theorem 1.1 is complete. □

4. Proof of Theorem 1.2

Before embarking into the details of our proof, let us recall that $\overline{\partial} f = \mu \partial f$ is called a Beltrami equation if μ is measurable and $\|\mu\|_\infty < 1$, where μ is called Beltrami coefficient.

Proof of Theorem 1.2. If $\log f' \in \mathcal{T}_K$, then f has an extension F with

$$|\mu_F(z)|^2 (|z|^2-1)^{-2} \, dA(z)$$

is a K-Carleson measure on $\mathbf{C} \setminus \overline{\mathbf{D}}$. Moreover, $\partial f(\mathbf{D})$ is a quasicircle. For the convenience of calculating, we assume that $\infty \in \partial f(\mathbf{D})$. Otherwise, it involves a Möbius transformation. By Theorem I.6.1 and Lemma I.6.2 in [10], $\partial f(\mathbf{D})$ admits a quasiconformal reflection which is defined as

$$\lambda(z) = F\left(1/\overline{f^{-1}(z)}\right), \quad z \in f(\mathbf{D}).$$

Hence $F\left(\frac{1}{\bar z}\right) = \lambda \circ f(z)$.

For an analytic function ψ on $\mathbf{D}$, set $\phi(z) = \int_0^z e^{\psi(w)} \, dw$. Then $\psi = \log \phi'$. It means that any analytic function has the form of $\log \phi'$. Since $\log f' \in \mathcal{T}_K$, by Theorem 1.1, $\log f' \in \mathcal{Q}_K$. For some enough small $\varepsilon > 0$, set

$$O = \{\log g' \colon \|\log g' - \log f'\|_{\mathcal{Q}_K} < \varepsilon\}.$$

To show $\mathcal{T}_K$ is open in $\mathcal{Q}_K$, it suffices to show $O \subset \mathcal{T}_K$. Clearly, if $\log g' \in O$, then $\log g' \in \mathcal{Q}_K$. Write $h = g \circ f^{-1}$. Hence,

$$\begin{aligned}
\|S_h\|_{f(\mathbf{D})} &= \sup_{z\in f(\mathbf{D})} |S_h(z)| \frac{(1-|f^{-1}(z)|^2)^2}{|(f^{-1}(z))'|^2} \\
&= \sup_{z\in f(\mathbf{D})} |S_{g\circ f^{-1}}(z) - S_{f\circ f^{-1}}(z)| \frac{(1-|f^{-1}(z)|^2)^2}{|(f^{-1}(z))'|^2} \\
&= \sup_{z\in f(\mathbf{D})} |S_g(f^{-1}(z)) - S_f(f^{-1}(z))||(f^{-1}(z))'|^2 \frac{(1-|f^{-1}(z)|^2)^2}{|(f^{-1}(z))'|^2} \\
&= \sup_{z\in \mathbf{D}} |S_g(z) - S_f(z)|(1-|z|^2)^2 \\
&\le \sup_{z\in \mathbf{D}} |(\log g' - \log f')''|(1-|z|^2)^2 \\
&\quad + \frac{1}{2}\sup_{z\in \mathbf{D}} |((\log g')')^2 - ((\log f')')^2|(1-|z|^2)^2.
\end{aligned}$$

By Lemma 1.3 in [12], we have $(1-|z|^2)|(\log f')'| \le 6$ since f is conformal on $\mathbf{D}$. Thus,

$$\begin{aligned}
\|\log g' + \log f'\|_{\mathcal{B}} &\le \|\log g' - \log f'\|_{\mathcal{B}} + 2\|\log f'\|_{\mathcal{B}} \\
&\le C\|\log g' - \log f'\|_{\mathcal{Q}_K} + 12 \le C\varepsilon + 12.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|S_h\|_{f(\mathbf{D})} &\le C\|\log g' - \log f'\|_{\mathcal{B}} + (C\varepsilon + 12)\sup_{z\in\mathbf{D}} |(\log g')' - (\log f')'|(1-|z|^2) \\
&\le (C\varepsilon + C + 12)\|\log g' - \log f'\|_{\mathcal{B}} \\
&\le (C\varepsilon + C)\|\log g' - \log f'\|_{\mathcal{Q}_K} \le (C\varepsilon + C)\varepsilon.
\end{aligned}$$

Note that ε is enough small. By Theorem II.4.1 in [10], h is conformal in $f(\mathbf{D})$ and there exists an extension H of h to the complex plane $\mathbf{C}$ with

$$|\mu_H(\lambda(z))| = |\mu_H(\zeta)| = \left|\frac{\overline{\partial} H(\zeta)}{\partial H(\zeta)}\right| = \left|\frac{S_h(z)(\zeta - z)^2\overline{\partial}\omega(\zeta)}{2 + S_h(z)(\zeta - z)^2\partial\omega(\zeta)}\right|$$

for all $z \in f(\mathbf{D})$, where $\zeta = \lambda(z)$ and $\omega = \lambda^{-1}$. Since $\infty \in \partial f(\mathbf{D})$, by formulas (I.6.1) and (I.6.4) in [10], we have

$$|\zeta - \omega(\zeta)| \le C\frac{1-|(f^{-1}(\omega(\zeta)))|^2}{|(f^{-1}(\omega(\zeta)))'|}$$

and

$$|\partial\omega(\zeta)| \le C.$$

Hence,

$$\begin{aligned}
|\mu_H(\lambda(z))| &\le \frac{|S_h(z)(\zeta - z)^2\overline{\partial}\omega(\zeta)|}{2 - |S_h(z)(\zeta - z)^2\partial\omega(\zeta)|} = \frac{|S_h(z)||\zeta - \omega(\zeta)|^2|\mu_\omega(\zeta)||\partial\omega(\zeta)|}{2 - |S_h(\omega(\zeta))||\zeta - \omega(\zeta)|^2|\partial\omega(\zeta)|} \\
&\le \frac{C|S_h(z)|\frac{(1-|f^{-1}(\omega(\zeta))|^2)^2}{|(f^{-1}(\omega(\zeta)))'|^2}}{2 - C\|S_h\|_{f(\mathbf{D})}} \le C|S_h(z)|\frac{(1-|f^{-1}(z)|^2)^2}{|(f^{-1}(z))'|^2}
\end{aligned}$$

for all $z \in f(\mathbf{D})$. Therefore, $g = h \circ f$ is conformal on $\mathbf{D}$ and has a quasiconformal extension $G = H \circ F$ with

$$|\mu_G| = \left| \frac{\mu_F + \mu_H(F)\overline{(\partial F)}/\partial F}{1 + \overline{\mu_F}\mu_H(F)\overline{(\partial F)}/\partial F} \right| \le \frac{|\mu_F| + |\mu_H(F)|}{1 - |\mu_F||\mu_H(F)|} \le C(|\mu_F| + |\mu_H(F)|),$$

where C depends only on $\|\mu_F\|_\infty$ and $\|\mu_H\|_\infty$. Since

$$\begin{aligned}|\mu_H(F(1/\bar z))| &= |\mu_H(\lambda \circ f(z))| \le C|S_h(f(z))| \frac{(1 - |f^{-1}(f(z))|^2)^2}{|(f^{-1})'(f(z))|^2} \\ &= C|S_{g\circ f^{-1}}(f(z)) - S_{f\circ f^{-1}}(f(z))| \frac{(1-|z|^2)^2}{|(f^{-1})'(f(z))|^2} \\ &= C|S_g(z) - S_f(z)|(1-|z|^2)^2,\end{aligned}$$

we have

$$|\mu_H(F(1/\bar z))|^2(1-|z|^2)^{-2} \le C|S_g(z) - S_f(z)|^2(1-|z|^2)^2.$$

Since $\log g'$ and $\log f'$ belong to $\mathcal{Q}_K$, by Theorem C, $|S_g(z) - S_f(z)|^2(1-|z|^2)^2 dA(z)$ is a K-Carleson measure on $\mathbf{D}$. Then $|\mu_H(F(1/\bar z))|^2(1-|z|^2)^{-2}dA(z)$ is also a K-Carleson measure on $\mathbf{D}$. Hence, for any arc I,

$$\begin{aligned}&\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} \frac{|\mu_H(F(\zeta))|^2}{(|\zeta|^2-1)^2} K\left(\frac{|\zeta|-1}{\ell(I)}\right) dA(\zeta) \\ &\le C \int_{S_{\mathbf{D}}(I)} \frac{|\mu_H(F(1/\bar z))|^2}{(|1/\bar z|^2-1)^2} K\left(\frac{1-|z|}{\ell(I)}\right) \frac{dA(z)}{|z|^4} \\ &= C \int_{S_{\mathbf{D}}(I)} \frac{|\mu_H(F(1/\bar z))|^2}{(1-|z|^2)^2} K\left(\frac{1-|z|}{\ell(I)}\right) dA(z) \le C,\end{aligned}$$

which deduces that $|\mu_H(F(z))|^2(|z|^2-1)^{-2}\, dA(z)$ is a K-Carleson measure on $\mathbf{C}\setminus\overline{\mathbf{D}}$. Therefore, $|\mu_G(z)|^2(|z|^2-1)^{-2}\, dA(z)$ is a K-Carleson measure on $\mathbf{C}\setminus\overline{\mathbf{D}}$. Thus $\log g' \in \mathcal{T}_K$ and $\mathcal{T}_K$ is open in $\mathcal{Q}_K$.

Now we consider the connectivity of $\mathcal{T}_K$. As the first step, let μ be a Beltrami coefficient with $\|\mu\|_\infty < 1$ and vanishing outside the unit disk $\mathbf{D}$. Then there exists a unique mapping $f = f^\mu$ which is conformal in $\mathbf{C}\setminus\overline{\mathbf{D}}$ with expansion $f(z) = z + b_1 z^{-1} + \cdots$ at ∞ and satisfies Beltrami equation $\overline{\partial} f = \mu\partial f$ in $\mathbf{D}$. Then $\partial f - 1 = H(\overline{\partial} f) = H(\mu\partial f)$, where H is the Hilbert transformation. Since H is an isometry on $L^2(\mathbf{C})$,

$$\|H(\mu\partial f)\|_2 = \|\mu\partial f\|_2 \le \|\mu\|_\infty \|\partial f\|_2,$$

where $\|\cdot\|_2$ denotes the L^2-norm on $\mathbf{D}$. Suppose that μ is such a coefficient and $|\mu(z)|^2(1-|z|^2)^{-2}dA(z)$ is a K-Carleson measure on $\mathbf{D}$. For convenience, denote by $g = f^{t\mu}$, $h = f^{s\mu}$, $0 \le s, t \le 1$. Checking the proof of Theorem 2 in [4], we have

$$S_g(z) - S_h(z) = -\frac{6}{\pi}(|z|^2-1)^{-2} \int_{\mathbf{D}} (\mu_{g\circ B}(\zeta)\partial\Phi_1(\zeta) - \mu_{h\circ B}(\zeta)\partial\Phi_2(\zeta))\, dA(\zeta),$$

where B is the Möbius transformation of $\mathbf{C}\setminus\overline{\mathbf{D}}$ sending ∞ to z, Φ_1 and Φ_2 are conformal on $\mathbf{C}\setminus\overline{\mathbf{D}}$, and $\overline{\partial}\Phi_1=\mu_{g\circ B}\partial\Phi_1$, $\overline{\partial}\Phi_2=\mu_{h\circ B}\partial\Phi_2$. We show that

$$\begin{aligned}\|\partial\Phi_1-\partial\Phi_2\|_2&=\|H(\mu_{g\circ B}\partial\Phi_1)-H(\mu_{h\circ B}\partial\Phi_2)\|_2=\|\mu_{g\circ B}\partial\Phi_1-\mu_{h\circ B}\partial\Phi_2\|_2\\&\le\|\mu_{g\circ B}\|_\infty\|\partial\Phi_1-\partial\Phi_2\|_2+\|\partial\Phi_2\|_2\|\mu_{g\circ B}-\mu_{h\circ B}\|_\infty\\&=\|\mu_g\|_\infty\|\partial\Phi_1-\partial\Phi_2\|_2+\|\partial\Phi_2\|_2\|\mu_g-\mu_h\|_\infty\\&=t\|\mu\|_\infty\|\partial\Phi_1-\partial\Phi_2\|_2+\|\partial\Phi_2\|_2|t-s|\|\mu\|_\infty.\end{aligned}$$

By the proof of Koebe area theorem in [10],

$$\begin{aligned}\int_{\mathbf{D}}|\partial\Phi_2(z)|^2dA(z)&\le(1-\|\mu_{h\circ B}\|_\infty^2)^{-1}\int_{\mathbf{D}}J_{\Phi_2}(z)\,dA(z)\\&=(1-\|\mu_h\|_\infty^2)^{-1}\int_{\Phi_2(\mathbf{D})}dA(z)\le\pi(1-\|\mu\|_\infty^2)^{-1},\end{aligned}$$

where J_{Φ_2} is the Jacobian of Φ_2. Thus

$$\|\partial\Phi_1-\partial\Phi_2\|_2\le\frac{\|\partial\Phi_2\|_2\|\mu\|_\infty}{1-t\|\mu\|_\infty}|t-s|\le C|t-s|,$$

where C depends only on μ. Therefore,

$$\begin{aligned}|S_g(z)-S_h(z)|^2&=\frac{36}{\pi^2}(|z|^2-1)^{-4}\left|\int_{\mathbf{D}}(\mu_{g\circ B}(\zeta)\partial\Phi_1(\zeta)-\mu_{h\circ B}(\zeta)\partial\Phi_2(\zeta))\,dA(\zeta)\right|^2\\&\le\frac{72}{\pi^2}(|z|^2-1)^{-4}\left\{\int_{\mathbf{D}}|\mu_{g\circ B}(\zeta)-\mu_{h\circ B}(\zeta)||\partial\Phi_1(\zeta)|\,dA(\zeta)\right\}^2\\&\quad+\frac{72}{\pi^2}(|z|^2-1)^{-4}\left\{\int_{\mathbf{D}}|\mu_{h\circ B}(\zeta)||\partial\Phi_1(\zeta)-\partial\Phi_2(\zeta)|\,dA(\zeta)\right\}^2\\&\le\frac{72}{\pi^2}(|z|^2-1)^{-4}\int_{\mathbf{D}}|\mu_{g\circ B}(\zeta)-\mu_{h\circ B}(\zeta)|^2\,dA(\zeta)\int_{\mathbf{D}}|\partial\Phi_1(\zeta)|^2\,dA(\zeta)\\&\quad+\frac{72}{\pi^2}(|z|^2-1)^{-4}\int_{\mathbf{D}}|\mu_{h\circ B}(\zeta)|^2\,dA(\zeta)\int_{\mathbf{D}}|\partial\Phi_1(\zeta)-\partial\Phi_2(\zeta)|^2\,dA(\zeta)\\&\le C(|z|^2-1)^{-2}\left\{\int_{\mathbf{D}}\frac{|\mu_g(\zeta)-\mu_h(\zeta)|^2}{|\zeta-z|^4}\,dA(\zeta)+\|\partial\Phi_1-\partial\Phi_2\|_2^2\int_{\mathbf{D}}\frac{|\mu_h(\zeta)|^2}{|\zeta-z|^4}\,dA(\zeta)\right\}\\&\le C(|z|^2-1)^{-2}|t-s|^2\int_{\mathbf{D}}\frac{|\mu(\zeta)|^2}{|\zeta-z|^4}\,dA(\zeta)\\&=C(|z|^2-1)^{-2}\int_{\mathbf{D}}\frac{|\mu_g(\zeta)-\mu_h(\zeta)|^2}{|\zeta-z|^4}\,dA(\zeta).\end{aligned}$$

For any $I\subset\mathbf{T}$,

$$\begin{aligned}&\sup_I\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)}(|z|^2-1)^2|S_g(z)-S_h(z)|^2K\left(\frac{|z|-1}{\ell(I)}\right)dA(z)\\&\le C\sup_I\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)}\int_{\mathbf{D}}\frac{|\mu_g(\zeta)-\mu_h(\zeta)|^2}{|\zeta-z|^4}\,dA(\zeta)K\left(\frac{|z|-1}{\ell(I)}\right)dA(z).\end{aligned}$$

Next, let $\log f'\in\mathcal{T}_K$. Then f has a quasiconformal extension F in $\mathbf{C}$ and its complex dilatation $\mu=\mu_F$ satisfies that $|\mu(z)|^2(|z|^2-1)^{-2}\,dA(z)$ is a K-Carleson measure on $\mathbf{C}\setminus\overline{\mathbf{D}}$. If $f^{t\mu}$ is the mapping with $\overline{\partial}f^{t\mu}=t\mu\partial f^{t\mu}$ in $\mathbf{C}$ and $(f^{t\mu})^{-1}(\infty)=$

$f^{-1}(\infty)$, in our second step mainly is to prove that $t \to \log(f^{t\mu})'$, $0 \le t \le 1$, is a continuous path in $\mathcal{Q}_K$. We also write $g = f^{t\mu}$, $h = f^{s\mu}$. By [10] or [4],

$$\|\log g' - \log h'\|_{\mathcal{B}} \le C|t-s|.$$

Since $|\mu(z)|^2(|z|^2-1)^{-2}\,dA(z)$ is a K-Carleson measure on $\mathbf{C}\setminus\overline{\mathbf{D}}$, a similar technique of Step 2 in the proof of Theorem 1.1 shows that $(1-|z|^2)^2|S_g(z)-S_h(z)|^2\,dA(z)$ is a K-Carleson measure on $\mathbf{D}$. We give some details as follows. Note that

$$\begin{aligned}
&(1-|z|^2)^2|S_g(z)-S_h(z)| \\
&= (1-|z|^2)^2\left|\left(\frac{g''}{g'}\right)'(z) - \left(\frac{h''}{h'}\right)'(z) - \frac{1}{2}\left(\left(\frac{g''}{g'}\right)^2(z) - \left(\frac{h''}{h'}\right)^2(z)\right)\right| \\
&\le (1-|z|^2)^2\left|(\log g')''(z) - (\log h')''(z)\right| \\
&\quad + \frac{1}{2}(1-|z|^2)^2\left|((\log g'(z))')^2 - ((\log h'(z))')^2\right| \\
&\le C\|\log g' - \log h'\|_{\mathcal{B}} + C\|\log g' - \log h'\|_{\mathcal{B}}(1-|z|^2)\left(|(\log g')'| + |(\log h')'|\right) \\
&\le C\|\log g' - \log h'\|_{\mathcal{B}} \le C|t-s|.
\end{aligned}$$

If $\ell(I) > \frac{1}{3}$, then

$$\begin{aligned}
&\int_{|z|\le\frac{3}{4}} (1-|z|^2)^2|S_g(z)-S_h(z)|^2 K\left(\frac{1-|z|}{\ell(I)}\right) dA(z) \\
&\le C\int_{|z|\le\frac{3}{4}} (1-|z|^2)^2|S_g(z)-S_h(z)|^2\,dA(z) \\
&\le C|t-s|^2\int_{|z|\le\frac{3}{4}} (1-|z|^2)^{-2}\,dA(z) \le C|t-s|^2.
\end{aligned}$$

Thus, for $\ell(I) > \frac{1}{3}$,

$$\begin{aligned}
&\int_{S_{\mathbf{D}}(I)} (1-|z|^2)^2|S_g(z)-S_h(z)|^2 K\left(\frac{1-|z|}{\ell(I)}\right) dA(z) \\
&\le \int_{\{z\in S_{\mathbf{D}}(I):|z|\le\frac{3}{4}\}} (1-|z|^2)^2|S_g(z)-S_h(z)|^2 K\left(\frac{1-|z|}{\ell(I)}\right) dA(z) \\
&\quad + \int_{\{z\in S_{\mathbf{D}}(I):|z|>\frac{3}{4}\}} (1-|z|^2)^2|S_g(z)-S_h(z)|^2 K\left(\frac{1-|z|}{\ell(I)}\right) dA(z) \\
&\le C|t-s|^2 + 8\pi \sup_{J\subset\mathbf{T},\ell(J)\le\frac{1}{4}} \int_{S_{\mathbf{D}}(J)} (1-|z|^2)^2|S_g(z)-S_h(z)|^2 K\left(\frac{1-|z|}{\ell(J)}\right) dA(z).
\end{aligned}$$

For $\ell(I) \le \frac{1}{3}$, using the first step and checking the proof of Theorem 1.1, we have

$$\begin{aligned}
&\int_{S_{\mathbf{D}}(I)} (1-|z|^2)^2|S_g(z)-S_h(z)|^2 K\left(\frac{1-|z|}{\ell(I)}\right) dA(z) \\
&\le C\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} (|w|^2-1)^2|S_{g\circ\psi}(w)-S_{h\circ\psi}(w)|^2 K\left(\frac{|w|-1}{\ell(I)}\right) dA(w) \\
&\le C\sup_I \int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)} \int_{\mathbf{D}} \frac{|\mu_{g\circ\psi}(\zeta)-\mu_{h\circ\psi}(\zeta)|^2}{|\zeta-w|^4}\,dA(\zeta) K\left(\frac{|w|-1}{\ell(I)}\right) dA(w)
\end{aligned}$$

$$
\begin{aligned}
&\le C\sup_{I}\int_{S_{\mathbf{D}}(I)}\frac{|\mu_{g\circ\psi}(\zeta)-\mu_{h\circ\psi}(\zeta)|^2}{(1-|\zeta|^2)^2}K\left(\frac{1-|\zeta|}{\ell(I)}\right)dA(\zeta)\\
&\le C\sup_{I\subset\mathbf{T}}\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)}\frac{|\mu_g(z)-\mu_h(z)|^2}{(|z|^2-1)^2}K\left(\frac{|z|-1}{\ell(I)}\right)dA(z)\\
&= C|t-s|^2\sup_{I\subset\mathbf{T}}\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)}\frac{|\mu(z)|^2}{(|z|^2-1)^2}K\left(\frac{|z|-1}{\ell(I)}\right)dA(z)\le C|t-s|^2,
\end{aligned}
$$

where $\psi(z)=\frac{1}{z}$. Therefore,

$$
\sup_{I\subset\mathbf{T}}\int_{S_{\mathbf{D}}(I)}(1-|z|^2)^2|S_g(z)-S_h(z)|^2K\left(\frac{1-|z|}{\ell(I)}\right)dA(z)\le C|t-s|^2.
$$

By Corollary 3.2 in [7], we have

$$
\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}(1-|z|^2)^2|S_g(z)-S_h(z)|^2K\left(1-|\varphi_a(z)|^2\right)dA(z)\le C|t-s|^2.
$$

Thus, for any $a\in\mathbf{D}$,

$$
\begin{aligned}
&\int_{\mathbf{D}}|(\log g'-\log h')'(z)|^2K(g(z,a))\,dA(z)\\
&\le C\int_{\mathbf{D}}|(\log g'-\log h')'(z)|^2K(1-|\varphi_a(z)|^2)\,dA(z)\\
&\le C\int_{\mathbf{D}}(1-|z|^2)^2|(\log g'-\log h')''(z)|^2K(1-|\varphi_a(z)|^2)\,dA(z)\\
&\le C\int_{\mathbf{D}}(1-|z|^2)^2|S_g(z)-S_f(z)|^2K(1-|\varphi_a(z)|^2)\,dA(z)\\
&\quad+C\int_{\mathbf{D}}(1-|z|^2)^2\left|\left(\frac{g''}{g'}\right)^2(z)-\left(\frac{h''}{h'}\right)^2(z)\right|^2K(1-|\varphi_a(z)|^2)\,dA(z)\\
&\le C|t-s|^2+C\|\log g'-\log h'\|_{\mathcal{B}}^2\int_{\mathbf{D}}\left|\frac{g''}{g'}(z)+\frac{h''}{h'}(z)\right|^2K(1-|\varphi_a(z)|^2)\,dA(z)\\
&\le C|t-s|^2+C|t-s|^2\int_{\mathbf{D}}\left|\frac{g''}{g'}(z)+\frac{h''}{h'}(z)\right|^2K(1-|\varphi_a(z)|^2)\,dA(z).
\end{aligned}
$$

By the proofs of Theorem C and Theorem 1.1,

$$
\begin{aligned}
&\int_{\mathbf{D}}\left|\frac{g''}{g'}(z)+\frac{h''}{h'}(z)\right|^2K(1-|\varphi_a(z)|^2)\,dA(z)\\
&\le C+C\sup_{I\subset\mathbf{T}}\int_{S_{\mathbf{D}}(I)}(1-|z|^2)^2|S_g(z)|^2K\left(\frac{1-|z|}{\ell(I)}\right)dA(z)\\
&\quad+C\sup_{I\subset\mathbf{T}}\int_{S_{\mathbf{D}}(I)}(1-|z|^2)^2|S_h(z)|^2K\left(\frac{1-|z|}{\ell(I)}\right)dA(z)
\end{aligned}
$$

$$\leq C + C\sup_{I\subset\mathbf{T}}\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)}\frac{|\mu_g(z)|^2}{(|z|^2-1)^2}K\left(\frac{|z|-1}{\ell(I)}\right)dA(z)$$
$$+ C\sup_{I\subset\mathbf{T}}\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)}\frac{|\mu_h(z)|^2}{(|z|^2-1)^2}K\left(\frac{|z|-1}{\ell(I)}\right)dA(z)$$
$$\leq C + C\sup_{I\subset\mathbf{T}}\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)}\frac{|\mu(z)|^2}{(|z|^2-1)^2}K\left(\frac{|z|-1}{\ell(I)}\right)dA(z).$$

Therefore,

$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}|(\log g'-\log h')'(z)|^2K(g(z,a))\,dA(z)$$
$$\leq C|t-s|^2 + C|t-s|^2\sup_{I\subset\mathbf{T}}\int_{S_{\mathbf{C}\setminus\overline{\mathbf{D}}}(I)}\frac{|\mu(z)|^2}{(|z|^2-1)^2}K\left(\frac{|z|-1}{\ell(I)}\right)dA(z)$$
$$\leq C|t-s|^2,$$

where the constant C depends only on μ and K. We obtain that

$$\|\log g'-\log h'\|_{\mathcal{Q}_K}\leq C|t-s|;$$

that is, $t\to\log(f^{t\mu})'$, $0\leq t\leq 1$, is a continuous path in $\mathcal{Q}_K$. Thus, we have shown that each $\log f'\in\mathcal{T}_K$ can be connected with a path to an element $\log\psi'\in\mathcal{Q}_K$, where $\psi=f^{0\mu}$ is a Möbius transformation. If $\psi(\mathbf{D})$ is unbounded, then $f(\zeta)=\psi(\zeta)$ for some $\zeta\in\mathbf{T}$. If $\psi(\mathbf{D})$ is bounded, then $r\to\log\psi'(rz)$, joins $\log\psi'$ to $0\in\mathcal{Q}_K$ and we know that there is a continuous path joins $\log f'$ and 0. Hence $\mathcal{T}_{K,b}$ and each $\mathcal{T}_{K,\theta}$, $\theta\in[0,2\pi]$, are connected. Since elements in different classes cannot be joined even in the Bloch topology [22], we obtain that $\mathcal{T}_{K,b}$ and the $\mathcal{T}_{K,\theta}$ are the connected components of $\mathcal{T}_K$. The proof of Theorem 1.2 is complete.

5. Results on $\mathcal{Q}_{K,0}$ spaces

Denote by $\mathcal{Q}_{K,0}$ the space of analytic functions f in $\mathbf{D}$ such that

$$\lim_{|a|\to 1}\int_{\mathbf{D}}|f'(z)|^2K(g(z,a))\,dA(z)=0.$$

By [6], $\mathcal{Q}_{K,0}$ is contained in the little Bloch space $\mathcal{B}_0$, which is defined as follows:

$$\mathcal{B}_0=\{f\in H(\mathbf{D})\colon \lim_{|z|\to 1}(1-|z|^2)|f'(z)|=0\}.$$

Moreover, a K-Carleson measure ν is vanishing if

$$\lim_{\ell(I)\to 0}\int_{S_G(I)}K\left(\frac{|1-|z||}{\ell(I)}\right)d\nu(z)=0.$$

Let f be conformal on $\mathbf{D}$. By classifying the Carleson boxes to *large* boxes, *bad* boxes and *father* boxes, Zhou proved Theorem C in [21]. Checking the proof of Theorem C, we find that the technique to prove (ii) ⇒ (i) in Theorem C in [21] can not be used to prove the similar result on $\mathcal{Q}_{K,0}$ spaces. This section is to present a short proof of the little version corresponding to Theorem C.

Theorem 5.1. *Let K satisfy (1.1). If f is conformal on $\mathbf{D}$, then the following are equivalent:*

(i) $\log f' \in \mathcal{Q}_{K,0}$;
(ii) $|S_f(z)|^2(1-|z|^2)^2\,dA(z)$ *is a vanishing K-Carleson measure on* $\mathbf{D}$.

Proof. Suppose $g = \log f' \in \mathcal{Q}_{K,0}$. Then both

$$|g'(z)|^2\,dA(z) \quad \text{and} \quad |g''(z)|^2(1-|z|^2)^2\,dA(z)$$

are vanishing K-Carleson measures (see [7] and [18]). Since $g \in \mathcal{Q}_{K,0} \subset \mathcal{B}$,

$$|g'(z)|^4(1-|z|^2)^2\,dA(z)$$

is also a vanishing K-Carleson measure. The facts above together with the inequality

$$|S_f(z)|^2 \le 2\Big(|g''(z)|^2 + \frac{1}{4}|g'(z)|^4\Big), \quad z \in \mathbf{D},$$

imply that $|S_f(z)|^2(1-|z|^2)^2dA(z)$ is a vanishing K-Carleson measure.

On the other hand, suppose that $|S_f(z)|^2(1-|z|^2)^2dA(z)$ is a vanishing K-Carleson measure on $\mathbf{D}$. First we will show that $g = \log f' \in \mathcal{B}_0$. For any $a \in \mathbf{D}$, let I be the arc with center $\frac{a}{|a|}$ and length $\ell(I) = 2(1-|a|)$. Note that $|S_f(z)|^2$ is a subharmonic function and for a fixed $r(0<r<1)$, the disk $E(a,r) = \{z\colon |z-a| < r(1-|a|)\}$ is contained in $S_{\mathbf{D}}(I)$. If $z \in E(a,r)$, then

$$(1-r)(1-|a|) \le 1-|z| \le (1+r)(1-|a|).$$

Therefore,

$$\begin{aligned} |S_f(a)|^2(1-|a|^2)^4 &\le C\int_{E(a,r)} |S_f(z)|^2(1-|z|^2)^2\,dA(z) \\ &\le C\int_{E(a,r)} |S_f(z)|^2(1-|z|^2)^2 K\left(\frac{1-|z|}{2(1-|a|)}\right) dA(z) \\ &\le C\int_{S_{\mathbf{D}}(I)} |S_f(z)|^2(1-|z|^2)^2 K\left(\frac{1-|z|}{\ell(I)}\right) dA(z), \end{aligned}$$

which deduces that $\lim_{|a|\to 1} |S_f(a)|(1-|a|^2)^2 = 0$. By Theorem 11.1 in [13], $g \in \mathcal{B}_0$.

Next, we prove that $g \in \mathcal{Q}_{K,0}$. Recall that $S_f = g'' - \frac{1}{2}(g')^2$, we have

$$\begin{aligned} I_a :&= \int_{\mathbf{D}} |g''(z)|^2(1-|z|^2)^2 K(g(z,a))\,dA(z) \\ &\le C\int_{\mathbf{D}} |g''(z)|^2(1-|z|^2)^2 K(1-|\varphi_a(z)|^2)\,dA(z) \\ &\le C\int_{\mathbf{D}} |S_f(z)|^2(1-|z|^2)^2 K(1-|\varphi_a(z)|^2)\,dA(z) \\ &\quad + C\int_{\mathbf{D}} |g'(z)|^4(1-|z|^2)^2 K(1-|\varphi_a(z)|^2)\,dA(z). \end{aligned}$$

Note that $g \in \mathcal{B}_0$. For any $\varepsilon > 0$, there exists $0 < r(\varepsilon) < 1$ such that if $|z| > r(\varepsilon)$, then $(1-|z|^2)|g'(z)| < \varepsilon$. Thus,

$$\begin{aligned}
&\int_{|z|>r(\varepsilon)} |g'(z)|^4(1-|z|^2)^2 K(1-|\varphi_a(z)|^2)\, dA(z) \\
&\le \varepsilon^2 \int_{|z|>r(\varepsilon)} |g'(z)|^2 K(1-|\varphi_a(z)|^2)\, dA(z) \\
&\le \varepsilon^2 C \int_{\mathbf{D}} |g''(z)|^2(1-|z|^2)^2 K(1-|\varphi_a(z)|^2)\, dA(z) \le \varepsilon^2 C I_a.
\end{aligned}$$

On the other hand, by Lemma 1.3 in [12],

$$\begin{aligned}
&\int_{|z|\le r(\varepsilon)} |g'(z)|^4(1-|z|^2)^2 K(1-|\varphi_a(z)|^2)\, dA(z) \\
&\le C \int_{|z|\le r(\varepsilon)} (1-|z|^2)^{-2} K(1-|\varphi_a(z)|^2)\, dA(z) \\
&\le CK\left(\frac{2(1-|a|^2)}{1-r(\varepsilon)}\right) \int_{|z|\le r(\varepsilon)} (1-|z|^2)^{-2}\, dA(z) \\
&\le CK\left(\frac{2(1-|a|^2)}{1-r(\varepsilon)}\right) \frac{1}{1-r(\varepsilon)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(1-\varepsilon^2 C) I_a \le C &\int_{\mathbf{D}} |S_f(z)|^2(1-|z|^2)^2 K(1-|\varphi_a(z)|^2)\, dA(z) \\
&+ CK\Big(\frac{2(1-|a|^2)}{1-r(\varepsilon)}\Big)\frac{1}{1-r(\varepsilon)}.
\end{aligned}$$

Fix ε such that $1-\varepsilon^2 C > 0$. Since $|S_f(z)|^2(1-|z|^2)^2\, dA(z)$ is a vanishing K-Carleson measure, by Corollary 3.2 in [7],

$$\lim_{|a|\to 1} \int_{\mathbf{D}} |S_f(z)|^2(1-|z|^2)^2 K(1-|\varphi_a(z)|^2)\, dA(z) = 0.$$

These facts together with $K(0) = 0$, we obtain that $I_a \to 0$ as $|a| \to 1$. By Corollary 3.2 in [7] again, $|g''(z)|^2(1-|z|^2)^2\, dA(z)$ is a vanishing K-Carleson measure which means that $g \in \mathcal{Q}_{K,0}$. □

6. Final remark

In fact, the little versions of Theorems 1.1 and 1.2 are also true. Let $\mathcal{T}_{K,0}$ denote the set of all functions $\log f'$ on $\mathbf{D}$ where f is conformal on $\mathbf{D}$ and admits a quasiconformal extension to $\mathbf{C}$ with a dilatation μ_f such that

$$|\mu_f(z)|^2(|z|^2-1)^{-2} dA(z)$$

is a vanishing K-Carleson measure on $\mathbf{C}\setminus\overline{\mathbf{D}}$. Let K satisfy (1.1) and (1.2). Checking the proof of Theorems 1.1 and 1.2, using Theorem 5.1, we obtain

Theorem 6.1. *Let K satisfy (1.1) and (1.2). Then $\mathcal{T}_{K,0}$ is a subset of $\mathcal{Q}_{K,0}$ space.*

Theorem 6.2. *Let K satisfy (1.1) and (1.2). Then $\mathcal{T}_{K,0}$ is open in $\mathcal{Q}_{K,0}$. Furthermore, $\mathcal{T}_{K,\,b,0} = \{\log f' \in \mathcal{T}_{K,0} \colon f(\mathbf{D})$ is bounded$\}$ and $\mathcal{T}_{K,\,\theta,0} = \{\log f' \in \mathcal{T}_{K,0} \colon f(e^{i\theta}) = \infty\}$, $\theta \in [0, 2\pi]$, are the connected components of $\mathcal{T}_{K,0}$.*

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