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MODIFIED LENGTH SPECTRUM METRIC ON THE TEICHMÜLLER SPACE OF A RIEMANN SURFACE WITH BOUNDARY

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Abstract. Let S_0 be a bordered Riemann surface of finite type, and let $T(S_0)$ (resp. $T^R(S_0)$) be the Teichmüller space (resp. reduced Teichmüller space) of S_0. The length spectrum function defines a metric on $T^R(S_0)$ but not on $T(S_0)$. In this paper, we introduce a modified length spectrum function that does define a metric on $T(S_0)$. Then we show that if two points of $T(S_0)$ are close in the Teichmüller metric then they are close in the modified length spectrum metric, but the converse is not true. We also prove that $T(S_0)$ is not complete under this modified length spectrum metric.

1. Introduction

Let S_0 be a Riemann surface. A marked Riemann surface is a pair (S, f), where $f\colon S_0 \to S$ is a quasiconformal mapping. Two pairs (S_1, f_1) and (S_2, f_2) are equivalent if there exists a conformal mapping $c\colon S_1 \to S_2$ such that $c \circ f_1$ is homotopic to f_2. The reduced Teichmüller space $T^R(S_0)$ is the set of the equivalence classes $[S, f]$. Furthermore, $c \circ f_1$ is homotopic to f_2 relative to boundary if $c \circ f_1$ agrees with f_2 on the boundary and there is a homotopy between them such that it takes the same image at every point on the boundary when the other variable of the homotopy changes. The set of the equivalence classes $[S, f]$ under such a homotopy is called the Teichmüller space $T(S_0)$. Clearly, if S_0 has no boundary, then $T^R(S_0) = T(S_0)$.

The Teichmüller metric on $T^R(S_0)$ (resp. $T(S_0)$) is defined by

$$d_T([S_1, f_1], [S_2, f_2]) = \log K(f),$$

where f is an extremal quasiconformal mapping in the homotopy class (resp. the homotopy class relative to boundary) of $f_2 \circ f_1^{-1}$ and $K(f)$ represents the maximal dilation of f.

By comparing hyperbolic lengths of closed curves and their images, another metric, called the length spectrum metric, is defined on reduced Teichmüller spaces. Let S be a Riemann surface and Σ'_S a collection of nontrivial closed curves on S such that none of them is homotopic to a puncture and no two of them are homotopic to each other. We assume that Σ'_S is maximal in the sense that every nontrivial closed curve

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on S that is not homotopic to a puncture is homotopic to an element of Σ'_S. For any closed curve γ on S, let $l_S(\gamma)$ denote the length of the geodesic in the homotopy class of γ with respect to the hyperbolic metric. The length spectrum metric on $T^R(S_0)$ is defined by

$$d_L([S_1,f_1],[S_2,f_2]) = \log \sup_{\gamma\in\Sigma'_{S_1}} \left\{ \frac{l_{S_2}(f_2\circ f_1^{-1}(\gamma))}{l_{S_1}(\gamma)}, \frac{l_{S_1}(\gamma)}{l_{S_2}(f_2\circ f_1^{-1}(\gamma))} \right\}.$$

This metric was introduced and studied by Sorvali [9] in 1972. In 1975, Sorvali [10] proved that the two metrics d_T and d_L are metrically equivalent on the Teichmüller space of a torus and posed a question as to whether or not this is true on the Teichmüller space of an arbitrary Riemann surface. In 1986, Li [5] showed that the two metrics induce the same topology on the Teichmüller spaces of compact Riemann surfaces. In 1999, Liu [7] generalized Li's result to the Teichmüller spaces of Riemann surfaces of finite topological type. Then in 2003, Li [6] gave a negative answer to Sorvali's question. In the same year, Shiga [8] proved that if S_0 is a Riemann surface of infinite topological type admitting a pants decomposition that is both upper-bounded and lower-bounded, then these two metrics define the same topology on $T^R(S_0)$. In the same paper, he also provided an example of a surface S_0 of infinite topological type such that on $T^R(S_0)$, d_T and d_L are not topologically equivalent.

When a Riemann surface S_0 has boundary, $T^R(S_0)\neq T(S_0)$. In this case, d_L does not define a metric on $T(S_0)$ since it does not separate points. For if $f\colon S_0\to S_0$ is given by a Dehn twist along a boundary geodesic β, i.e., β is a geodesic homotopic to some boundary component of S_0, then $d_T([S_0,f],[S_0,id])>0$ in $T(S_0)$. Thus $[S_0,f]\neq[S_0,id]$ in $T(S_0)$. However, since there is no closed geodesic crossing β, the length of every closed geodesic γ is not changed under the Dehn twist and then $d_L([S_0,f],[S_0,id])=0$.

Assume that S_0 has a boundary. In this paper, we first introduce a modified length spectrum that does define a metric on $T(S_0)$. Then we study properties of this new metric and its relationship with the Teichmüller metric on $T(S_0)$ when S_0 is of finite topological type.

Let S be a Riemann surface with boundary and Σ''_S a collection of arcs connecting boundary components of S such that none of them is homotopic to a boundary segment relative to endpoints and no two of them are homotopic to each other relative to endpoints. We assume that Σ''_S is maximal in the sense that every arc connecting boundary components of S that is not homotopic to a boundary segment relative to endpoints is homotopic to an element of Σ''_S relative to endpoints. For any arc γ joining two boundary components, there exists a unique geodesic arc α homotopic to γ relative to endpoints. Let β_1 and β_2 be the two closed geodesics homotopic to the boundary components containing the endpoints of α (possibly $\beta_1=\beta_2$), namely, the boundary geodesics of the corresponding boundary components. If $\beta_1\neq\beta_2$, then α crosses each of them exactly once; if $\beta_1=\beta_2$, then α crosses β_1 exactly twice, probably at the same point. Let $l_S(\gamma)$ be the length of the geodesic segment of α between β_1 and β_2. We define the modified length spectrum on $T(S_0)$ by

$$d_{ML}([S_1,f_1],[S_2,f_2]) = \log \sup_{\gamma\in\Sigma_{S_1}} \left\{ \frac{l_{S_2}(f_2\circ f_1^{-1}(\gamma))}{l_{S_1}(\gamma)}, \frac{l_{S_1}(\gamma)}{l_{S_2}(f_2\circ f_1^{-1}(\gamma))} \right\},$$

where $\Sigma_{S_0}=\Sigma'_{S_0}\bigcup\Sigma''_{S_0}$.

In this paper, we first prove the following:

Theorem 1. *Assume that S_0 is a Riemann surface with boundary. Then the modified length spectrum function d_{ML} defines a metric on $T(S_0)$.*

Then we assume that S_0 is a Riemann surface of type (g,m,k), where g, m and k are the genus, the number of punctures and the number of ideal boundaries, respectively, with $k > 0$ and $6g-6+m+3k > 0$. Under these assumptions, we show the following results.

Theorem 2. *The identity map*

$$\mathrm{id}\colon (T(S_0), d_T) \to (T(S_0), d_{ML})$$

is continuous, but the inverse map is not.

Corollary 1. *The topologies induced by d_{ML} and d_T on $T(S_0)$ are not equivalent.*

Theorem 3. *The metric space $(T(S_0), d_{ML})$ is not complete.*

2. Modified length spectrum

In this section, we prove Theorem 1. Notice that by using the definition, it is easy to verify that d_{ML} is nonnegative and symmetric, and satisfies the triangle inequality. The main work is to show that it separates points. We first introduce some notation.

(1) Let $L_{x,y}$ denote the geodesic in the unit disk $\mathbf{D}$ or the upper half-plane $\mathbf{H}$ with respect to the hyperbolic metric that connects two points x and y on the boundary of $\mathbf{D}$ or $\mathbf{H}$.

(2) If a geodesic L intersects two geodesics L_1 and L_2, then $l(L;L_1,L_2)$ denotes the length, in the hyperbolic metric, of the segment of L between L_1 and L_2.

Proposition 1. *Let L_1 and L_2 be two disjoint geodesics in $\mathbf{H}$ without any common endpoint and let L_{x_0,y_0} be their common orthogonal. Then $l(L_{x_0,y};L_1,L_2) > l(L_{x_0,y_0};L_1,L_2)$ for any geodesic $L_{x_0,y}$ crossing L_1 and L_2 with $y_0 \neq y$. Moreover, for any given value $l_0 > l(L_{x_0,y_0};L_1,L_2)$ there exist exactly two geodesics L_{x_0,y_1} and L_{x_0,y_2} crossing L_1 and L_2 such that $l(L_{x_0,y_1};L_1,L_2) = l(L_{x_0,y_2};L_1,L_2) = l_0$. These geodesics are contained in different connected components of $\mathbf{H}\setminus L_{x_0,y_0}$.*

Proof. Without loss of generality, we may assume that $L_{x_0,y_0} = L_{0,\infty}$, $L_1 = L_{-1,1}$, $L_2 = L_{-b,b}$ for some $b > 1$. Let c be a positive number with $2c > b$. Then the intersection points of $L_{0,2c}$ with L_1 and L_2 are the solutions of the following systems respectively:

$$\begin{cases} x^2+y^2 &= 1, \\ (x-c)^2+y^2 &= c^2, \end{cases}$$

and

$$\begin{cases} x^2+y^2 &= b^2, \\ (x-c)^2+y^2 &= c^2. \end{cases}$$

Solving these systems, we see that the x-coordinates of the intersection points of $L_{0,2c}$ with L_1 and L_2 are given by $1/2c$ and $b^2/2c$ respectively. Now we parameterize the segment of $L_{0,2c}$ between L_1 and L_2 by the equation

$$\gamma_{0,2c}(t) = t + i\sqrt{c^2-(t-c)^2} = t + i\sqrt{2ct-t^2},\ t \in [1/2c, b^2/2c].$$

Then

$$l(L_{0,2c};L_1,L_2)=\int_{\frac{1}{2c}}^{\frac{b^2}{2c}}\frac{|\gamma'_{0,2c}(t)|}{\operatorname{Im}(\gamma_{0,2c}(t))}\,dt=\int_{\frac{1}{2c}}^{\frac{b^2}{2c}}\frac{1}{\sqrt{2ct-t^2}}\sqrt{1+\frac{(c-t)^2}{2ct-t^2}}\,dt$$
$$=\int_{\frac{1}{2c}}^{\frac{b^2}{2c}}\frac{c}{(2c-t)t}\,dt=\ln b+\frac{1}{2}\ln\frac{4c^2-1}{4c^2-b^2}.$$

Let

$$h(c)=l(L_{0,2c};L_1,L_2)=\ln b+\frac{1}{2}\ln\frac{4c^2-1}{4c^2-b^2}.$$

The derivative of $h(c)$ is given by

$$h'(c)=\frac{4c}{4c^2-1}-\frac{4c}{4c^2-b^2}.$$

Since $2c>b$ and $b>1$, $h'(c)<0$. It follows that h is a strictly decreasing function of c as soon as $c>b/2>1/2$, which implies that the length decreases as c goes to ∞. In fact, $h(\infty)=\ln b=l(L_{0,\infty};L_1,L_2)$.

Similarly, for $c<0$ with $2c<-b$, $g(c)=h(-c)$ is the length of $\gamma_{0,2c}$. Since $g'(c)=-h'(-c)>0$, it follows that for $c<-b/2$, g is an strictly increasing function of c with $g(-\infty)=\ln b=l(L;L_{0,\infty},L_2)$. □

Now we prove that d_{ML} separates points. Given any two points $[S_1,f_1]$ and $[S_2,f_2]$ in $T(S_0)$, we use the same symbols to denote their equivalent classes in $T^R(S_0)$. Then

$$d_L([S_1,f_1],[S_2,f_2])\le d_{ML}([S_1,f_1],[S_2,f_2]).$$

Suppose that $d_{ML}([S_1,f_1],[S_2,f_2])=0$. Then $d_L([S_1,f_1],[S_2,f_2])=0$. Since d_L is a metric in $T^R(S_0)$, it follows that $f_2\circ f_1^{-1}$ is homotopic to a conformal mapping $c\colon S_1\to S_2$. We need to prove that $f=f_2\circ f_1^{-1}$ is homotopic to c relative to boundary.

The following is a classical theorem which can be found in [4].

Theorem 4. *Let S_1 and S_2 be two hyperbolic Riemann surfaces and let $f_i\colon S_1\to S_2$, $i=1,2$, be two quasiconformal mappings. Assume that the Fuchsian group G_1 representing S_1 is non-elementary. Then*

(1) *f_1 is homotopic to f_2 if and only if they can be lifted to mappings of $\mathbf{H}$ which agree on the limit set of G_1; and*
(2) *f_1 is homotopic to f_2 relative to boundary if and only if they can be lifted to mappings of $\mathbf{H}$ which agree on $\widehat{\mathbf{R}}$.*

Given a Riemann surface S with boundary, let G be the group uniformizing S. The universal covering $\pi\colon\mathbf{D}\to\operatorname{int}(S)$ extends to a covering $\pi\colon\overline{\mathbf{D}}\setminus\Lambda(G)\to S$, where $\Lambda(G)$ is the limiting set of G. Then a quasiconformal mapping f from S to S lifts to a mapping $F\colon\overline{\mathbf{D}}\setminus\Lambda(G)\to\overline{\mathbf{D}}\setminus\Lambda(G)$, which extends to a homeomorphism of $\overline{\mathbf{D}}$. In this paper, in order to avoid repeating the details of the maps π and F on the boundary of $\mathbf{D}$, we say in brief that $\pi\colon\mathbf{D}\to S$ is a universal covering map for S and $F\colon\mathbf{D}\to\mathbf{D}$ is a lifting of f.

We continue to show that d_{ML} separates points in $T(S_0)$. Following the notation introduced in this section and using the previous theorem, we let $F\colon\mathbf{H}\to\mathbf{H}$ and $C\colon\mathbf{H}\to\mathbf{H}$ be liftings of f and c respectively that agree on the limit set Λ of the Fuchsian group G_1 uniformazing S_1. Clearly, $F\circ C^{-1}$ maps each connected component of $\mathbf{S}^1 \setminus \Lambda$ onto itself. It remains to show $F \circ C^{-1}$ is the identity on each connected component. Let L be a lifting of a boundary geodesic β of S_1. Then one of the arcs bounded by the endpoints of L on $\mathbf{S}^1$ is a connected component of $\mathbf{S}^1 \setminus \Lambda$ and each connected component is bounded by the endpoints of a lifting of a boundary geodesic of S_1.

Let L_1 and L_2 be two different liftings of a boundary geodesic β of S_1, and let L_{x_0,y_0} be their common perpendicular. The mapping C, being conformal, maps L_{x_0,y_0} to the common perpendicular between $C(L_1)$ and $C(L_2)$. By assumption, $l_{S_2}(f(\gamma)) = l_{S_1}(\gamma)$ for every $\gamma \in \Sigma''_{S_0}$. On the other hand, since c is an isometry, we must have $l_{S_1}(\gamma) = l_{S_2}(c(\gamma))$ for every $\gamma \in \Sigma''_{S_0}$. It follows that

$$l(L_{F(x_0),F(y_0)}; C(L_1), C(L_2)) = l(L_{x_0,y_0}; L_1, L_2) = l(L_{C(x_0),C(y_0)}; C(L_1), C(L_2)).$$

Since the common perpendicular segment is the unique segment of the smallest length among all segments connecting two geodesics, it follows that $F(x_0) = C(x_0)$ and $F(y_0) = C(y_0)$.

Let I_i be the interval on the real line bounded by the endpoints of L_i. It projects to the component of the ideal boundary of S_0 homotopic to β. Assume $x_0 \in I_1$ and $y_0 \in I_2$. For any point $x \in I_2 \setminus \{y_0\}$, consider the geodesic $L_{x_0,x}$. Then

$$l(L_{F(x_0),F(x)}; C(L_1), C(L_2)) = l(L_{x_0,x}; L_1, L_2) = l(L_{C(x_0),C(x)}; C(L_1), C(L_2)).$$

It follows from Proposition 1 that $F(x) = C(x)$. This argument can be applied to any point $x \in \widehat{\mathbf{R}}$ that is not in the limit set Λ of G_1. Thus both maps agree on the whole boundary of $\mathbf{H}$. It follows from Theorem 4 that their projections to the surfaces are homotopic to each other modulo boundary. Thus, d_{ML} separates points in $T(S_0)$ and then Theorem 1 follows.

3. Proofs of main results

The following is a well known result due to Wolpert (see [1]).

Lemma 1. *Let S_1 and S_2 be two homeomorphic hyperbolic Riemann surfaces. If $f\colon S_1 \to S_2$ is a quasiconformal mapping, then*

$$K(f) \geq \frac{l_{S_2}(f(\gamma))}{l_{S_1}(\gamma)} \quad \text{for any } \gamma \in \Sigma'_{S_1}.$$

As an immediate consequence we obtain that for any two points $\tau_1, \tau_2 \in T^R(S_0)$, $d_L(\tau_1, \tau_2) \leq d_T(\tau_1, \tau_2)$. It follows that the identity map

$$\mathrm{id}\colon (T^R(S_0), d_T) \to (T^R(S_0), d_L)$$

is continuous. Theorem 2 is an analogy of the previous statement to $T(S_0)$ under the Teichmüller metric d_T and the modified length spectrum d_{ML}. Before proving this theorem, we introduce some lemmas.

Lemma 2. *Let $L_{a,b}$, $a < b$, be a geodesic in $\mathbf{H}$ and I a closed interval contained in (a, b). Assume that $\{L_{a_n,b_n}\}$, $b < a_n < b_n$, is a sequence of geodesics in $\mathbf{H}$ such that the hyperbolic distance between L_{a_n,b_n} and $L_{a,b}$ is bounded below by some $\epsilon > 0$ for every n. Then*

$$\inf_{x \in I, y_n \in (a_n, b_n)} l(L_{x,y_n}; L_{a,b}, L_{a_n,b_n}) \to \infty \quad \text{as } n \to \infty$$

provided that $a_n, b_n \to b$ as $n \to \infty$.

Proof. For each n, we use the Möbius transformation

$$T_n(z) = \frac{z-b}{z-a}\frac{b_n-a}{b_n-b}$$

to map $L_{a,b}$ and L_{a_n,b_n} to $L_{\infty,0}$ and $L_{T_n(a_n),1}$ respectively. By the assumptions, the distance between $L_{\infty,0}$ and $L_{T_n(a_n),1}$ is bounded below by some $\epsilon > 0$. Then there exists $r > 0$ such that $T_n(a_n) \geq r$ for every n. For any $x \in I$ and $y_n \in (a_n, b_n)$, we have

$$l(L_{x,y_n}; L_{a,b}, L_{a_n,b_n}) = l(L_{T_n(x),T_n(y_n)}; L_{\infty,0}, L_{T_n(a_n),1}) \geq l(L_{T_n(x),T_n(y_n)}; L_{\infty,0}, L_{r,1}).$$

Since $T_n(x) \to -\infty$ uniformly for $x \in I$ as $n \to \infty$, it follows that

$$\inf_{x\in I, y_n\in(a_n,b_n)} l(L_{T_n(x),T_n(y_n)}; L_{\infty,0}, L_{r,1}) \to \infty$$

as $n \to \infty$. □

Let x and y be two distinct points on the unit circle $\mathbf{S}^1$. Denote by $\lceil x, y\rceil$ the circular arc on $\mathbf{S}^1$ connecting x to y in counterclockwise direction.

For any $d > 0$, choose $b = b(d) \in \mathbf{S}^1$ such that $0 < \arg(b) < \pi/2$ and the hyperbolic distance between $L_{-i,i}$ and $L_{\overline{b},b}$ is d. We call $L_{\overline{b},b}$ the *d-standard geodesic.* Now we assume that a positive number s is sufficiently small (depending on $b(d)$). Let $I_s = \lceil x_s, y_s\rceil \subseteq \lceil i, -i\rceil$ and $J_{s,b} = \lceil z_s, w_s\rceil \subseteq \lceil \overline{b}, b\rceil$ be the arcs on $\mathbf{S}^1$ such that the length of each of the arcs $\lceil i, x_s\rceil$, $\lceil y_s, -i\rceil$, $\lceil \overline{b}, z_s\rceil$, and $\lceil w_s, b\rceil$ is equal to s.

Lemma 3. *Assume that $0 < d_0 < d_1$. For any $D > d_1$, there exists $s_0 > 0$ such that for any $d_0 \leq d \leq d_1$, if $l(L_{p,q}; L_{-i,i}, L_{\overline{b(d)},b(d)}) \leq D$, then $p \in I_{s_0}$ and $q \in J_{s_0,b(d)}$.*

Proof. Let $D > d_1$. For any $d \in [d_0, d_1]$, there exists a maximal s, denoted by $s(d)$, such that if $l(L_{p,q}; L_{-i,i}, L_{\overline{b(d)},b(d)}) \leq D$ then $p \in I_s$ and $q \in J_{s,b(d)}$. The function $d \mapsto s(d)$ is a continuous function defined on the compact interval $[d_0, d_1]$. Then $s_0 = \min_{d\in[d_0,d_1]} s(d)$ satisfies the conclusion of the lemma. □

Lemma 4. *Assume that $0 < d_0 < d_1$ and also assume that s_0 is a positive number small enough (only depending on d_1). Then for any $\epsilon > 0$, there exists $\delta > 0$ depending on d_0, d_1, s_0 and ϵ such that*

1. *for every $d \in [d_0, d_1]$,*
2. *for every $x, y, z, w \in \mathbf{S}^1$,*
3. *for every $p_1, p_2 \in I_{s_0}$, and*
4. *for every $q_1, q_2 \in J_{s_0,b(d)}$,*

if each of the numbers $|p_1 - p_2|, |q_1 - q_2|, |x - i|, |y + i|, |z - \overline{b(d)}|$ and $|w - b(d)|$ is less than δ, then

$$\left|\log \frac{l(L_{p_1,q_1}; L_{-i,i}, L_{\overline{b(d)},b(d)})}{l(L_{p_2,q_2}; L_{x,y}, L_{z,w})}\right| < \epsilon.$$

Proof. Suppose the lemma is false. Then there exists $\epsilon > 0$ such that for every $\delta_n = 1/n$, there exist

1. $d_n \in [d_0, d_1]$,
2. $x_n, y_n, z_n, w_n \in \mathbf{S}^1$ with $|x_n - i|, |y_n + i|, |z_n - \overline{b(d_n)}|$ and $|w - b(d_n)| < 1/n$,
3. $p_1^{(n)}, p_2^{(n)} \in I_{s_0}$ with $|p_1^{(n)} - p_2^{(n)}| < 1/n$, and
4. $q_1^{(n)}, q_2^{(n)} \in J_{s_0,b(d_n)}$ with $|q_1^{(n)} - q_2^{(n)}| < 1/n$

such that

$$\left|\log \frac{l(L_{p_1^{(n)},q_1^{(n)}}; L_{-i,i}, L_{\overline{b(d_n)},b(d_n)})}{l(L_{p_2^{(n)},q_2^{(n)}}; L_{x_n,y_n}, L_{z_n,w_n})}\right| \geq \epsilon. \tag{3.1}$$

We may assume, by passing to subsequences, that

(1) $d_n \to d^{(0)}$, which implies $b(d_n) \to b(d^{(0)})$;
(2) $x_n \to i$, $y_n \to -i$, $z_n \to b(d^{(0)})$, $w_n \to \overline{b(d^{(0)})}$; and
(3) $p_1^{(n)} \to p_1^{(0)}$, $q_1^{(n)} \to q_1^{(0)}$ and thus $p_2^{(n)} \to p_1^{(0)}$, $q_2^{(n)} \to q_1^{(0)}$.

Since $p_1^{(n)} \in I_{s_0}$ and $q_1^{(n)} \in J_{s_0,b(d_n)}$, it follows that $p_1^{(0)}$ and $q_1^{(0)}$ are contained in the interior of $\lceil i,-i \rceil$ and $\lceil \overline{b(d^{(0)})}, b(d^{(0)}) \rceil$ respectively. Then we can choose n sufficiently large so that

$$\left|\log \frac{l(L_{p_1^{(n)},q_1^{(n)}}; L_{-i,i}, L_{\overline{b(d_n)},b(d_n)})}{l(L_{p_1^{(0)},q_1^{(0)}}; L_{-i,i}, L_{\overline{b(d^{(0)})},b(d^{(0)})})}\right| < \epsilon/2$$

and

$$\left|\log \frac{l(L_{p_2^{(n)},q_2^{(n)}}; L_{x_n,y_n}, L_{z_n,w_n})}{l(L_{p_1^{(0)},q_1^{(0)}}; L_{-i,i}, L_{\overline{b(d^{(0)})},b(d^{(0)})})}\right| < \epsilon/2.$$

Combining both inequalities, we obtain

$$\left|\log \frac{l(L_{p_1^{(n)},q_1^{(n)}}; L_{-i,i}, L_{\overline{b(d_n)},b(d_n)})}{l(L_{p_2^{(n)},q_2^{(n)}}; L_{x_n,y_n}, L_{z_n,w_n})}\right| < \epsilon$$

for n sufficiently large. This is a contradiction to the inequality (3.1). □

For the rest of this paper, we consider Riemann surfaces S of type (g,m,k), where g, m and k are the genus, the number of punctures and the number of ideal boundaries, respectively, with $k > 0$ and $6g - 6 + m + 3k > 0$. Let S^d be the double of S. Then S^d is of type $(2g + k - 1, 2m, 0)$ and the boundary curves of S become closed geodesics on S^d. The intrinsic metric on S is defined to be the restriction to S of the hyperbolic metric on S^d. The Nielsen kernel $\widetilde{S}$ of S is the Riemann surface of the same type obtained by removing from S the k funnels formed by the boundary geodesics and the ideal boundary of S. The surface S is called the Nielsen extension of $\widetilde{S}$ and one of them is uniquely determined by the other. For more details about the Nielsen kernel of a Riemann surface, the reader is referred to [2].

Bers [2] proved the following result.

Lemma 5. *The intrinsic metric on $\widetilde{S}$ is equal to the restriction to $\widetilde{S}$ of the hyperbolic metric on S.*

Before we prove our theorems, we prepare a couple of more lemmas.

Lemma 6. *For any curve $\gamma \in \Sigma''_S$,*

$$\frac{l_{\widetilde{S}^d}(\widetilde{\gamma}^d)}{2} \leq l_S(\gamma) \leq \frac{l_{\widetilde{S}^d}(\widetilde{\gamma}^d)}{2} + 2M, \tag{3.2}$$

where $\widetilde{\gamma}$ is the restriction of γ to $\widetilde{S}$, $\widetilde{\gamma}^d$ is the double of $\widetilde{\gamma}$, and

$$M = \max\{l_S(\beta) \colon \beta \text{ is a boundary geodesic in } S\}.$$

Proof. For any curve α in $\widetilde{S}^d$, let $\widetilde{l}_{\widetilde{S}^d}(\alpha)$ denote the length of the curve α in the hyperbolic metric on $\widetilde{S}^d$. Let γ be an arc in Σ''_S; without loss of generality we may assume that it is a geodesic arc. By Lemma 5, $\widetilde{l}_{\widetilde{S}^d}(\widetilde{\gamma}) = l_S(\gamma)$. Since $\widetilde{l}_{\widetilde{S}^d}(\widetilde{\gamma}) = \widetilde{l}_{\widetilde{S}^d}(\widetilde{\gamma}^d)/2 \geq l_{\widetilde{S}^d}(\widetilde{\gamma}^d)/2$, the left-hand side inequality follows.

Recall that the geodesic arc γ either crosses two distinct boundary geodesics exactly once or one exactly twice. Let $\beta_1, \beta_2 \in \Sigma'_S$ be the ones crossed by γ at the points p_1 and p_2 respectively. If $\beta_1 = \beta_2$, then p_1 and p_2 belong to the same geodesic boundary and in this case p_1 may be equal to p_2. Let β be the closed geodesic on $\widetilde{S}^d$ in the homotopy class of $\widetilde{\gamma}^d$. Then β crosses β_1 and β_2 in a similar fashion as γ. Denote these two crossing points by q_1 and q_2 respectively. Let β'_i be one of the two segments of β_i joining p_i to q_i, $i = 1, 2$. Then

$$\begin{aligned} l_S(\gamma) &= \widetilde{l}_{\widetilde{S}^d}(\widetilde{\gamma}) \leq \widetilde{l}_{\widetilde{S}^d}(\beta'_1) + \widetilde{l}_{\widetilde{S}^d}(\beta \cap \widetilde{S}) + \widetilde{l}_{\widetilde{S}^d}(\beta'_2) = \widetilde{l}_{\widetilde{S}^d}(\beta'_1) + \tfrac{1}{2}\widetilde{l}_{\widetilde{S}^d}(\beta) + \widetilde{l}_{\widetilde{S}^d}(\beta'_2) \\ &\leq {}_{\widetilde{S}^d}(\beta_1) + \tfrac{1}{2}\widetilde{l}_{\widetilde{S}^d}(\beta) + l_{\widetilde{S}^d}(\beta_2) \leq \tfrac{1}{2}\widetilde{l}_{\widetilde{S}^d}(\beta) + 2M = \tfrac{1}{2} l_{\widetilde{S}^d}(\widetilde{\gamma}^d) + 2M, \end{aligned}$$

where the second equality follows from the fact that β is homotopic to $\widetilde{\gamma}^d$, which is symmetric with respect to the geodesics on $\tilde{S}^d$ coming from the boundary geodesics of S. The right-hand inequality of (3.2) now follows. □

Lemma 7. *Let $f_n \colon S_0 \to S_n$, $n = 1, 2, \cdots$, be a sequence of quasiconformal mappings such that $K(f_n) \to 1$ as $n \to \infty$. Then there exists a sequence of quasiconformal mappings $h_n \colon S_0 \to S_n$ such that* (i) *if n is sufficiently large, then h_n is homotopic to f_n relative to boundary and h_n preserves the Nielsen kernel of S_n, i.e., $h_n(\widetilde{S}_0) = \widetilde{S}_n$, and* (ii) *$K(h_n) \to 1$ as $n \to \infty$.*

Proof. For each $n = 0, 1, 2, \cdots$, let G_n be the group uniformizing S_n, let $\pi_n \colon \mathbf{D} \to S_n$ be the universal covering map, and let $F_n \colon \mathbf{D} \to \mathbf{D}$ be a lifting of f_n normalized to fix three points. Then $K(F_n) \to 1$ and F_n converges uniformly to the identity map.

Let $D_0 \subseteq \mathbf{D}$ be a Dirichlet fundamental domain for G_0. Then $F_n(D_0)$ is a fundamental domain for G_n. From now on, we assume that n is sufficiently large. Then F_n is very close to the identity map. It follows that the vertices of D_0 are moved very little by F_n. Then for each edge e of D_0, replace $F_n(e)$ by the geodesic segment connecting the endpoints of $F_n(e)$. These new edges bound a domain D_n, which is a new fundamental domain for G_n. We briefly explain why it is so in two steps.

Note that when n is sufficiently large, nonadjacent edges of D_n do not intersect.

Step 1: We show that each orbit under the action of G_n intersects D_n. Let $p \in \mathbf{D}$. Then there exists $g \in G_n$ such that $g(p) \in F_n(D_0)$. Suppose $g(p) \notin D_n$. Then $g(p)$ belongs to a connected component of $F_n(D_0) \setminus D_n$. This connected component is bounded by a segment of an edge (or probably a full edge) of D_n and a segment of $F_n(e)$ (or probably the full curve $F_n(e)$), where e is an edge of D_0. The curve $F_n(e)$ is paired to another curve $F_n(e')$ by an element $g_2 \in G_n$, where e' is an edge of D_0. Then $(g_2 \circ g)(p)$ is contained in $D_n \setminus F_n(D_0)$. Thus, $(g_2 \circ g)(p) \in D_n$.

Step 2: We prove that the interior $\mathrm{int}(D_n)$ of D_n contains at most one point from each orbit under G_n. Suppose that there are two points p_1 and p_2 of $\mathrm{int}(D_n)$ that lie on the same orbit. Assume that $p_1 \in \mathrm{int}(D_n) \setminus F_n(D_0)$. Using an argument similar to the one in Step 1, we can show that there exists $g_1 \in G_n$ such that

$g_1(p_1)$ belongs to the interior of $F_n(D_0)$. Clearly, there exists $g_2 \in G_n$ such that $g_2(p_2) \in F_n(D_0)$. The positions of $g_1(p_1)$ and $g_2(p_2)$ make a contradiction since they stay on the same orbit. Therefore, $p_1 \in F_n(D_0)$. With the same reasoning, we know $p_2 \in F_n(D_0)$. Since $F_n(D_0)$ is a fundamental domain for G_n, it follows that there exist two edges e_1 and e_2 of D_0 and an element $g \in G_n$ such that $p_1 \in F_n(e_1)$, $p_2 \in F_n(e_2)$, $g(F_n(e_1)) = F_n(e_2)$ and $g(p_1) = p_2$. All these force p_1 and p_2 to stay on a pair of edges of D_n; otherwise, one of them belongs to $\mathrm{int}(D_n)$ and the other has to be outside of D_n. Both situations contradict the assumption. Therefore, $\mathrm{int}(D_n)$ contains at most one point from each orbit.

Now for each such large n, let $\widetilde{D}_n = \pi_n^{-1}(\widetilde{S}_n) \bigcap D_n$. The region $\widetilde{D}_n$ is a convex polygon whose vertices are either in $\mathbf{D}$ or on $\partial\mathbf{D}$ and it projects to the Nielsen kernel $\widetilde{S}_n$ of S_n. Each $\underline{D_n}$ is the convex hull of its vertices, thus we can construct a piecewise hyperbolic affine map $H_n \colon \widetilde{D}_0 \to \widetilde{D}_n$ mapping vertices to vertices. In order to extend H_n to D_0, we foliate each connected component of $D_0 \setminus \widetilde{D}_0$ by geodesic rays starting at $\partial\widetilde{D}_0$ and ending at $\partial\mathbf{D} \bigcap \overline{D}_0$, where $\overline{D}_0$ is the closure of D_0 in the closed unit disk with respect to the Euclidean metric. For each such geodesic ray with endpoints $z \in \partial\widetilde{D}_0$ and $x \in \partial\mathbf{D} \bigcap \overline{D}_0$, we let H_n map this ray onto the geodesic ray starting at $H_n(z)$ and ending at $F_n(x)$ such that the distances from the starting points are preserved. Finally we extend H_n to the whole hyperbolic disk by using the actions of G_0 and G_n on $\mathbf{D}$. By Theorem 4, H_n projects to a mapping $h_n \colon S_0 \to S_n$ which is homotopic to f_n relative to boundary and by the construction $h_n(\widetilde{S}_0) = \widetilde{S}_n$. Moreover, since F_n converges uniformly to the identity, the vertices of D_n approach the vertices of D_0 as $n \to \infty$. Thus, $H_n \to \mathrm{id}$ and $K(H_n) \to 1$ as $n \to \infty$. Hence $K(h_n) \to 1$ as $n \to \infty$. □

Theorem 5. *The identity function* $\mathrm{id} \colon (T(S_0), d_T) \to (T(S_0), d_{ML})$ *is continuous.*

Proof. Let $\{\tau_n\}$ be a sequence of points in $T(S_0)$ converging to a point τ in the Teichmüller metric and let $\epsilon > 0$ be given. Without loss of generality, we may assume that $\tau = [S_0, \mathrm{id}]$. Let $\tau_n = [S_n, f_n]$, by Lemma 7, we may assume that $f_n(\widetilde{S}_0) = \widetilde{S}_n$ and $K(f_n) \to 1$ as $n \to \infty$. Consider $[S_n, f_n]$ and $[S_0, \mathrm{id}]$ as elements of $T^R(S_0)$. As we mention in the introduction, it is proved in [7] that d_L and d_T are topologically equivalent in $T^R(S_0)$. Thus, there exists N_1 such that

$$\left|\log \frac{l_{S_n}(f_n(\gamma))}{l_{S_0}(\gamma)}\right| < \epsilon \tag{3.3}$$

for every $n > N_1$ and every $\gamma \in \Sigma'_{S_0}$. For each $n = 0, 1, 2, \cdots$, let

$$M_n = \max\{l_{S_n}(\beta) \colon \beta \text{ is a boundary geodesic in } S_n\}.$$

Then the previous property (3.3) implies that M_n converges to M_0 as $n \to \infty$. Hence there exists a constant $M' > 0$ such that $M_n \leq M'$ for every n.

Let $\gamma \in \Sigma''_{S_0}$. Since f_n maps $\widetilde{S}_0$ to $\widetilde{S}_n$, it follows that $\widetilde{f_n(\gamma)} = \widetilde{f}_n(\widetilde{\gamma})$ and $(\widetilde{f}_n(\widetilde{\gamma}))^d = \widetilde{f}_n^d(\widetilde{\gamma}^d)$, where $\widetilde{f}_n = f|_{\widetilde{S}_0}$ and $\widetilde{f}_n^d \colon \widetilde{S}_0^d \to \widetilde{S}_n^d$ is the double mapping of $\widetilde{f}_n$. Then by lemma 6 and the fact that $M_n \leq M'$ for any $n = 0, 1, 2, \cdots$, we conclude that

$$\frac{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)}{2} \leq l_{S_0}(\gamma) \leq \frac{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)}{2} + 2M' \tag{3.4}$$

and

$$\frac{l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))}{2} \le l_{S_n}(f_n(\gamma)) \le \frac{l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))}{2} + 2M'. \tag{3.5}$$

Combining inequalities (3.4) and (3.5), we obtain

$$\frac{\frac{1}{2}l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))}{\frac{1}{2}l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d) + 2M'} \le \frac{l_{S_n}(f_n(\gamma))}{l_{S_0}(\gamma)} \le \frac{\frac{1}{2}l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d)) + 2M'}{\frac{1}{2}l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)}$$

or

$$\frac{\frac{l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))}{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)}}{1 + \frac{4M'}{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)}} \le \frac{l_{S_n}(f_n(\gamma))}{l_{S_0}(\gamma)} \le \frac{l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))}{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)} + \frac{4M'}{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)}.$$

Lemma 1 and the fact that $K(\widetilde{f}_n^d) = K(\widetilde{f}_n) \le K(f_n)$ imply

$$\frac{1}{K(f_n)} \le \frac{l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))}{l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)} \le K(f_n).$$

Thus we can choose D and N_2 sufficiently large such that if $n > N_2$ and $l_{S_0}(\gamma) > D$, then $l_{\widetilde{S}_n^d}(\widetilde{f}_n^d(\widetilde{\gamma}^d))/l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)$ is sufficiently close to 1 and $4M'/l_{\widetilde{S}_0^d}(\widetilde{\gamma}^d)$ is sufficiently small. More precisely, we choose D and N_2 large enough such that

$$\left|\log \frac{l_{S_n}(f_n(\gamma))}{l_{S_0}(\gamma)}\right| < \epsilon \tag{3.6}$$

for every $n > N_2$ and every $\gamma \in \Sigma''_{S_0}, l_{S_0}(\gamma) > D$. It remains to consider all arcs $\gamma \in \Sigma''_{S_0}$ with $l_{S_0}(\gamma) \le D$.

Let G_0 be the Fuchsian group uniformizing S_0 and let $\pi\colon \mathbf{D} \to S_0$ be the universal covering map. Let B be a boundary component of S_0 and β the corresponding boundary geodesic. Assume that β^* is a lifting of β in $\mathbf{D}$. Then β^* separates the unit circle $\mathbf{S}^1$ into two open circular arcs and one of them is a cover of B under π. We denote it by B^*. Let I be a closed segment of B^* such that I minus one of its endpoint covers B exactly once. Without loss of generality, we assume that all elements in Σ''_{S_0} are geodesic arcs. Let $\mathcal{F}$ be the collection of the liftings γ^* of the elements $\gamma \in \Sigma''_{S_0}$ with $l_{S_0}(\gamma) \le D$ such that they have one endpoint on I. We claim that besides β^*, there are only finitely many lifting geodesics of the boundary geodesics of S_0 such that they intersect at least one $\gamma^* \in \mathcal{F}$. Let $\beta_1^*, \beta_2^*, \cdots, \beta_r^*$ be such lifting geodesics (different from β^*).

We show the claim. Suppose there are infinitely many such lifting geodesics $\{\beta_j^*\}_{j=1}^\infty$ besides β^*. For each j, let $\gamma_j^* \in \mathcal{F}$ such that $\gamma_j^* \cap \beta_j^* \ne \emptyset$. If there is a subsequence $\{\beta_{j_k}^*\}$ that does not accumulate at one of the endpoints of β^*, then it is easy to see that $l(\gamma_{j_k}^*; \beta^*, \beta_{j_k}^*) \to \infty$ as $k \to \infty$, which contradicts the fact that $l(\gamma_{j_k}^*; \beta^*, \beta_{j_k}^*) \le D$. Suppose now that there is a subsequence $\{\beta_{j_k}^*\}$ accumulating at one of the endpoints of β^*. By the Collar Lemma [3], the distance between β^* and β_j^* is bounded from below by a constant $d_0 > 0$. Then by Lemma 2, we also obtain $l(\gamma_{j_k}^*; \beta^*, \beta_{j_k}^*) \to \infty$ as $k \to \infty$. Again a contradiction to $l(\gamma_{j_k}^*; \beta^*, \beta_{j_k}^*) \le D$ for all k.

As we mentioned above, by the Collar Lemma, there exists $d_0 > 0$ such that the hyperbolic distance between β^* and β_j^*, $j = 1, 2, \cdots, r$, is at least d_0. On the other hand, $l(\gamma^*; \beta^*, \beta_j^*) \le D$ for each γ^* in $\mathcal{F}$. Let d_1 be the maximum of the hyperbolic distance between β^* and β_j^* for $j = 1, 2, \cdots, r$. Then $d_0 < d_1 \leq D$. For each $j = 1, 2, \cdots, r$, we can normalize the group G_0 so that $\beta^* = L_{-i,i}$ and $\beta_j^* = L_{\overline{b},b}$, i.e., β_j^* is the d-standard geodesic for some $d \in [d_0, d_1]$. Corresponding to each normalization, every map F_n is changed to $F_{n,j}$. Let $s_0 > 0$ be the constant to guarantee the conclusion of Lemma 3. Choose $\delta > 0$ so small that the conclusion of Lemma 4 follows. Recall that F_n converges uniformly to the identity map. In fact, for each fixed $1 \leq j \leq r$, $F_{n,j}$ also converges uniformly to the identity map as $n \to \infty$. Thus we can choose $N(j)$ sufficiently large so that for each $n > N(j)$, $|F_{n,j}(x) - x| < \delta$. It follows from Lemma 4 that for every $n > N(j)$ and for every geodesic $\gamma^* = L_{p,q} \in \mathcal{F}$ crossing $L_{-i,i}$ and $L_{\overline{b},b}$, we obtain

$$\left|\log \frac{l(L_{p,q}; L_{-i,i}, L_{\overline{b(d)},b(d)})}{l(L_{F_{n,j}(p),F_{n,j}(q)}; L_{F_{n,j}(-i),F_{n,j}(i)}, L_{F_{n,j}(\overline{b}),F_{n,j}(b)})}\right| < \epsilon. \tag{3.7}$$

After applying the same argument to any geodesic β_j^* for each $1 \leq j \leq r$ and then the same arguments of this part to any boundary component B of S_0 (finitely many), we conclude that there exists N_3 such that for every $n > N_3$ and every $\gamma \in \Sigma''_{S_0}$ with $l_{S_0}(\gamma) \leq D$,

$$\left|\log \frac{l_{S_n}(f_n(\gamma))}{l_{S_0}(\gamma)}\right| < \epsilon. \tag{3.8}$$

Letting $N = \max\{N_1, N_2, N_3\}$ and combining inequalities (3.3), (3.6) and (3.8), we obtain for every $n > N$,

$$\log \sup_{\gamma \in \Sigma_{S_0}} \left\{ \frac{l_{S_n}(f_n(\gamma))}{l_{S_0}(\gamma)}, \frac{l_{S_0}(\gamma)}{l_{S_n}(f_n(\gamma))} \right\} < \epsilon;$$

that is, $d_{ML}([S_n, fn], [S_0, \mathrm{id}]) \to 0$ as $n \to \infty$. Thus the map

$$\mathrm{id}\colon (T(S_0), d_T) \to (T(S_0), d_{ML})$$

is continuous. □

In fact, the techniques used in the proof of Theorem 5 also show the following result.

Corollary 2. *Let $\{[S_n, f_n]\}$ be a sequence in $T(S_0)$ satisfying*

(1) *for each n, $f_n(\widetilde{S}_0) = \widetilde{S}_n$,*
(2) *$K(f_n|_{\widetilde{S}_0}) \to 1$ as $n \to \infty$, and*
(3) *for each n, there is a lifting $F_n\colon \mathbf{D} \to \mathbf{D}$ of f_n such that the sequence $\{F_n\}$ converges uniformly to the identity on $\mathbf{S}^1$.*

Then $d_{ML}([S_n, f_n], [S_0, \mathrm{id}]) \to 0$ as $n \to \infty$.

Unlike the case of $T^R(S_0)$, the metrics d_T and d_{ML} do not define the same topology on $T(S_0)$.

Theorem 6. *There exists a sequence $\{\tau_n\}$ in $T(S_0)$ such that*

$$d_{ML}(\tau_n, \tau) \to 0 \quad \text{and} \quad d_T(\tau_n, \tau) \to \infty$$

as $n \to \infty$, where $\tau = [S_0, \mathrm{id}]$.

Proof. For each $n = 1, 2, 3, \cdots$, we construct a mapping $F_n\colon \mathbf{D} \to \mathbf{D}$ that projects to a mapping $f_n\colon S_0 \to S_0$ such that the sequence $\{[S_0, f_n]\}$ satisfies the hypothesis of Corollary 2.

Let G_0 be the Fuchsian group uniformizing S_0 and $\pi\colon \mathbf{D} \to S_0$ the covering map. Suppose $D_0 \subseteq \mathbf{D}$ is a Dirichlet fundamental domain for G_0 and let I be an arc contained in $\overline{D}_0 \cap \mathbf{S}^1$ that projects to a segment on a boundary component of S_0. Let T be a Möbius transformation from $\mathbf{D}$ onto the upper half plane that maps I to the interval $[0,1]$, and for each n, let $b_n = 1/(2^{n+1}-1), c_n = 1/2^n$. Define $F_n|_I$ to be the mapping $T^{-1} \circ H_n \circ T$, where $H_n\colon [0,1] \to [0,1]$ is the piecewise linear map that sends $0, b_n, c_n$ and 1 to $0, (2^n-1)/2^{2n}$ and $c_n, 1$ respectively. Clearly, H_n and hence $F_n|_I$ converge to the identity map on I uniformly as $n \to \infty$. Denote by β the boundary geodesic on S_0 homotopic to the boundary component containing $\pi(I)$ and let A be the connected component of $\overline{D}_0 \setminus \pi^{-1}(\beta)$ containing I. Define F_n to be the identity on $(\overline{D}_0 \cap \mathbf{S}^1 \setminus I) \cup (D_0 \setminus A)$. Finally, foliate A by geodesic rays starting at points in $D_0 \cap \pi^{-1}(\beta)$ and ending at points in $\overline{A} \cap \mathbf{S}^1$. For every geodesic ray in the foliation starting at z and ending at $x \in \mathbf{S}^1$, let F_n map this ray onto the one starting at z and ending at $F_n(x)$ by preserving the hyperbolic distance to z. The mapping $F_n\colon D_0 \to D_0$ can be extended to the whole hyperbolic plane by pre-composing and post-composing by elements of G_0. Since $F_n|_{\overline{D}_0}$ converges uniformly to the identity, it follows that $F_n\colon \mathbf{D} \to \mathbf{D}$ and $F_n\colon \mathbf{S}^1 \to \mathbf{S}^1$ converge uniformly to the identity as well. Moreover, each F_n can be projected to a map $f_n\colon S_0 \to S_0$ such that $f_n|_{\widetilde{S}_0} = \mathrm{id}_{\widetilde{S}_0}$.

Let $\tau_n = [S_0, f_n]$. Then by Corollary 2,

$$d_{ML}(\tau_n, \tau) \to 0 \quad \text{as } n \to \infty,$$

where $\tau = [S_0, \mathrm{id}]$.

For any four points $x, y, z, w \in \mathbf{R}$, let $\mathrm{cr}(x,y,z,w)$ denote the cross ratio

$$\mathrm{cr}(x,y,z,w) = \frac{(y-x)(w-z)}{(z-y)(w-z)}.$$

Notice that by construction,

$$\mathrm{cr}(0, b_n, c_n, 1) = 1$$

and

$$\mathrm{cr}(F_n(0), F_n(b_n), F_n(c_n), F_n(1)) = 2^n - 2 + \frac{1}{2^{n-1}} \to \infty$$

as $n \to \infty$. This implies that $K(f_n) \to \infty$ as $n \to \infty$, i.e.,

$$d_T(\tau, \tau_n) \to \infty \quad \text{as } n \to \infty. \qquad \square$$

Theorems 5 and 6 together imply Theorem 2. Now we prove Theorem 3.

Proof. Let G_0, D_0 and $I = \lceil a, b \rceil$ be as in the proof of Theorem 6. Let T be a Möbius transformation from $\mathbf{D}$ onto the upper half plane $\mathbf{H}$ such that I is mapped to $[0,1]$, with $T(a) = 0$ and $T(b) = 1$. For each n, we construct a mapping $F_n\colon \mathbf{H} \to \mathbf{H}$ in the same fashion as in the proof of Theorem 6 except that the map H_n used to define $F_n|_I = T^{-1} \circ H_n \circ T$ is replaced by the piecewise linear map that maps $0, 1/2$ and 1 to $0, 1/2^n$ and 1, respectively. Note that for any $n > m$,

$$H_n \circ H_m^{-1}(x) = \begin{cases} 2^{m-n}x & \text{if } 0 \le x \le \frac{1}{2^m}, \\ \frac{2^m - 2^{m+n}}{2^n - 2^{n+m}}(x-1) + 1 & \text{if } \frac{1}{2^m} \le x \le 1. \end{cases}$$

Thus

$$\max_{x \in [0,1]} |H_n \circ H_m^{-1}(x) - x| \le \frac{1}{2^m} - \frac{1}{2^n} \le \frac{1}{2^m}.$$

It follows that $H_n \circ H_m^{-1}$ is uniformly close to the identity if $n > m$ and m is large, and hence so does $F_n \circ F_m^{-1}$. Let f_n be the projection of F_n to the surface S_0. Clearly, $f_n \circ f_m^{-1}$ is the identity on the Nielsen kernel $\widetilde{S}_0$. Now we apply Corollary 2 to conclude that

$$d_{ML}([S_0, f_n], [S_0, f_m]) = d_{ML}([S_0, f_n \circ f_m^{-1}], [S_0, \mathrm{id}]) \to 0$$

as $n, m \to \infty$. Thus $\{[S_0, f_n]\}$ is a Cauchy sequence under the metric d_{ML}. Now we show that this sequence cannot have a limit in $T(S_0)$ in this metric. Suppose not, then there is $\tau = [S, f] \in T(S_0)$ such that

$$d_{ML}([S_0, f_n], [S, f]) \to 0 \tag{3.9}$$

as $n \to \infty$. Then

$$d_L([S_0, f_n], [S, f]) \to 0$$

in the reduced Teichmüller space $T^R(S_0)$ as $n \to \infty$.

Our construction shows that $[S_0, f_n] = [S_0, \mathrm{id}]$ in $T^R(S_0)$ for each n. It follows that $[S, f]$ and $[S_0, \mathrm{id}]$ determine the same point in the reduced Teichmüller space and hence S_0 is conformally equivalent to S. By post-composing by an appropriate conformal mapping, we may assume that $\tau = [S_0, f]$ and f_n is homotopic to f for each n.

The maps F_n are liftings of the maps f_n and they agree with each other on the limit set of G_0. Let F be the lifting of f that agrees with F_n on the limit set. Let β_1^* and β_2^* be two liftings of the boundary geodesic β that is homotopic to the boundary component of S_0 to which $I = \lceil a, b \rceil$ projects. We can choose I to be an interval such that it shares one endpoint with the common perpendicular geodesic γ_0^* of β_1^* and β_2^*. That is, we may assume $\gamma_0^* = L_{a,y}$ for some $y \in \mathbf{S}^1$.

By the construction, each F_n fixes a and y. Then $L_{a,y} = L_{F_n(a), F_n(y)}$ for all n. It follows that $l_{S_0}(f_n(\gamma_0)) = l_{S_0}(\gamma_0)$ for all n, where γ_0 is the projection of γ_0^*. Then, condition (3.9) implies $l_{S_0}(f(\gamma_0)) = l_{S_0}(\gamma_0)$. Since the common perpendicular is the unique shortest segment between β_1^* and β_2^*, we must have $F(a) = a$ and $F(y) = y$.

Now let $\gamma_1^* = L_{T^{-1}(1/2), y}$. By the construction,

$$F_n(T^{-1}(1/2)) \to a \quad \text{as} \quad n \to \infty.$$

Thus

$$l(L_{F_n(T^{-1}(1/2)), F_n(y)}; \beta_1^*, \beta_2^*) \to l(L_{a,y}; \beta_1^*, \beta_2^*) \quad \text{as} \quad n \to \infty;$$

that is

$$l_{S_0}(f_n(\gamma_1)) \to l_{S_0}(\gamma_0) \quad \text{as} \quad n \to \infty,$$

where γ_1 is the projection of γ_1^*. By condition (3.9), we obtain $l_{S_0}(f(\gamma_1)) = l_{S_0}(\gamma_0)$. Using again the uniqueness of the common perpendicular as the shortest segment between β_1^* and β_2^*, we conclude that

$$F(T^{-1}(1/2)) = a.$$

Since $F(a) = a$, we obtain a contradiction to the injectivity of F on $\overline{\mathbf{D}}$. □

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