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HÖLDER CONTINUITY OF DEGENERATE p-HARMONIC FUNCTIONS

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Abstract. We prove a partial Hölder continuity result for finite energy solutions of degenerate elliptic equations. The function that measures the degeneracy of the problem is assumed to belong to a suitable Sobolev class. Moreover, we prove an analogous result for infinite energy solutions provided their gradients have a suitable degree of integrability.

1. Introduction

Let us consider the equation

$$\operatorname{div} A(x, Du) = 0 \tag{1.1}$$

in a bounded domain Ω of $\mathbf{R}^n$. We suppose that $A\colon \Omega \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies the following growth and ellipticity conditions

$$|A(x,\xi)| \le k(x)|\xi|^{p-1}, \tag{1.2}$$

$$\langle A(x,\xi),\xi\rangle \ge \frac{1}{k(x)}|\xi|^p \tag{1.3}$$

for almost every $x \in \Omega$, all $\xi \in \mathbf{R}^n$ and an exponent $1 < p \le n$.

We shall say that $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is a solution of the equation (1.1) if $A(x, Du)$ is locally integrable in Ω and u satisfies the equation in the sense of distributions, that is,

$$\int_\Omega \langle A(x,Du), D\varphi\rangle\, dx = 0$$

for every $\varphi \in C^\infty_0(\Omega)$. A function u will be called a locally finite energy solution if, in addition, $\langle A(x, Du), Du\rangle$ is locally integrable in Ω.

If k is bounded, the equation is uniformly elliptic and finite energy solutions are those belonging to the class $W^{1,p}_{\mathrm{loc}}(\Omega)$. The regularity of such solutions has been widely investigated and, for an exhaustive treatment of the argument, we refer the interested reader to [10, 11, 12] and references therein.

Anyway, we recall that the study of the continuity properties of the solutions started, in the case $n = 2$, with the pioneering papers by Morrey [19, 20] and, in higher dimensions, by De Giorgi [3] and Nash [18] and later on by Ladyzhenskaya and Ural'tseva [16].

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More recently, also the regularity of infinite energy solutions attracted the interest of many authors. It has been proven in [14, 17] that, if the gradient of a solution u is assumed to be integrable with an exponent sufficiently close to the natural exponent p, then it is actually a finite energy solution.

In the case k unbounded, the equation is degenerate elliptic and the study of the regularity of (finite and infinite energy) solutions started with the celebrated paper by Iwaniec and Sbordone ([15]) and, later on, developed in different directions by many authors (see for example [1, 8, 13, 21]).

It is worth pointing out that the higher integrability of the gradient of the solutions has been usually established assuming that the function k, which measures the degree of degeneracy of the equation, is exponentially integrable, that is $\exp(\beta k) \in L^1(\Omega)$, for some $\beta > 0$. In fact, in this case, finite energy solutions have a degree of integrability not too far from the natural one. More recently, the attention has been given also to the case k subexponentially integrable, i.e., $\exp(P(k)) \in L^1(\Omega)$, for a suitable Orlicz function P which is diverging at ∞ (see for example [6, 9]).

As far as we know, the continuity of finite energy solutions under the assumption of exponentially integrable degeneracy, conjectured by De Giorgi in [4], is still open. More precisely

Conjecture 1.1. (De Giorgi) *Let u be a finite energy solution of the equation*

$$\operatorname{div}(A(x)\nabla u) = 0 \quad \text{in } \Omega \subset \mathbf{R}^n,$$

where

$$\frac{1}{k(x)}|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq |\xi|^2$$

for almost every $x \in \Omega$ and all $\xi \in \mathbf{R}^n$. If

$$\exp(k(x)) \in L^1(\Omega),$$

then u is continuous.

Many authors gave their contribution to prove this conjecture, but the exponentially integrability assumption on the degeneracy k has been used together with stronger requirements. Among the others, we recall [22], where the continuity is obtained in the case $p = n = 2$ and [24], where the author proves the continuity of local solutions to the linear non uniformly elliptic equation of the type

$$\operatorname{div}(a(x)\nabla u) = 0$$

assuming that $a^p \in W^{1,n}$, $n \geq 3$, for some p opportunely related to n and $a(x) \geq a_0 > 0$.

In this paper we establish the Hölder continuity of finite energy solutions of (1.1), under the assumption

$$k \in W^{1,n}(\Omega). \tag{1.4}$$

In fact, our main result is the following

Theorem 1.2. *Let u be a finite energy solution of equation* (1.1) *and suppose that* (1.2) *and* (1.3) *hold for a function $k \in W^{1,n}(\Omega)$. Then there exists an open set $\Omega_0 \subset \Omega$ with full measure such that*

$$u \in C^{0,\alpha}(\Omega_0)$$

for every $\alpha \in (0, 1)$. Moreover, $\dim_H(\Omega \setminus \Omega_0) = 0$.

In order to prove Theorem 1.2, the idea is to combine the isoperimetric type inequality, obtained in a more general setting in [7], with the methods of [9]. It is well known that the solutions of equation (1.1) are Hölder continuous if the operator $A(x, \xi)$ depends continuously on the variable x. Therefore, the assumption $k \in W^{1,n}$ could not appear adequate. Actually, we take advantage from this assumption since, via the isoperimetric type inequality and from the Sobolev imbedding theorem on spheres, we deal with k as a continuous function on $\mathbf{R}^{n-1}$.

We pay the possible discontinuity of k restricting ourselves to the set Ω_0 where the mean value of k is bounded (compare with [2]). We'd like to point out that, with respect to the above mentioned papers, here we are able to deal with non linear equations in any dimension n.

Since infinite energy solutions of the equation (1.1), whose degeneracy $k(x)$ is exponentially integrable, are actually finite energy solutions provided their degree of integrability is not too far from L^p, as a consequence of Theorem 1.2 and by virtue of the results in [6, 8], we have the following

Corollary 1.3. *Let u be a infinite energy solution of equation* (1.1) *and suppose that* (1.2) *and* (1.3) *hold for a function $k \in W^{1,n}(\Omega)$. Then there exist a positive exponent $\beta_0 = \beta_0(p, n, ||k||_{W^{1,n}(\Omega)})$ and an open set $\Omega_0 \subset \Omega$ with full measure such that if*

$$Du \in \frac{L^p}{\log^\beta L}(\Omega)$$

for every $\beta > \beta_0$, then

$$u \in C^{0,\alpha}(\Omega_0)$$

for every $\alpha \in (0,1)$. Moreover $\dim_H(\Omega \setminus \Omega_0) = 0$.

We conclude by noting that the assumption $k \in W^{1,n}$ has been previously employed in [23] where an higher differentiability result for the gradient of the solution has been obtained. Therefore, the Hölder continuity can be deduced from the result of [23] by the use of Sobolev imbedding theorem, if p and n are opportunely related.

2. Notation and preliminary results

In this section we recall some standard definitions and collect several results that we will use to establish our main result. We indicate with $B_R(x) \equiv B(x, R)$ the open ball $\{y \in \mathbf{R}^n : |x - y| < R\}$ centered at the point $x \in \mathbf{R}^n$ and having radius $R > 0$. We omit the center of the ball when no confusion arises. All the balls considered are concentric, unless differently specified.

If u is an integrable function defined on $B_R(x)$, we indicate with

$$u_{x,R} = \fint_{B_R(x)} u(x)\,dx = \frac{1}{\omega_n R^n} \int_{B_R(x)} u(x)\,dx$$

the integral average of the function u over the ball $B_R(x)$, where ω_n is the Lebesgue measure of $B_1(0)$. We also adopt the convention of writing u_R instead of $u_{x,R}$ when the center of the ball is clear from the context.

Next result is a technical iteration lemma proven in [10, Lemma 2.1].

Lemma 2.1. *Let $\bar{R}$, a, b, α, β be positive constants with $\beta < \alpha$. There exist $A, \bar{t} > 0$ such that if f is a nonnegative, nondecreasing function such that*

$$f(\rho) \le a\left[\left(\frac{\rho}{R}\right)^{\alpha} + t\right] f(R) + bR^{\beta}$$

for some $t < \bar{t}$ and for all $0 < \rho < R \le \bar{R}$, then we have

$$f(\rho) \le A\left(\frac{\rho}{R}\right)^{\beta}\left[f(R) + bR^{\beta}\right]$$

for every $0 < \rho < R \le \bar{R}$.

We shall need the following Sobolev inequality on spheres as formulated by Gehring in [5].

Theorem 2.2. *Let k be a function in the Sobolev class $W^{1,n\vartheta}_{\mathrm{loc}}(\Omega)$, $\frac{n-1}{n} < \vartheta \le 1$. Then*

$$\sup_{\partial B_t} k - \inf_{\partial B_t} k \le c(n)t\left(\fint_{\partial B_t} |Dk|^{n\vartheta}\, d\mathcal{H}^{n-1}\right)^{\frac{1}{n\vartheta}} \tag{2.1}$$

for almost every radius $t \in (0, R)$, where $B_R \Subset \Omega$.

Next theorem relates the decay estimate for the gradient of a Sobolev function with its Hölder regularity properties (see [12, Theorem 7.19])

Theorem 2.3. (Morrey's Lemma) *Let $u \in W^{1,1}(\Omega)$ and suppose that there exist two positive constants K, α, with $\alpha \le 1$, such that*

$$\int_{B_r} |\nabla u|\, dx \le K r^{n-1+\alpha}$$

for all balls $B_r \subset \Omega$. Then $u \in C^{0,\alpha}(\Omega)$ and for every ball $B_r \subset \Omega$

$$\mathrm{osc}_{B_r} u \le CKr^{\alpha}.$$

Next result has been proven in [9, Lemma 3.5].

Lemma 2.4. *Let u be a finite energy solution of equation* (1.1) *and suppose that* (1.2) *and* (1.3) *hold for a function $k \in W^{1,n}(\Omega)$. Then we have*

$$\int_{\mathbf{R}^n} \langle A(x, Du), \eta Du\rangle\, dx \le \int_{\mathbf{R}^n} |A(x, Du)||D\eta||u - c|\, dx \tag{2.2}$$

for every $\eta \in C^{\infty}_0(\mathbf{R}^n)$ and for every constant c.

From previous lemma we deduce the following isoperimetric type inequality, already proven in a slightly different version in [9] (see also [7]). The proof is given for the reader's convenience.

Proposition 2.5. *Let u be a finite energy solution of equation* (1.1) *and suppose that* (1.2) *and* (1.3) *hold for a function $k \in W^{1,n}$. Then, for every $x_0 \in \Omega$,*

$$\int_{B(x_0,r)} \frac{1}{k(x)}|Du|^p\, dx \le c(n)\left(\int_{\partial B(x_0,r)} k(x)|Du|^{p-1}|u - u_{\partial B_r}|\, d\mathcal{H}^{n-1}\right) \tag{2.3}$$

for almost every radius $0 < r < \mathrm{dist}(x_0, \partial\Omega)$.

Proof. Let us set $B(x_0, t) = B_t$ and let us define on B_r the function

$$\eta_{\varepsilon}(x) = \min\left\{1, \frac{r - |x|}{\varepsilon}\right\}.$$

Choosing $\eta(x) = \eta_\varepsilon(x)$ in (2.2), we get

$$\int_{B_{r-\varepsilon}} \langle A(x,Du), Du\rangle \, dx \le \left| \int_{B_r \setminus B_{r-\varepsilon}} \langle A(x,Du), Du\rangle \frac{r-|x|}{\varepsilon}\, dx \right| + \int_{B_r \setminus B_{r-\varepsilon}} |A(x,Du)| \left| D\left(\frac{r-|x|}{\varepsilon}\right)\right| |u-c|\, dx$$

If we observe that $\dfrac{r-|x|}{\varepsilon} < 1$ in $B_r \setminus B_{r-\varepsilon}$ and that the second integral in the right hand side can be written as

$$\frac{1}{\varepsilon}\int_{r-\varepsilon}^{r} \left(\int_{\partial B_\rho} |A(x,Du)|\, |D(r-|x|)|\, |u-c|\, d\mathcal{H}^{n-1} \right) d\rho,$$

taking the limit as $\varepsilon \to 0$, we obtain

$$\int_{B_r} \langle A(x,Du), Du\rangle \, dx \le \int_{\partial B_r} |A(x,Du)||u-c|\, d\mathcal{H}^{n-1}.$$

Hence, by using assumption (1.2) in the right hand side and (1.3) in the left hand side of previous estimate and choosing $c = u_{\partial B_r}$, we obtain

$$\int_{B_r} \frac{1}{k(x)} |Du|^p \, dx \le c(n) \left(\int_{\partial B_r} k(x) |Du|^{p-1} |u - u_{\partial B_r}|\, d\mathcal{H}^{n-1} \right),$$

i.e., the conclusion. □

3. The proof of the main result

This section is devoted to the proof of Theorem 1.2. The starting point will consist in a decay estimate for the energy integral. In order to shorten the notation, in what follows we shall denote by

$$\mathcal{K}_R := \left(\int_{B_R} |Dk|^n \, dx \right)^{\frac{1}{n}} \quad \text{with } B_R \Subset \Omega.$$

Theorem 3.1. *Let u be a finite energy solution of equation* (1.1)*. Assume that* (1.2) *and* (1.3) *hold for a function $k \in W^{1,n}(\Omega)$. Let B_{2R} be a ball contained in the set*

$$\Omega_0 = \left\{ x \in \Omega \colon \limsup_{r\to 0} \fint_{B_r} k\, dx < +\infty \right\}. \tag{3.1}$$

Then there exist a constant $\tilde{A} > 0$ and a positive exponent $\beta = \beta(n, \mathcal{K}_R)$ such that

$$\int_{B_r} \frac{1}{k} |Du|^p \, dx \le \tilde{A} \left(\frac{r}{R}\right)^\beta \int_{B_R} \frac{1}{k} |Du|^p \, dx, \tag{3.2}$$

whenever $0 < r < R$.

Proof. For every $i \in \mathbf{N}$, let us consider the interval

$$\Delta_i := \left(\frac{R}{2^i}, \frac{R}{2^{(i-1)}} \right)$$

and the annulus

$$A_i := B_{\frac{R}{2^{(i-1)}}} \setminus B_{\frac{R}{2^i}}.$$

Let us define the sets $E_i = E_i^1 \cap E_i^2 \cap E_i^3$ with

$$E_i^1 := \left\{ t \in \Delta_i \colon \int_{\partial B_t} |Dk|^n \, d\mathcal{H}^{n-1} \le \frac{12}{|\Delta_i|} \int_{A_i} |Dk|^n \, dx \right\},$$
$$E_i^2 := \left\{ t \in \Delta_i \colon \int_{\partial B_t} k \, d\mathcal{H}^{n-1} \le \frac{12}{|\Delta_i|} \int_{A_i} k \, dx \right\},$$
$$E_i^3 := \left\{ t \in \Delta_i \colon \int_{\partial B_t} \left(\frac{1}{k(x)} |Du|^p \right)^{\frac{n-1}{n}} d\mathcal{H}^{n-1} \le \frac{12}{|\Delta_i|} \int_{A_i} \left(\frac{1}{k(x)} |Du|^p \right)^{\frac{n-1}{n}} dx \right\}.$$

By Fubini's Theorem we have that

$$|\mathcal{C}(E_i^1)| \le \frac{|\Delta_i|}{12}, \quad |\mathcal{C}(E_i^2)| \le \frac{|\Delta_i|}{12} \quad \text{and} \quad |\mathcal{C}(E_i^3)| \le \frac{|\Delta_i|}{12},$$

and therefore

$$|E_i| \ge \frac{|\Delta_i|}{4} > 0.$$

Choosing $r \in E_i$ so that inequalities (2.3) and (2.1) hold, we get

$$\begin{aligned}
\int_{B_{\frac{R}{2^i}}} \frac{1}{k(x)} |Du|^p \, dx &\le \int_{B_r} \frac{1}{k(x)} |Du|^p \, dx \le c(n) \int_{\partial B_r} k(x) |Du|^{p-1} |u - u_{\partial B_r}| \, d\mathcal{H}^{n-1} \\
&\le c(n) \sup_{\partial B_r} k(x) \int_{\partial B_r} |Du|^{p-1} |u - u_{\partial B_r}| \, d\mathcal{H}^{n-1} \\
&\le c(n) \sup_{\partial B_r} k(x) \left(\int_{\partial B_r} |Du|^{\frac{p(n-1)}{n}} \, d\mathcal{H}^{n-1} \right)^{\frac{p-1}{p} \frac{n}{n-1}} \\
&\quad \cdot \left(\int_{\partial B_r} |u - u_{\partial B_r}|^{\frac{p(n-1)}{n-p}} \, d\mathcal{H}^{n-1} \right)^{\frac{n-p}{p(n-1)}} \\
&\le c(n) \sup_{\partial B_r} k(x) \left(\int_{\partial B_r} |Du|^{\frac{p(n-1)}{n}} \, d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}},
\end{aligned} \tag{3.3}$$

where, in the last line, we also used Sobolev–Poincaré inequality on spheres for the function u. From estimate (3.3) we deduce that

$$\begin{aligned}
\int_{B_{\frac{R}{2^i}}} \frac{1}{k(x)} |Du|^p \, dx &\le c(n) \sup_{\partial B_r} k(x) \left(\int_{\partial B_r} |Du|^{\frac{p(n-1)}{n}} \, d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}} \\
&= c(n) \sup_{\partial B_r} k(x) \left(\int_{\partial B_r} k^{\frac{n-1}{n}} \left(\frac{1}{k} |Du|^p \right)^{\frac{n-1}{n}} d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}} \\
&\le c(n) \left(\sup_{\partial B_r} k(x) \right)^2 \left(\int_{\partial B_r} \left(\frac{1}{k} |Du|^p \right)^{\frac{n-1}{n}} d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}} \\
&\le c(n) \left(\sup_{\partial B_r} k(x) - \inf_{\partial B_r} k(x) \right)^2 \left(\int_{\partial B_r} \left(\frac{1}{k} |Du|^p \right)^{\frac{n-1}{n}} d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}} \\
&\quad + c(n) \left(\inf_{\partial B_r} k(x) \right)^2 \left(\int_{\partial B_r} \left(\frac{1}{k} |Du|^p \right)^{\frac{n-1}{n}} d\mathcal{H}^{n-1} \right)^{\frac{n}{n-1}}
\end{aligned} \tag{3.4}$$

Using Theorem 2.2 in (3.4), we get

$$\int_{B_{\frac{R}{2^i}}} \frac{1}{k(x)}|Du|^p\,dx \le c(n)r^{\frac{2}{n}}\left(\int_{\partial B_r}|Dk|^n d\mathcal{H}^{n-1}\right)^{\frac{2}{n}}\left(\int_{\partial B_r}\left(\frac{1}{k}|Du|^p\right)^{\frac{n-1}{n}} d\mathcal{H}^{n-1}\right)^{\frac{n}{n-1}}$$
$$+ c(n)\left(\fint_{\partial B_r} k\,d\mathcal{H}^{n-1}\right)^2\left(\int_{\partial B_r}\left(\frac{1}{k}|Du|^p\right)^{\frac{n-1}{n}} d\mathcal{H}^{n-1}\right)^{\frac{n}{n-1}},$$

where we used the obvious inequality

$$\inf_{\partial B_r} k(x) \le \fint_{\partial B_r} k(x)\,d\mathcal{H}^{n-1}.$$

Since $r \in E_i$, from previous inequality we obtain

$$\int_{B_{\frac{R}{2^i}}} \frac{1}{k(x)}|Du|^p\,dx \le c(n)\left[r^{\frac{2}{n}}\left(\frac{1}{|\Delta_i|}\int_{A_i}|Dk|^n\,dx\right)^{\frac{2}{n}} + \left(\frac{1}{r^{n-1}|\Delta_i|}\int_{A_i}k\,dx\right)^2\right]$$
$$\cdot\left(\frac{1}{|\Delta_i|}\int_{A_i}\left(\frac{1}{k}|Du|^p\right)^{\frac{n-1}{n}} dx\right)^{\frac{n}{n-1}}. \tag{3.5}$$

In order to shorten the notation, set

$$\Gamma := \left[r^{\frac{2}{n}}\left(\frac{1}{|\Delta_i|}\int_{A_i}|Dk|^n\,dx\right)^{\frac{2}{n}} + \left(\frac{1}{r^{n-1}|\Delta_i|}\int_{A_i}k\,dx\right)^2\right]$$

so that (3.5) can be written as

$$\int_{B_{\frac{R}{2^i}}} \frac{1}{k(x)}|Du|^p\,dx \le c(n)\Gamma\left(\frac{1}{|\Delta_i|}\int_{A_i}\left(\frac{1}{k}|Du|^p\right)^{\frac{n-1}{n}} dx\right)^{\frac{n}{n-1}}. \tag{3.6}$$

By using Hölder's inequality and the fact that, since

$$|A_i| = C(n)\frac{(2^n-1)R^n}{2^{ni}}, \quad |\Delta_i| = \frac{R}{2^i},$$

we have

$$\frac{|A_i|^{\frac{1}{n-1}}}{|\Delta_i|^{\frac{n}{n-1}}} = C(n),$$

from (3.6), we deduce that

$$\int_{B_{\frac{R}{2^i}}} \frac{1}{k(x)}|Du|^p\,dx \le c(n)\Gamma\left(\frac{1}{|\Delta_i|}\right)^{\frac{n}{n-1}}|A_i|^{\frac{1}{n-1}}\int_{A_i}\frac{1}{k}|Du|^p\,dx$$
$$\le c(n)\Gamma\frac{R}{2^i}\frac{1}{|\Delta_i|}\int_{A_i}\frac{1}{k}|Du|^p\,dx. \tag{3.7}$$

In order to estimate Γ, we take into account that

$$|\Delta_i| = \frac{R}{2^i} \quad\text{and}\quad \frac{R}{2^i} < r < \frac{R}{2^{i-1}}$$

and that, by virtue of the assumption $B_{2R} \subset \Omega_0$, we may suppose the existence of a constant $L > 1$ such that

$$\fint_{B_R} k\,dx \le L.$$

Hence, we have

$$\Gamma \leq 2^{\frac{2}{n}} \left(\int_{A_i} |Dk|^n \, dx \right)^{\frac{2}{n}} + 2^{2in} \left(\frac{1}{R^n} \int_{B_R} k \, dx \right)^2$$
$$\leq c(n,L) \left[\left(\int_{A_i} |Dk|^n \, dx \right)^{\frac{2}{n}} + 2^{2in} \right]. \tag{3.8}$$

Inserting estimate (3.8) in (3.7), it follows

$$\int_{B_{\frac{R}{2^i}}} \frac{1}{k(x)} |Du|^p \, dx \leq c(n,L) \frac{R2^{-i}}{|\Delta_i|} \left[\left(\int_{A_i} |Dk|^n \, dx \right)^{\frac{2}{n}} + 2^{2in} \right] \int_{A_i} \frac{1}{k} |Du|^p \, dx \tag{3.9}$$

Now, for $t \in \Delta_i$, we set

$$v_i(t) := \int_{B_{\frac{R}{2^i}}} \frac{1}{k(x)} |Du|^p \, dx + c(n,L) \frac{t - R2^{-i}}{|\Delta_i|} \int_{A_i} \frac{1}{k(x)} |Du|^p \, dx$$

and

$$v(t) := v_1(t) \chi_{\left[\frac{R}{2}, R\right]}(t) + \sum_{i=2}^{\infty} v_i(t) \chi_{\left[\frac{R}{2^i}, \frac{R}{2^{(i-1)}}\right)}(t).$$

Estimate (3.9) implies that

$$v_i(t) \leq c(n,L) \frac{1}{|\Delta_i|} \left\{ \frac{R}{2^i} \left[\left(\int_{A_i} |Dk|^n \, dx \right)^{\frac{2}{n}} + 2^{2in} \right] + t - \frac{R}{2^i} \right\} \int_{A_i} \frac{1}{k(x)} |Du|^p \, dx.$$

Since

$$v_i'(t) = c(n,L) \frac{1}{|\Delta_i|} \int_{A_i} \frac{1}{k(x)} |Du|^p \, dx,$$

we obtain

$$v_i(t) \leq \left\{ \frac{R}{2^i} \left[\left(\int_{A_i} |Dk|^n \, dx \right)^{\frac{2}{n}} + 2^{2in} \right] + t - \frac{R}{2^i} \right\} v_i'(t) \tag{3.10}$$

for all $t \in \Delta_i$. Since $t - \frac{R}{2^i} \leq \frac{R}{2^{(i-1)}} - \frac{R}{2^i} = \frac{R}{2^i}$, from (3.10) we deduce

$$v_i(t) \leq \frac{R}{2^i} \left[\left(\int_{A_i} |Dk|^n \, dx \right)^{\frac{2}{n}} + 2^{2in} + 1 \right] v_i'(t)$$
$$\leq \frac{2R}{2^i} \left[\left(\int_{B_R} |Dk|^n \, dx \right)^{\frac{2}{n}} + 2^{2in} \right] v_i'(t). \tag{3.11}$$

Using that $\frac{R}{2^i} \leq t \leq \frac{R}{2^{(i-1)}}$ in (3.11), it follows that

$$v_i(t) \leq 2^{2n+1} t \left[\mathcal{K}_R^2 + \frac{R^{2n}}{t^{2n}} \right] v_i'(t)$$

and hence, summing on i and observing that $v(t)$ is a piecewise affine function, we get

$$v(t) \leq 2^{2n+1} t \left[\mathcal{K}_R^2 + \frac{R^{2n}}{t^{2n}} \right] v'(t) \tag{3.12}$$

for every $0 < t < R$. Therefore

$$\frac{v'(t)}{v(t)} \geq \frac{1}{2^{2n+1}} \frac{t^{2n-1}}{\mathcal{K}_R^2 t^{2n} + R^{2n}}.$$

Now, we easily obtain for every $\rho < R$

$$\int_\rho^R \frac{v'(t)}{v(t)}\, dt \geq \frac{1}{2^{2n+1}} \int_\rho^R \frac{t^{2n-1}}{\mathcal{K}_R^2 t^{2n} + R^{2n}}\, dt = \frac{1}{n2^{2n+2}\mathcal{K}_R^2} \log \frac{\mathcal{K}_R^2 R^{2n} + R^{2n}}{\mathcal{K}_R^2 \rho^{2n} + R^{2n}}$$

and therefore

$$\log \frac{v(R)}{v(\rho)} \geq \frac{1}{n2^{2n+2}\mathcal{K}_R^2} \log \frac{\mathcal{K}_R^2 R^{2n} + R^{2n}}{\mathcal{K}_R^2 \rho^{2n} + R^{2n}} = \log \left(\frac{\mathcal{K}_R^2 R^{2n} + R^{2n}}{\mathcal{K}_R^2 \rho^{2n} + R^{2n}} \right)^{\alpha(n,R)}, \tag{3.13}$$

where we used the notation

$$\alpha(n,R) := \frac{1}{n2^{2n+2}\mathcal{K}_R^2}.$$

Inequality (3.13) yields

$$v(\rho) \leq \left(\frac{\rho^{2n}\mathcal{K}_R^2 + R^{2n}}{R^{2n}\mathcal{K}_R^2 + R^{2n}} \right)^{\alpha(n,R)} v(R) \leq c(n,R) \left(\left(\frac{\rho}{R} \right)^{2n\alpha(n,R)} + 1 \right) v(R), \tag{3.14}$$

for every $0 < \rho < R$. By Lemma 2.1 and estimate (3.14), we infer that there exists a positive constant A such that

$$v(\rho) \leq A \left(\frac{\rho}{R} \right)^\beta v(R) \tag{3.15}$$

for every $\rho < R$ and for every

$$\beta < 2n\alpha(n,R) = \frac{1}{2^{2n+1}\mathcal{K}_R^2}.$$

Since $\rho < R$, there exists $j \in \mathbf{N}$ such that $\rho \in [R2^{-j}, R2^{-j+1})$. Then, by the definition of the function $v(t)$, from (3.15) it follows

$$\int_{B_{\frac{R}{2^j}}} \frac{1}{k} |Du|^p\, dx + c(n,L) \frac{\rho - R2^{-j}}{|\Delta_j|} \left[\int_{A_j} \frac{1}{k} |Du|^p\, dx \right] \leq A \left(\frac{\rho}{R} \right)^\beta \int_{B_R} \frac{1}{k} |Du|^p\, dx$$

which obviously implies

$$\int_{B_{\frac{R}{2^j}}} \frac{1}{k} |Du|^p\, dx \leq A \left(\frac{\rho}{R} \right)^\beta \int_{B_R} \frac{1}{k} |Du|^p\, dx.$$

At this point, choosing $r = \frac{\rho}{2} < \frac{R}{2}$, we get

$$\int_{B_r} \frac{1}{k} |Du|^p\, dx \leq \tilde{A} \left(\frac{r}{R} \right)^\beta \int_{B_R} \frac{1}{k} |Du|^p\, dx,$$

which concludes the proof. □

Now, we are ready to embark in the core of the proof of our main result.

Proof of Theorem 1.2. Let us fix a ball B_{2R} contained in the set Ω_0 defined at (3.1). The assumption $k \in W^{1,n}(\Omega)$ implies, through the Sobolev imbedding

Theorem that $k \in L^q(\Omega)$, for every $q > 1$. Hence, for every $0 < r < \frac{R}{2}$, Hölder's inequality yields

$$\int_{B_r} |Du|\, dx \le \left(\int_{B_r} \frac{1}{k} |Du|^p\, dx \right)^{\frac{1}{p}} \left(\int_{B_r} k^{\frac{1}{p-1}} \right)^{\frac{p-1}{p}} \le c(||k||_{W^{1,n}(\Omega)}) \left(\int_{B_r} \frac{1}{k} |Du|^p\, dx \right)^{\frac{1}{p}}.$$

Therefore, by the assumption (1.3) and estimate (3.2), we obtain

$$\int_{B_r} |Du|\, dx \le C r^{\frac{\beta}{p}}.$$

In order to use Morrey's Lemma in Theorem 2.3, we need that β satisfies

$$0 < \frac{\beta}{p} - n + 1 \le 1 \qquad \Longleftrightarrow \qquad (n-1)p < \beta \le np.$$

Since $\beta < \frac{1}{2^{2n+1}\mathcal{K}_R^2}$, such choice is possible if

$$\mathcal{K}_R^2 < \frac{1}{p 2^{2n+1}(n-1)}.$$

In order to obtain the Hölder continuity, it is sufficient to choose a radius $R < R_0$ such that

$$\left(\int_{B_{R_0}} |Dk|^n\, dx \right)^{\frac{2}{n}} < \frac{1}{p 2^{2n+1}(n-1)},$$

that is possible thanks to the absolute continuity of the integral.

Observe now that, for a function belonging to the Sobolev space $W^{1,p}(\Omega)$, with $1 < p < n$, one has $\dim_H(\Omega \setminus \Omega_0) \le n - p$, where Ω_0 is the set defined at (3.1) (see for example Theorem 3.2 in [11]). Since $Dk \in L^n(\Omega)$, we have $\dim_H(\Omega \setminus \Omega_0) \le n - p$ for all $p < n$. Hence, taking the limit as p goes to n, we have $\dim_H(\Omega \setminus \Omega_0) = 0$. □

We conclude with the

Proof of Corollary 1.3. It suffices to observe that, through the classical Moser Trudinger inequality [12], the function k is exponentially integrable, i.e.,

$$\int_\Omega \exp(\lambda k^{\frac{n}{n-1}})\, \mathrm{d}x < +\infty$$

for some constant $\lambda > 0$ depending on the $W^{1,n}$-norm of k. By virtue of the results in [6, 8], we have that there exists a positive exponent $\beta_0 = \beta_0(p, n, ||k||_{W^{1,n}(\Omega)})$ such that, if

$$Du \in \frac{L^p}{\log^\beta L}(\Omega)$$

for every $\beta > \beta_0$, then u is a finite energy solution. Once we have that u is a finite energy solution, we may use Theorem 1.2 to conclude that there exists an open set $\Omega_0 \subset \Omega$ with full measure such that $u \in C^{0,\alpha}(\Omega_0)$, for every $\alpha \in (0, 1)$. □

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