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SOME RESULTS ON THE INVERTIBILITY OF TOEPLITZ PLUS HANKEL OPERATORS

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Abstract. The paper deals with the invertibility of Toeplitz plus Hankel operators $T(a)+H(b)$ acting on classical Hardy spaces on the unit circle $\mathbf{T}$. It is supposed that the generating functions a and b satisfy the condition $a(t)a(1/t) = b(t)b(1/t)$, $t \in \mathbf{T}$. Special attention is paid to the case of piecewise continuous generating functions. In some cases the dimensions of null spaces of the operator $T(a) + H(b)$ and its adjoint are described.

1. Introduction

Fredholm properties of Toeplitz plus Hankel operators $T(a)+H(b)$ with piecewise continuous generating functions a and b have been studied for many years. These operators are considered in various Banach and Hilbert spaces, and the results obtained show that the structure of the algebras generated by such operators is much more complicated than the structure of the algebras generated by one-dimensional singular integral operators with piecewise continuous coefficients defined on closed smooth curves. Moreover, calculating the index of Toeplitz plus Hankel operators, one encounters even more difficult problems, and the difficulties grow if one attempts to study invertibility or one-sided invertibility of such operators. It is worth noting that, in general, Fredholm operators $T(a) + H(b)$ are not one-sided invertible. Nevertheless, a few works where one-sided invertibility was discussed, have appeared in literature recently. They are mainly concerned with two special cases of Toeplitz plus Hankel operators—viz. with the operators having the form $M(a) := T(a) + H(a)$ or the form $\widetilde{M}(a) := T(a)+H(\widetilde{a})$, where $\widetilde{a}(t) := a(1/t)$, $t \in \mathbf{T}$ and $\mathbf{T} := \{t \in \mathbf{C} \colon |t| = 1\}$ is the counterclockwise oriented unit circle [1, 4, 5]. Thus one-sided invertibility of Fredholm operators $M(a)$ was established in [1]. Similar problems are studied in [4, 5] but a different approach is employed. Moreover, in order to describe the kernel and cokernel dimensions of Fredholm Toeplitz plus Hankel operators $T(a)+H(b)$ the method of asymmetric factorization has been proposed in [1, 7]. This method is similar to the Wiener-Hopf factorization used in the theory of Toeplitz operators. Still, for $a \neq b$ the papers [1, 7] do not offer effectively verifiable conditions of one-sided invertibility of the operators under consideration. On the other hand, in recent paper

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[2], Toeplitz plus Hankel operators with piecewise continuous generating functions are considered on the classical Hardy spaces H^p, $1 < p < \infty$ under the assumption

$$a\widetilde{a} = b\widetilde{b}. \tag{1.1}$$

The approach of [2] is based on a careful study of two auxiliary functions. Thus assuming that the operator under consideration is Fredholm, the authors establish modified Wiener–Hopf factorizations of the auxiliary functions. These factorizations contain integer valued parameters which are used in the evaluation of the defect numbers of the operator $T(a) + H(b)$. It is worth noting that in certain cases, the results depend not only on the parameters mentioned but also on the factorization factors.

The present paper also deals with Toeplitz plus Hankel operators $T(a) + H(b)$, the generating functions of which satisfy condition (1.1). However, our method is completely different from that of [2]. It avoids Wiener–Hopf factorization technique, and in many cases it is much easier to apply. Besides, although the focus here is mainly on piecewise continuous generating functions, many results remain true for general generating functions. We also compute the defect numbers of the operator $T(a) + H(b)$ but in different terms. It is even more important that the method used allows us to obtain a complete description of the kernels and cokernels of certain Toeplitz plus Hankel operators (see Example (4.13)). Moreover, our approach is not limited to Toeplitz plus Hankel operators considered below. The results remains true for Toeplitz plus Hankel operators with generating functions $a, b \in PC_p$ and acting on the spaces $l^p(\mathbf{Z})$, $1 < p < \infty$, and also for Wiener–Hopf plus Hankel operators acting on $L^p(\mathbf{R}_+)$. The corresponding operators on weighted spaces with appropriately chosen weights can be studied, as well. Nevertheless, for the sake of simplicity, here we only consider the case of H^p spaces.

2. Spaces and operators

Let X be a Banach space. By $\mathcal{L}(X)$ we denote the Banach algebra of all linear continuous operators on X. An operator $A \in \mathcal{L}(X)$ is called Fredholm if the range $\operatorname{im} A := \{Ax \colon x \in X\}$ of the operator A is a closed subset of X and the null spaces $\ker A := \{x \in X \colon Ax = 0\}$ and $\ker A^* := \{h \in X^* \colon A^*h = 0\}$ of the operator A and the adjoint operator A^* are finite-dimensional. For the sake of convenience, the null space of the adjoint operator A^* is called the cokernel of A and it is denoted by $\operatorname{coker} A$. Further, if an operator $A \in \mathcal{L}(X)$ is Fredholm, then the number

$$\kappa := \dim \ker A - \dim \operatorname{coker} A$$

is referred to as the index of the operator A. Note that by $\dim Y$ we denote the dimension of the linear space Y. Let $\mathcal{K}(X)$ denote the set of all compact operators from $\mathcal{L}(X)$. Then the Fredholmness of an operator $A \in \mathcal{L}(X)$ is equivalent to the invertibility of the coset $A + \mathcal{K}(X)$ in the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$.

Let us now introduce some operators and spaces we need. As usual, let $L^\infty(\mathbf{T})$ stand for the C^*-algebra of all essentially bounded Lebesgue measurable functions on $\mathbf{T}$, and let $L^p = L^p(\mathbf{T})$, $1 \le p \le \infty$ denote the Banach space of all Lebesgue measurable functions f such that

$$\|f\|_p := \left(\int_{\mathbf{T}} |f(t)|^p \, dt \right)^{1/p}, \ 1 \le p < \infty; \quad \|f\|_\infty := \operatorname*{ess\,sup}_{t \in \mathbf{T}} |f(t)|,$$

is finite. Further, let H^p and $\overline{H^p}$ refer to the Hardy spaces of all functions $f \in L^p$ the Fourier coefficients of which

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \, d\theta$$

vanish for all $n < 0$ and $n > 0$, respectively. It is a classical result that for $p \in (1, \infty)$ the Riesz projection P, defined by

$$P\colon \sum_{n=-\infty}^{\infty} f_n e^{in\theta} \mapsto \sum_{n=0}^{\infty} f_n e^{in\theta},$$

is bounded on the space L^p and its range is the whole space H^p. The operator $Q := I - P$,

$$Q\colon \sum_{n=-\infty}^{\infty} f_n e^{in\theta} \mapsto \sum_{n=-\infty}^{-1} f_n e^{in\theta}$$

is also a projection and its range is a subspace of the codimension one in $\overline{H^p}$.

We also consider the flip operator $J\colon L^p \mapsto L^p$,

$$(Jf)(t) := \overline{t} f(\overline{t}), \quad t \in \mathbf{T},$$

where the bar denotes the complex conjugation. Note that the operator J changes the orientation, satisfies the relations

$$J^2 = I, \quad JPJ = Q, \quad JQJ = P,$$

and for any $a \in L^\infty$,

$$JaJ = \widetilde{a} I.$$

Now let us introduce Toeplitz and Hankel operators $T(a)$ and $H(a)$. The operator $T(a)\colon H^p \mapsto H^p$ is defined for all $a \in L^\infty$ and $1 < p < \infty$ by

$$T(a)\colon f \mapsto Paf.$$

This operator is obviously bounded and

$$\|T(a)\| \le c_p \|a\|_\infty,$$

where c_p is the norm of the Riesz projection on L^p. This operator is called Toeplitz operator generated by the function a. Toeplitz operators with matrix-valued generating functions acting on $H^p \times H^p$ are defined similarly.

For $a \in L^\infty$, the Hankel operator $H(a)\colon H^p \mapsto H^p$ is defined by

$$H(a)\colon f \mapsto PaQJf.$$

It is clear that this operator is also bounded but, in contrast to Toeplitz operators, the corresponding generating function a is not uniquely defined by the operator itself. Note that if a belongs to the space of all continuous functions $C = C(\mathbf{T})$, then the Hankel operator $H(a)$ is compact on the space H^p. Moreover, Hankel operators are never Fredholm, whereas if a Toeplitz operator $T(a)\colon H^p \to H^p$ is Fredholm, then $T(a)$ is one-sided invertible. On the other hand, Fredholm Toeplitz operators with matrix-valued generating functions are not necessarily one-sided invertible, and this fact causes numerous difficulties in their studies.

A function $a \in L^\infty$ is called piecewise continuous if for every $t \in \mathbf{T}$ the one-sided limits $a(t+0)$ and $a(t-0)$ exist. The set of all piecewise continuous functions is denoted by $PC(\mathbf{T})$ or simply by PC. It is well-known that PC is a closed subalgebra of L^∞. Any function from PC has an at most countable set of jumps. Moreover, for each $\delta > 0$ the set $S := \{t \in \mathbf{T} : |a(t+0) - a(t-0)| > \delta\}$ is finite.

If $\mathcal{B}$ is a unital subalgebra of L^∞ containing the algebra C, then we denote by $T(\mathcal{B})$ and $TH(\mathcal{B})$ the smallest closed subalgebras of $\mathcal{L}(H^p)$ containing, respectively, all Toeplitz operators $T(a)$ and all Toeplitz plus Hankel operators $T(a) + H(b)$ with generating functions from $\mathcal{B}$. In the case where $\mathcal{B} = C$ the algebras $TH(C)$ and $T(C)$ coincide since the algebra $T(C)$ already contains all compact operators. Additionally, an operator $T(a) + H(b)$ is Fredholm if and only if $a(t) \neq 0$ for all $t \in \mathbf{T}$ and

$$\operatorname{ind}(T(a) + H(b)) = \operatorname{ind} T(a) = -\operatorname{wind} a,$$

where $\operatorname{wind} a$ is the winding number of the contour $\Gamma := \{a(t) : t \in \mathbf{T}\}$ with respect to the origin. The algebras $T(PC)$ and $TH(PC)$ are much more complicated and will be discussed in Section 4.

The theory of Toeplitz plus Hankel operators is heavily based upon the relations

$$T(ab) = T(a)T(b) + H(a)H(\widetilde{b}), \tag{2.1}$$

$$H(ab) = T(a)H(b) + H(a)T(\widetilde{b}), \tag{2.2}$$

which show that Toeplitz and Hankel operators are closely related even for general $a, b \in L^\infty$. A consequence of (2.1) is that for $a = c_- c c_+$ with $c_- \in \overline{H^\infty}, c \in L^\infty, c_+ \in H^\infty$, the representation $T(a) = T(c_-)T(c)T(c_+)$ holds. This representation is often used in the forthcoming sections. More facts about Toeplitz and Hankel operators can be found in [3, 10, 11].

3. A method to study invertibility in $TH(L^\infty)$

To study the invertibility of the operators $T(a) + H(b)$ acting on the space H^p, $1 < p < \infty$ we shall make use of a well-known classical formula. Let X and Y be arbitrary operators acting on the space L^p and let J be the above defined flip operator. Then the relation

$$\begin{pmatrix} X & Y \\ JYJ & JXJ \end{pmatrix} = \frac{1}{2} \begin{pmatrix} I & I \\ J & -J \end{pmatrix} \begin{pmatrix} X+YJ & 0 \\ 0 & X-YJ \end{pmatrix} \begin{pmatrix} I & J \\ I & -J \end{pmatrix} \tag{3.1}$$

holds, and the factors

$$\frac{1}{2} \begin{pmatrix} I & I \\ J & -J \end{pmatrix}, \quad \begin{pmatrix} I & J \\ I & -J \end{pmatrix}.$$

are the inverses to each other. This identity is widely used to study singular integral operators with conjugation and also equations with Carleman shifts preserving orientation.

Note that the operators $T(a) \pm H(b) : H^p \mapsto H^p$ are invertible or Fredholm if and only if so are the operators $T(a) \pm H(b) + Q = (PaP + Q) \pm PbQJ : L^p \mapsto L^p$. Thus we can study the modified operator $T(a) \pm H(b) + Q$ instead of the initial one, and if we set $X = PaP + Q$, $Y = PbQ$, then relation (3.1) leads to the representation

$$\begin{aligned} \mathcal{A} := \begin{pmatrix} PaP+Q & PbQ \\ Q\widetilde{b}P & Q\widetilde{a}Q + P \end{pmatrix} &= \begin{pmatrix} PaP+Q & PbQ \\ JPbQJ & J(PaP+Q)J \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} I & I \\ J & -J \end{pmatrix} \begin{pmatrix} PaP+Q+PbQJ & 0 \\ 0 & PaP+Q-PbQJ \end{pmatrix} \begin{pmatrix} I & J \\ I & -J \end{pmatrix}. \end{aligned} \tag{3.2}$$

Thus the operator $\operatorname{diag}(T(a)+H(b), T(a)-H(b))$ acting on the space $H^p \times H^p$ has the same Fredholm properties as the operator $\mathcal{A}$ acting on $L^p \times L^p$, $1<p<\infty$. In particular, the kernels of these two operators have the same dimension and so are the cokernels. Let us also recall that the shift J changes the orientation and this fact causes essential complications, for example, it can happen that one of the operators $T(a) \pm H(b)$ is Fredholm but the other one is not. Examples where such situation occurs are presented in Section 4, cf. Examples 4.13–4.16.

From now on we always assume that the generating function a is invertible. This does not lead to loss of generality since Fredholmness of the operator $T(a)+H(b)$ implies the invertibility of a in L^∞, [1, 15]. Now we can formulate the first result.

Theorem 3.1. *The operators* $\operatorname{diag}(T(a)+H(b), T(a)-H(b))\colon H^p \times H^p \mapsto H^p \times H^p$ *and* $T(U(a,b))\colon H^p\times H^p \mapsto H^p \times H^p$,

$$U(a,b) = \begin{pmatrix} a - b\widetilde{b}\widetilde{a}^{-1} & -b\widetilde{a}^{-1} \\ \widetilde{b}\widetilde{a}^{-1} & \widetilde{a}^{-1} \end{pmatrix} \tag{3.3}$$

are simultaneously Fredholm or not. If they are Fredholm, then

$$\operatorname{ind}\operatorname{diag}(T(a)+H(b), T(a)-H(b)) = \operatorname{ind} T(U(a,b)). \tag{3.4}$$

Moreover, the kernel and cokernel dimensions of these operators coincide.

Proof. Let us first represent the operator $\mathcal{A}$ in a different form. To this end we recall the following fact. If p is an idempotent in a unital Banach algebra $\mathcal{C}$ with identity e, then the elements $ap+(e-p)$ and $pap+(e-p)$ have identical invertibility properties. Consequently, if $\mathcal{C}=\mathcal{L}(Z)$, where Z is a Banach space, then $ap+(e-p)$ and $pap+(e-p)$ have identical Fredholm properties. This immediately follows from the relation

$$ap+(e-p) = (pap+(e-p))(e+(e-p)ap), \tag{3.5}$$

and from the invertibility of the element $e+(e-p)ap$ with $((e+(e-p)ap)^{-1} = e-(e-p)ap$. Consider now the idempotent $p \in \mathcal{L}(L^p \times L^p)$ defined by $p := \operatorname{diag}(P,Q)$. Then the operator $\mathcal{A}$ of (3.2) can be represented in the form [8],

$$\mathcal{A} = p \begin{pmatrix} a & b \\ \widetilde{b} & \widetilde{a} \end{pmatrix} p + (\operatorname{diag}(I,I) - p).$$

The relation (3.5) implies that the operators $\mathcal{A}$ and $\widehat{\mathcal{A}}$,

$$\widehat{\mathcal{A}} = \begin{pmatrix} a & b \\ \widetilde{b} & \widetilde{a} \end{pmatrix} \operatorname{diag}(P,Q) + \operatorname{diag}(Q,P)$$

have identical Fredholm properties. On the other hand, the operator $\widehat{\mathcal{A}}$ can also be written as

$$\widehat{\mathcal{A}} = \begin{pmatrix} a & 0 \\ \widetilde{b} & 1 \end{pmatrix} \operatorname{diag}(P,P) + \begin{pmatrix} 1 & b \\ 0 & \widetilde{a} \end{pmatrix} \operatorname{diag}(Q,Q).$$

Now we note that the invertibility of the function a implies the invertibility of the matrix

$$R := \begin{pmatrix} 1 & b \\ 0 & \widetilde{a} \end{pmatrix}.$$

Therefore, applying the relation (3.5) again but to the operator $R^{-1}\widehat{\mathcal{A}}$, we obtain the claim. $\square$

Remark 3.2. If the operator $T(U(a,b))$ is Fredholm, then its kernel and cokernel dimensions can be expressed via the partial indices of the matrix $U(a,b)$. However, the computation of the partial indices is a very complicated task, and nowadays there are only a few classes of matrices the partial indices of which can be determined [12]. In the present paper we derive some results on one-sided invertibility of Toeplitz plus Hankel operators but our method is not directly linked to the partial indices approach.

Let us make an important assumption, viz. let us assume that the generating functions of the corresponding Toeplitz plus Hankel operator $T(a)+H(b)$ satisfy the condition

$$a(t)\widetilde{a}(t) = b(t)\widetilde{b}(t) = 1, \quad t \in \mathbf{T}. \tag{3.6}$$

Equation (3.6) is called the matching condition, and the corresponding pair of functions (a,b) is called matching pair. Note that the set of all matching pairs $(a,b) \in L^\infty \times L^\infty$ endowed with the operation $(a_1,b_1)(a_2,b_2) := (a_1a_2, b_1b_2)$ is a group. For any matching pair (a,b), consider the functions $c := ab^{-1}$ and $d := b\widetilde{a}^{-1}$ and call them the matching functions for (a,b) or, simply, the matching functions. Moreover, the duo (c,d) is referred to as the subordinated matching pair for the pair (a,b). Obviously, any matching function possesses the property

$$c(t)\widetilde{c}(t) = d(t)\widetilde{d}(t) = 1, \quad t \in \mathbf{T}. \tag{3.7}$$

On the other hand, if a function c satisfies the equation (3.7), then it is the matching function for any pair (ac, a) and for any pair $(a, a\widetilde{c})$, $a \in L^\infty$. Therefore, in the following any function satisfying (3.7) is called matching function. Moreover, if c_1, c_2 are matching functions, then the product c_1c_2 is the one as well.

An example of a matching function is the function $c = c(t) = t^n, t \in \mathbf{T}, n \in \mathbf{Z}$. The set of matching functions is quite large. For instance, let an element $g_0 \in L^\infty$ be continuous at the points $t = \pm 1$, invertible in L^∞ and such that $g_0(\pm 1) \in \{-1, 1\}$. Let $\mathbf{T}_+ := \{t \in \mathbf{T} : \operatorname{Im} t \geq 0\}$ be the upper half-circle. Set

$$g(t) = \begin{cases} g_0(t) & \text{if } t \in \mathbf{T}_+, \\ g_0^{-1}(\overline{t}) & \text{if } t \in \mathbf{T} \setminus \mathbf{T}_+. \end{cases}$$

Then $g\widetilde{g} = 1$ for all $t \in \mathbf{T}$, so g is a matching function.

In passing note that if (a,b) is a matching pair, then the matrix $U(a,b)$ takes the form

$$U(a,b) = \begin{pmatrix} 0 & -b\widetilde{a}^{-1} \\ \widetilde{b}^{-1}\widetilde{a}^{-1} & \widetilde{a}^{-1} \end{pmatrix} = \begin{pmatrix} 0 & -d \\ c & \widetilde{a}^{-1} \end{pmatrix}.$$

Thus, in this case $U(a,b)$ is a triangular matrix and the corresponding Toeplitz plus Hankel operator can be studied in more detail.

Now we need a few more facts concerning the connections between the Fredholmness of certain matrix operators and their entries. Let $\mathcal{R}$ be an associative ring with the unit e. Denote by $\mathcal{R}^{2\times 2}$ the set of all 2×2 matrices with entries from $\mathcal{R}$. This set, equipped with usual matrix operations, forms a non-commutative ring with the unit $E = \operatorname{diag}(e,e)$.

Proposition 3.3. *Let $\alpha, \beta, \gamma \in \mathcal{R}$. Then:*

(i) *If the matrix*

$$\Upsilon := \begin{pmatrix} 0 & -\beta \\ \alpha & \gamma \end{pmatrix}$$

is invertible, then the elements α and β are, respectively, invertible from the left and from the right. If, in addition, one of these two elements is invertible, then so is the other.

(ii) *If both elements α and β are invertible, then the matrix Υ is invertible too.*

Proof. The matrix Υ can be represented in the form

$$\begin{pmatrix} 0 & -\beta \\ \alpha & \gamma \end{pmatrix} = \begin{pmatrix} \beta & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} 0 & -e \\ e & \gamma \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & e \end{pmatrix},$$

which immediately leads to the result desired since the middle factor in the left-hand side is invertible. □

To prepare the next theorem we need a few more notation. If $u, v \in L^\infty$ are functions such that the operators $T(u)$ and $T(v)$ are Fredholm, and hence one-sided invertible, we denote by (κ_1, κ_2) the pair $(\operatorname{ind} T(v), \operatorname{ind} T(u))$. Let $\mathbf{Z}_+$ and $\mathbf{Z}_-$ be, respectively, the sets of all non-negative and negative integers. Finally, if $T(\psi)$, $\psi \in L^\infty$ is Fredholm, then ψ can be represented in the form $\psi = \psi_0 t^n$, where the operator $T(\psi_0)$ is invertible and $n = -\operatorname{ind} T(\psi)$. Thus for $n \le 0$, a right inverse for the operator $T(\psi) = T(t^n)T(\psi_0)$ is $T_r^{-1}(\psi) = T^{-1}(\psi_0)T(t^{-n})$. If $n \ge 0$, a left inverse for $T(\psi) = T(\psi_0)T(t^n)$ is $T_l^{-1}(\psi) = T(t^{-n})T^{-1}(\psi_0)$. Given functions $u, v, w \in L^\infty$, let us consider the matrix function G defined by

$$G := \begin{pmatrix} 0 & -v \\ u & w \end{pmatrix}.$$

Theorem 3.4. *Suppose that one of the operators $T(u)$ or $T(v)$ is Fredholm. Then the Toeplitz operator $T(G)$ is Fredholm if and only if both operators $T(u)$ and $T(v)$ are Fredholm. If they are Fredholm, then*

$$\operatorname{ind} T(G) = \operatorname{ind} \operatorname{diag}(T(v), T(u)) = \operatorname{ind} T(v) + \operatorname{ind} T(u).$$

Moreover:

(i) *If (κ_1, κ_2) belongs to one of the sets $\mathbf{Z}_+ \times \mathbf{Z}_+$, $\mathbf{Z}_- \times \mathbf{Z}_+$, $\mathbf{Z}_- \times \mathbf{Z}_-$, then*

$$\dim\ker T(G) = \dim\ker \operatorname{diag}(T(v), T(u)). \tag{3.8}$$

(ii) *If $(\kappa_1, \kappa_2) \in \mathbf{Z}_+ \times \mathbf{Z}_-$, then*

$$\begin{aligned} \dim\ker T(G) &= \dim\ker \left(P_{-\kappa_2-1} T^{-1}(u_0) T(w) \big|_{\ker T(v)}\right) \\ &= \dim\ker \left(P_{-\kappa_2-1} T^{-1}(u_0) T(w) T^{-1}(v_0) \big|_{\operatorname{im} P_{\kappa_1-1}}\right) \end{aligned} \tag{3.9}$$

with $P_{m-1} := I - T(t^m)T(t^{-m})$, $m \ge 1$.

Proof. It is clear that if $T(G)$ and one of the operators $T(u)$, $T(v)$ is Fredholm, then so is $\operatorname{diag}(T(v), T(u))$. Really, since the operator $T(G)$ can be represented in the form

$$T(G) = \begin{pmatrix} T(v) & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & T(w) \end{pmatrix} \begin{pmatrix} T(u) & 0 \\ 0 & I \end{pmatrix}, \tag{3.10}$$

and the middle factor in the right-hand side of (3.10) is invertible, the claim follows. On the other hand, if $T(G)$ is not Fredholm then, according to (3.10), one of the operators $T(u)$, $T(v)$ is also not Fredholm. In order to establish identity (3.8), one has to employ the representation (3.10) once more.

The proof of the relation (3.9) is more complicated. To this end one has to study the set $S_1 := \ker(\operatorname{diag}(T(v), I)) \cap \operatorname{im} A_1$, where

$$A_1 = \begin{pmatrix} 0 & -I \\ I & T(w) \end{pmatrix} \begin{pmatrix} T(u) & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & -I \\ T(u) & T(w) \end{pmatrix}.$$

Let $T_l^{-1}(u)$ denote one of the left inverses of the operator $T(u)$, and let A_2 be the matrix operator defined by

$$A_2 := \begin{pmatrix} T_l^{-1}(u)T(w) & I \\ -I & 0 \end{pmatrix}.$$

Then

$$A_1A_2 = \begin{pmatrix} I & 0 \\ (T(u)T_l^{-1}(u) - I)T(w) & T(u) \end{pmatrix},$$

and

$$\begin{pmatrix} I & 0 \\ 0 & T_l^{-1}(u) \end{pmatrix} A_1A_2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

so the operator A_1A_2 is left-invertible. Since the operator A_2 is invertible, one obtains that $\operatorname{im} A_1 = \operatorname{im}(A_1A_2)$. It follows from the theory of one-sided invertible operators [10] that the operator R,

$$R := A_1A_2 \operatorname{diag}(I, T_l^{-1}(u)) = \begin{pmatrix} I & 0 \\ (T(u)T_l^{-1}(u) - I)T(w) & T(u)T_l^{-1}(u) \end{pmatrix}$$

is a projection onto $\operatorname{im} A_1A_2 = \operatorname{im} A_1$. Moreover, the projection R has the same kernel as the operator $\operatorname{diag}(I, T_l^{-1}(u))$. Note that $\ker(\operatorname{diag}(T(v), I)) = (\ker T(v), 0)^t$, where $(\alpha, \beta)^t$ denotes the vector-column with the entries α and β. For $x \in \ker T(v)$ the element $(x, 0)^t$ belongs to the set $\operatorname{im} A_1$ if and only if $R((x, 0)^t) = (x, 0)^t$, i.e. if

$$x \in \ker\left((T(u)T_l^{-1}(u) - I)T(w)\,|_{\ker T(v)}\right).$$

Using now the above mentioned representation $T_l^{-1}(u) = T(t^{\kappa_2})T^{-1}(u_0)$, one obtains

$$((T(u)T_l^{-1}(u) - I)T(w)\,|_{\ker T(v)} = T(u_0)((T(t^{-\kappa_2})T(t^{\kappa_2}) - I)T^{-1}(u_0)T(w))\,|_{\ker T(v)}.$$

Because $T(u_0)$ is invertible, we get that $(x, 0)^t$, $x \in \ker T(v)$ belongs to $\operatorname{im} A_1$ if and only if

$$x \in \ker\left(P_{-\kappa_2-1}T^{-1}(u_0)T(w)\,|_{\ker T(v)}\right) = \ker\left(P_{-\kappa_2-1}T^{-1}(u_0)T(w)T^{-1}(v_0)\,|_{\operatorname{im} P_{\kappa_1-1})}\right).$$

The theorem is proved. □

Set now $u = c$, $v = d$ and $w = \widetilde{a}^{-1}$.

Corollary 3.5. *Let $(a, b) \in L^\infty \times L^\infty$ be a matching pair with matching functions c and d. Then:*

(i) *If $\operatorname{ind} T(c) \geq 0$ and $\operatorname{ind} T(d) \geq 0$, then both operators $T(a) + H(b)$ and $T(a) - H(b)$ are right-invertible.*
(ii) *If $\operatorname{ind} T(c) \leq 0$ and $\operatorname{ind} T(d) \leq 0$, then both operators $T(a) + H(b)$ and $T(a) - H(b)$ are left-invertible.*

Moreover, employing Theorems 3.1 and 3.4, one can also describe the kernel dimensions of Toeplitz plus Hankel operators $T(a) \pm H(b)$. Thus for PC-generating functions a and b the following theorem is true.

Theorem 3.6. *If $a, b \in PC$ form a matching pair with matching functions c and d, then the operators* $\operatorname{diag}(T(a)+H(b), T(a)-H(b))$ *and* $\operatorname{diag}(T(d), T(c))$ *are simultaneously Fredholm or not. If they are Fredholm, then*

$$\begin{aligned} \operatorname{ind}\operatorname{diag}(T(a)+H(b), T(a)-H(b)) &= \operatorname{ind}\operatorname{diag}(T(d), T(c)) \\ &= \operatorname{ind} T(d) + \operatorname{ind} T(c). \end{aligned} \tag{3.11}$$

Moreover, if $\kappa_1 := \operatorname{ind} T(d)$ and $\kappa_2 := \operatorname{ind} T(c)$, then the kernels of the above mentioned operators are connected in the following way.

(i) *If (κ_1, κ_2) belongs to one of the sets $\mathbf{Z}_+ \times \mathbf{Z}_+$, $\mathbf{Z}_- \times \mathbf{Z}_+$, $\mathbf{Z}_- \times \mathbf{Z}_-$, then*

$$\dim\ker(\operatorname{diag}(T(a)+H(b), T(a)-H(b))) = \dim\ker(\operatorname{diag}(T(d), T(c))). \tag{3.12}$$

(ii) *If $(\kappa_1, \kappa_2) \in \mathbf{Z}_+ \times \mathbf{Z}_-$, then*

$$\begin{aligned} &\dim\ker\operatorname{diag}(T(a)+H(b), T(a)-H(b)) \\ &= \dim\ker\left(P_{-\kappa_2+1} T^{-1}(ct^{-\kappa_2}) T(\widetilde{b}^{-1} c) T^{-1}(dt^{\kappa_1})\,\big|_{\operatorname{im} P_{\kappa_1 - 1}}\right) \end{aligned} \tag{3.13}$$

Proof. Let us first point out that semi-Fredholm Toeplitz operators with piecewise continuous generating functions are indeed Fredholm operators. The latter statement can be easily derived from Proposition 3.1 in Section 9.3 of [11] in conjunction with Lemma 1 in Section 18 of [13]. The remaining part of the proof immediately follows from Theorems 3.1 and 3.4.

The last theorem will be used later in the study of invertibility of Toeplitz plus Hankel operators with piecewise continuous generating functions. Note that necessary information on properties of such operators, including an index formula, will be provided in Section 4. □

4. *PC*-generating functions

For the following, we need additional results concerning Toeplitz and Toeplitz plus Hankel operators with piecewise continuous generating functions acting on the space H^p, $1 < p < \infty$. For, let us introduce the functions

$$\nu_p(y) := \frac{1}{2}\left(1 + \coth\left(\pi\left(y + \frac{i}{p}\right)\right)\right), \quad h_p(y) := \sinh^{-1}\left(\pi\left(y + \frac{i}{p}\right)\right),$$

where $y \in \overline{\mathbf{R}}$, and $\overline{\mathbf{R}}$ refers to the two-point compactification of $\mathbf{R}$. Note that for given points $u, w \in \mathbf{C}$, $u \neq w$ the set $\mathcal{A}_p(u,w) := \{z \in \mathbf{C}\colon z = u(1-\nu_p(y)) + w\nu_p(y), y \in \overline{\mathbf{R}}\}$ forms a circular arc which starts at u and ends at w as y runs through $\overline{\mathbf{R}}$. This arc $\mathcal{A}_p(u,w)$ has the property that from any point of the arc, the line segment $[u,w]$ is seen at the angle $2\pi/(\max\{p,q\})$, $1/p + 1/q = 1$. Moreover, if $2 < p < \infty$ $(1 < p < 2)$ the arc $\mathcal{A}_p(u,w)$ is located on the right-hand side (left-hand side) of the straight line passing through the points u and w and directed from u to w. If $p = 2$, the set $\mathcal{A}_p(u,w)$ coincides with the line segment $[u,w]$.

Let us list some properties of the function h_p. From the relation

$$\sinh\left(\pi\left(y + \frac{i}{p}\right)\right) = \cos\left(\frac{\pi}{p}\right)\sinh(\pi y) + i\sin\left(\frac{\pi}{p}\right)\cosh(\pi y),$$

one easily obtains the following result.

(i) The real (imaginary) part of h_p is an odd (even) function.

(ii) The function $h_p\colon \mathbf{R} \to \mathbf{C}$ takes values in the lower half-plane $\Pi^- := \{z \in \mathbf{C}\colon \operatorname{Im} z < 0\}$ and $h_p(\pm\infty) = 0$.

(iii) If y runs from $-\infty$ to $+\infty$, the values of h_p trace out an oriented closed continuous curve. For $1 < p \le 2$,

$$h_p(y) \in \begin{cases} \Omega_1 & \text{if } y \in [-\infty, 0], \\ \Omega_2 & \text{if } y \in [0, \infty], \end{cases}$$

where $\Omega_1 := \{z \in \mathbf{C}\colon \operatorname{Im} z \le 0 \text{ and } \operatorname{Re} z \ge 0\}$, $\Omega_2 := \{z \in \mathbf{C}\colon \operatorname{Im} z \le 0 \text{ and } \operatorname{Re} z \le 0\}$. On the other hand, for $2 \le p < \infty$,

$$h_p(y) \in \begin{cases} \Omega_2 & \text{if } y \in [-\infty, 0], \\ \Omega_1 & \text{if } y \in [0, \infty]. \end{cases}$$

Thus for $p = 2$ the point $h_p(y), y \in \mathbf{R}$ moves along the imaginary axis from the origin to the point $z_0 = -i$ and then goes back to the origin.

(iv) For all $y \in \overline{\mathbf{R}}$ the function h_p satisfies the inequality

$$|\operatorname{Im} h_p(y)| \le \frac{1}{\sin(\pi/p)}, \tag{4.1}$$

and $y = 0$ is the only point where equality holds.

As was already mentioned, $T(PC)$ and $TH(PC)$ are the smallest closed subalgebras of $\mathcal{L}(H^p)$ containing all Toeplitz operators $T(a)$ and Toeplitz plus Hankel operators $T(a) + H(b)$ with $a, b \in PC$, respectively. An index formula for Fredholm operators from $T(PC) \subset \mathcal{L}(H^p)$ is well-known and goes back to Gohberg and Krupnik [3, 10]. Let us now recall some results from [3] and [14]. First of all, we note that the algebra $T(PC)$ contains the set $\mathcal{K}(H^p)$ of all compact operators from $\mathcal{L}(H^p)$, and the Banach algebra $T^\pi(PC) := T(PC)/\mathcal{K}(H^p)$ is a commutative subalgebra of the Calkin algebra the maximal ideal space of which $\mathcal{M}(T^\pi(PC))$ can be identified with the cylinder $\mathbf{T} \times [0, 1]$ equipped with an exotic topology. However, in this paper we prefer to identify the space of maximal ideal of the above algebra with $\mathbf{T} \times \overline{\mathbf{R}}$, and by $\operatorname{smb} T^\pi(PC)$ we denote the Gelfand transform of the algebra $T^\pi(PC)$.

Theorem 4.1. *Let $T^\pi(PC)$ be the above defined Banach algebra. Then*

(i) *On the generators $T^\pi(a) := T(a) + \mathcal{K}(H^p)$, $a \in PC$ the Gelfand transform $\operatorname{smb}\colon T^\pi(PC) \mapsto C(\mathcal{M}(T^\pi(PC)))$ is given by*

$$\operatorname{smb} T^\pi(a)(t, y) = a(t+0)\nu_p(y) + a(t-0)(1 - \nu_p(y)), \quad (t, y) \in \mathbf{T} \times \overline{\mathbf{R}}.$$

(ii) *The operator $A \in T(PC)$ is Fredholm if and only if*

$$(\operatorname{smb} A^\pi)(t, y) \ne 0 \quad \text{for all } (t, y) \in \mathbf{T} \times \overline{\mathbf{R}}.$$

(iii) *If $A \in T(PC)$ is Fredholm, then*

$$\operatorname{ind} A = -\operatorname{wind} \operatorname{smb} A^\pi.$$

Remark 4.2. The function $\operatorname{smb} A$ traces out an oriented curve, the orientation of which is inherited from those of $\overline{\mathbf{R}}$ and $\mathbf{T}$. Thus if $\operatorname{smb} A$ does not cross the origin, the winding number $\operatorname{wind} \operatorname{smb} A$ is well-defined. We shall also agree on writing $\operatorname{smb} A$

for $\operatorname{smb} A^\pi$, so we can think about the homomorphism smb as a function acting on the whole algebra $T(PC)$. In this case, $\operatorname{smb} A = 0$ for any compact operator A.

Later on we will see that for generating functions $b \in PC$ having discontinuity points on $\mathbf{T} \setminus \{-1, 1\}$, the Hankel operator $H(b)$ does not belong to the algebra $T(PC)$. On the other hand, we need a non-trivial result that for $b \in PC \cap C(\mathbf{T} \setminus \{-1, 1\})$ the operator $H(b) \in T(PC)$. Let us give an idea of the proof for this fact. First of all, one can observe that it suffices to show that for the characteristic function χ_+ of the upper half-circle $\mathbf{T}_+$, the corresponding Hankel operator $H(\chi_+)$ belongs to the Toeplitz algebra $T(PC)$. However, the last inclusion could be verified following localization arguments from [14, pp. 245–247] and also using Theorem 4.4.5(iii) from there. Moreover, it is established in [14] that the Gelfand transform of the coset $H^\pi(b)$, $b \in PC \cap C(\mathbf{T} \setminus \{-1, 1\})$ is

$$\operatorname{smb} H^\pi(b)(t, y) = \begin{cases} \dfrac{b(1+0) - b(1-0)}{2}\, h_p(y) & \text{if } t = 1, \\ -\dfrac{b(-1+0) - b(-1-0)}{2}\, h_p(y) & \text{if } t = -1, \\ 0 & \text{if } t \in \mathbf{T} \setminus \{-1, 1\}, \end{cases}$$

and

$$\operatorname{ind}(T(a) + H(b)) = -\operatorname{wind} \operatorname{smb}(T(a) + H(b)).$$

The Fredholm theory for operators belonging to $TH(PC)$ is considerably more complicated than that for $T(PC)$. This is due to the fact that the Calkin image $TH^\pi(PC) := TH(PC)/\mathcal{K}(H^p)$ of the algebra $TH(PC)$ is not commutative. Let $\overset{\circ}{\mathbf{T}}_+ := \{t \in \mathbf{T}_+ : \operatorname{Im} t > 0\}$.

Theorem 4.3. *If $a, b \in PC$, then*

(i) *The operator $T(a) + H(b)$ is Fredholm if and only if the matrix*

$$\operatorname{smb}(T(a) + H(b))(t, y) :=$$
$$\begin{pmatrix} a(t+0)\nu_p(y) + a(t-0)(1-\nu_p(y)) & \dfrac{b(t+0) - b(t-0)}{2i}\, h_p(y) \\ \dfrac{b(\bar t - 0) - b(\bar t + 0)}{2i}\, h_p(y) & a(\bar t + 0)\nu_p(y) + a(\bar t - 0)(1 - \nu_p(y)) \end{pmatrix}$$

is invertible for every $(t, y) \in \overset{\circ}{\mathbf{T}}_+ \times \overline{\mathbf{R}}$ and the function

$$\begin{aligned} \operatorname{smb}(T(a) + H(b))(t, y) :={}& a(t+0)\nu_p(y) + a(t-0)(1-\nu_p(y)) \\ &+ t\, \frac{b(t+0) - b(t-0)}{2}\, h_p(y) \end{aligned}$$

does not vanish on $\{-1, 1\} \times \overline{\mathbf{R}}$.

(ii) *The mapping* smb *defined in assertion (i) extends for each $(t, y) \in \mathbf{T}_+ \times \overline{\mathbf{R}}$ to an algebra homomorphism on the whole $TH(PC)$.*

(iii) *An operator $A \in TH(PC)$ is Fredholm if and only if the function*

$$\operatorname{smb} A \colon \mathbf{T}_+ \times \overline{\mathbf{R}} \mapsto \mathbf{C}$$

is invertible.

Remark 4.4. (i) Theorem 4.3 indicates that the algebra $TH^\pi(PC)$ is not commutative. In particular, this means that Hankel operators with generating functions from $PC(\mathbf{T})$ with discontinuity points on $\mathbf{T} \setminus \{-1, 1\}$ do not belong to the algebra $T(PC)$.

(ii) This theorem also shows that the roles of the points $t = \pm 1$ and $t \in \overset{\circ}{\mathbf{T}}_+$ are very different. In connection with this, let us remaind that $t = \pm 1$ are the only fixed points of the operator J.

(iii) There is an index formula for Fredholm operators $T(a) + H(b)$ considered on the space $l^p(\mathbf{Z}_+)$ [17]. The approach of [17] can be modified in order to get a similar result for Toeplitz plus Hankel operators acting on H^p-spaces. On the other hand, the index formula presented in Proposition 4.8 below, is relatively simple and sufficiently effective to handle all problems considered in this work.

(iv) If $a \in PC$, then the symbols of the operator $T(a)$ defined by Theorems 4.1 and 4.3 coincide.

Let us note that for the first time Theorem 4.3 appeared in [16], see also [14]. We proceed with additional preparatory material.

Theorem 4.5. *If $a, b \in PC$ are continuous at the points $t = -1$ and $t = 1$, then the operators $T(a) + H(b)$ and $T(a) - H(b)$ are simultaneously Fredholm or not. If one of this operators is Fredholm, then*

$$\operatorname{ind}(T(a) + H(b)) = \operatorname{ind}(T(a) - H(b)).$$

Moreover,

$$\operatorname{ind}(T(a) + H(b)) = \frac{1}{2}\operatorname{ind} T(U),$$

where $T(U)$ is the block Toeplitz operator on $H^p \times H^p$ with generating matrix-function $U = U(a, b)$ defined by (3.3).

The first assertion can be proved analogously to [17], see also [6], and the second one directly follows from the relation (3.4).

Remark 4.6. In the case at hand, the index computation is reduced to the similar problem for a block Toeplitz operator the generating matrix of which has piecewise continuous entries. The index formula for such operators is known [3, Chapter 5].

As the next step we prove the following proposition.

Proposition 4.7. *If $a, b \in PC$ and the operator $T(a) + H(b)$ is Fredholm, then*

$$(T(a) + H(b)) - (T(g) + H(b_0))(T(ag^{-1}) + H(b_1 g^{-1})) \in \mathcal{K}(H^p), \tag{4.2}$$

where b_0 is a function continuous on $\mathbf{T} \setminus \{-1, 1\}$ such that $b_0(1 \pm 0) = b(1 \pm 0)$, $b_0(-1 \pm 0) = b(-1 \pm 0)$, $b_1 := b - b_0$, and g is a function on $\mathbf{T}$, which is invertible, continuous on $\mathbf{T} \setminus \{-1, 1\}$ and $a(1 \pm 0) = g(1 \pm 0)$, $a(-1 \pm 0) = g(-1 \pm 0)$. Moreover, the operators $T(g) + H(b_0)$ and $T(ag^{-1}) + H(b_1 g^{-1})$ are also Fredholm and $T(g) + H(b_0)$ belongs to the algebra $T(PC)$.

Proof. By Theorem 4.3, the function a is invertible in PC. The existence of functions b_0 and g with the properties prescribed is obvious. Notice that ag^{-1} is continuous at the points $t = \pm 1$ and $ag^{-1}(\pm 1) = 1$. Using the relations (2.1)–(2.2) and Corollary 5.33 from [3], one arrives at the representation (4.2). The Fredholmness of all the Toeplitz plus Hankel operators appeared is clear, and the inclusion $T(g)+H(b_0)\in T(PC)$ for such a kind operators has been mentioned before. $\square$

Now we are able to establish a transparent index formula for the operators under consideration.

Proposition 4.8. *If $T(a)+H(b)$ is Fredholm, then*

$$\operatorname{ind}(T(a)+H(b)) = -\operatorname{wind}\operatorname{smb}(T(g)+H(b_0)) + \frac{1}{2}\operatorname{ind}T(U_1),$$

where

$$U_1 = \begin{pmatrix} a_2 - b_2\widetilde{b}_2\widetilde{a}_2^{-1} & -b_2\widetilde{a}_2^{-1} \\ \widetilde{b}_2\widetilde{a}_2^{-1} & \widetilde{a}_2^{-1} \end{pmatrix}$$

and $a_2 = ag^{-1}$, $b_2 = b_1 g^{-1}$.

Proof. Remark that the function b_2 vanishes at the points $t=-1$ and $t=1$. The result now follows from the representation (4.2) and Theorem 4.5. $\square$

Remark 4.9. Proposition 4.8 indicates that if both operators $T(a)\pm H(b)$ are Fredholm, the difference between their indices depends only on the index difference for the operators $T(g)+H(b_0)$ and $T(g)-H(b_0)$. However, one can observe that

$$\operatorname{ind}(T(g)+H(b_0)) - \operatorname{ind}(T(g)-H(b_0)) \in \{-2,-1,0,1,2\}.$$

Theorem 4.10. *Let $(a,b)\in PC(\mathbf{T})\times PC(\mathbf{T})$ be a matching pair with matching functions c and d. Then the following assertions hold.*

(i) *The operators $T(a)\pm H(b)$ are Fredholm if and only if so are the Toeplitz operators $T(d)$ and $T(c)$.*

(ii) *The operators $T(a)\pm H(b)$ are invertible from the right if*

$$\operatorname{ind}T(c)\geq 0 \quad \text{and} \quad \operatorname{ind}T(d)\geq 0.$$

(iii) *The operators $T(a)\pm H(b)$ are invertible from the left if*

$$\operatorname{ind}T(c)\leq 0 \quad \text{and} \quad \operatorname{ind}T(d)\leq 0.$$

(iv) *If both numbers $m=\dim\ker T(U(a,b))$ and $n=\dim\operatorname{coker}T(U(a,b))$ are not equal to zero but $\operatorname{ind}(T(a)+H(b))=\operatorname{ind}(T(a)-H(b))$, then at least one of the operators $T(a)+H(b)$ or $T(a)-H(b)$ is not one-sided invertible.*

Notice than certain deficiency of Theorem 4.10 consists in its inability to handle one-sided invertibility in the case where one of the operators, say $T(a)+H(b)$, is Fredholm but $T(a)-H(b)$ is not. This situation can still be studied but to do so we need a result of Shneiberg [18].

Let A be an operator defined on each space L_p, $1<p<\infty$. Then the set $A_F := \{p\in(1,\infty)\colon A \quad \text{is Fredholm}\}$ is open. Moreover, for each connected component of the set A_F, the index of A is constant.

Let us now assume that on a space H^p the operator $T(a)+H(b)$ is Fredholm but $T(a)-H(b)$ is not. For the operator $T(a)+H(b)$, consider that connected component of the set $(T(a)+H(b))_F$ which contains p. Consider also the operators $T(c)$ and $T(d)$. If $T(a)-H(b)$ is not Fredholm, then at least one of the operators $T(c)$ and $T(d)$ is not Fredholm. Replacing p by s, where $s>p$ and is close enough to p, one obtains that $T(c)$ and $T(d)$ are already Fredholm on H^s. Consequently, there is an interval (p,p'), $p<p'$ such that $T(c)$ and $T(d)$ are Fredholm operators for all $s\in(p,p')$.

A result of I. Shneiberg ensures us that $\operatorname{ind} T(c)\,|_{H^s\mapsto H^s}$ and $\operatorname{ind} T(d)\,|_{H^s\mapsto H^s}$ are constant functions in s. Applying Theorem 3.6, one obtains that this is also true for the operators $T(a)+H(b)$ and $T(a)-H(b)$ acting on H^s, $s\in(p,p')$. Moreover, the above argumentation shows that this statement is true even for $T(a)+H(b)$ acting on H^s for all $s\in(p'',p')$ where $p''<p$ and is close enough to p.

Let us now formulate the final result.

Theorem 4.11. *Let $(a,b)\in PC(\mathbf{T})\times PC(\mathbf{T})$ be a matching pair with subordinated pair (c,d), and let the operator $T(a)+H(b)$ be Fredholm on a space $H^p, 1<p<\infty$. Then:*

(i) *The operator $T(a)+H(b)$ is invertible from the right if*

$$\lim_{s\to p+} \operatorname{ind} T(c)\geq 0 \quad \textit{and} \quad \lim_{s\to p+} \operatorname{ind} T(d)\,|_{H^s\mapsto H^s}\geq 0\,.$$

(ii) *The operator $T(a)+H(b)$ is invertible from the left if*

$$\lim_{s\to p+} \operatorname{ind} T(c)\leq 0 \quad \textit{and} \quad \lim_{s\to p+} \operatorname{ind} T(d)\,|_{H^s\mapsto H^s}\leq 0\,.$$

Proof. If $s\in(p,p')$, one can apply Theorem 4.10 to obtain that on H^s the operator $T(a)+H(b)$ is invertible from the left, respectively, from the right if the condition (i), respectively, (ii) is satisfied. Because $\operatorname{ind}(T(a)+H(b))\,|_{H^s\mapsto H^s}$ is constant on the interval $[p,p')$, one can employ the following result [9, p. 226]. Let X_1, X_2 be Banach spaces such that X_1 is continuously and densely embedded into X_2, and let A be a linear continuous Fredholm operator on both X_1 and X_2 which has the same index on each space. Then

$$\ker A\,|_{X_1\mapsto X_1} = \ker A\,|_{X_2\mapsto X_2}$$

and

$$\operatorname{coker} A\,|_{X_1\mapsto X_1} = \operatorname{coker} A\,|_{X_2\mapsto X_2}$$

Therefore, according to Theorem 4.10, the operator $T(a)+H(b)$ is also one-sided invertible on H^p, which completes the proof. □

Example 4.12. Let $a\in PC$ and the operator $T(a)+H(a)\colon H^p\mapsto H^p$ be Fredholm. The matching condition is trivial, and the last theorem shows that the operator $T(a)+H(a)$ is one-sided invertible. Hence, we recovered a result of Basor and Ehrhardt for $a\in PC$. In the following the index of certain operators will be computed. These computations use Theorem 4.1 and the results presented after that theorem.

Example 4.13. Consider the function $a=\exp(i\xi/4), \xi\in(0,2\pi)$. This function is continuous on $\mathbf{T}\setminus\{1\}$ and has a jump at the point $t=1$, viz. $a(1+0)=1$ and $a(1-0)=i$. The operators $T(a)+H(at)$ and $T(a)-H(at)$ are well-defined on every space $H^p, 1<p<\infty$. Since $b=at$ the matching condition (3.6) is satisfied with the matching functions $c(t)=t^{-1}$ and $d=a\widetilde{a}^{-1}t$. Note that $a\widetilde{a}^{-1}=-i\exp(i\xi/2)$. By Theorem 4.1 the operator $T(a\widetilde{a}^{-1})$ is not Fredholm on H^2 because $a\widetilde{a}^{-1}(1+0)=-i$, $a\widetilde{a}^{-1}(1-0)=i$, and the interval connecting these two points includes the origin. In view of Theorem 3.6, at least one of the operators $T(a)+H(at)$ or $T(a)-H(at)$ is not Fredholm. Since $H(ta)\in T(CP)$, it follows from Theorem 4.1 that $T(a)-H(at)$ is Fredholm but $T(a)+H(at)$ is not. Moreover, $\operatorname{ind}(T(a)-H(at))=0$, and easy arguments show that $T(a)-H(at)$ is Fredholm on each space H^p: If $p\neq 2$, then $T(a\widetilde{a}^{-1})$ is Fredholm with index -1 for $2<p<\infty$ and with index 0 for $1<p<2$.

Then, according to Theorem 3.6, the operator $T(a) - H(at)$ is Fredholm for each $p \in (1,\infty)$. The above mentioned theorem of Shneiberg then entails that $T(a)-H(at)$ has index zero for all $p \in (1,\infty)$. Thus we have

$$\operatorname{ind} T(d) = \begin{cases} -1 & \text{if } 1 < p < 2, \\ -2 & \text{if } 2 < p < \infty, \end{cases} \tag{4.3}$$

whereas

$$\operatorname{ind} T(c) = \operatorname{ind} T(t^{-1}) = 1 \quad \text{for all } p \in (1,\infty).$$

From Theorem 3.6 one concludes that

$$\dim \ker \operatorname{diag}\,(T(a) + H(at), T(a) - H(at)) = 1 \quad \text{for all } p \in (1,\infty) \setminus \{2\},$$

and

$$\dim \operatorname{coker} \operatorname{diag}\,(T(a) + H(at), T(a) - H(at)) = \begin{cases} -1 & \text{if } 1 < p < 2, \\ -2 & \text{if } 2 < p < \infty. \end{cases}$$

Moreover, the constant function $x(t) = 1, t \in \mathbf{T}$ belongs to the kernel of the operator $T(a) - H(at)$. Therefore,

$$\dim \operatorname{coker}\,(T(a) - H(at)) = 1,$$

so this operator is not one-sided invertible. This leads to the conclusion that $T(a) + H(at)$ is invertible if $1 < p < 2$, and it is invertible from the left with $\dim \operatorname{coker}\,(T(a)+H(at)) = 1$ if $2 < p < \infty$.

In conclusion of this example, we would like to mention a quite remarkable fact. It turns out that the approach presented here allows one to find the kernels of Toeplitz plus Hankel operators in various situations. Thus, let us consider the operators $T(a) + H(at^n)$ and $T(a) - H(at^n)$ where a is the above defined function and $n \in \mathbf{Z}_+$. The reader can verify that if $n = 2m+1, m \in \mathbf{Z}_+$, then

$$\begin{aligned} \ker(T(a) + H(at^n)) &= \left\{t^{m+k} - t^{m-k},\ k = 1,2,\ldots,m\right\}, \\ \ker(T(a) - H(at^n)) &= \left\{t^m, t^{m+k} + t^{m-k},\ k = 1,2,\ldots,m\right\}, \end{aligned}$$

and if $n = 2m, m \in \mathbf{Z}_+$, then

$$\begin{aligned} \ker(T(a) + H(at^n)) &= \left\{t^{m-k-1} - t^{m+k},\ k = 0,1,2,\ldots,m-1\right\}, \\ \ker(T(a) - H(at^n)) &= \left\{t^{m-k-1} + t^{m+k},\ k = 0,1,2,\ldots,m-1\right\}. \end{aligned}$$

It is also worth noting that for arbitrary function $a \in PC$, the above presented functions belong to the kernel of the corresponding operators $T(a) + H(at^n)$ and $T(a) - H(at^n)$ but they may not exhaust it.

Example 4.14. Consider the operator $T(a) + H(at^{-1})$ where a is the function given in Example 4.13. It is clear that (a, at^{-1}) is a matching pair with the matching functions $c(t) = t$ and $d = a\widetilde{a}^{-1}t^{-1}$. Using (4.3), one obtains

$$\operatorname{ind} T(d) = \begin{cases} 1 & \text{if } 1 < p < 2, \\ 0 & \text{if } 2 < p < \infty. \end{cases}$$

Besides for $p = 2$ the operator $T(a) - H(at^{-1})$ is Fredholm but $T(a) + H(at^{-1})$ is not. Further, since $\operatorname{ind}\,(T(a) - H(at^{-1}))\,|_{H^2 \mapsto H^2} = 0$, the operator $T(a) - H(at^{-1})$ is Fredholm on all spaces H^p, $1 < p < \infty$ with the index 0, cf. Example 4.13. Now

Theorem 3.6(i) entails that if $2 < p < \infty$, then the operator $T(a) - H(at^{-1})\colon H^p \mapsto H^p$ is invertible whereas $T(a) + H(at^{-1})$ is left-invertible with codimension one.

To study the properties of the operators $T(a) \pm H(at^{-1})$ in more detail for $1 < p < 2$, let us first note that we are in the situation described in Theorem 3.6, case (ii). Further, on each space H^p, $1 < p < 2$ the operators $T(\widetilde{a}^{-1})$ and $T(a\widetilde{a}^{-1})$ are Fredholm with the index zero, so they are invertible.

Note that $P_0 T(\widetilde{a}^{-1})T(a\widetilde{a}^{-1})P_0\colon \operatorname{im} P_0 \to \operatorname{im} P_0$ is a non-zero operator. Really, let us show that

$$P_0 T(\widetilde{a}^{-1})T(a\widetilde{a}^{-1})P_0\colon \operatorname{im} P_0(1) \neq 0. \tag{4.4}$$

The proof of (4.4) can be given from the Wiener–Hopf factorization of power functions. Definition and properties of such a factorization can be found in [3, 10, 11]. Here we will only sketch the proof. The details are left to the reader.

Let us introduce a few more notation. From now on, the argument $\arg z$ of a complex number $z \neq 0$ is always chosen so that $\arg z \in [0, 2\pi)$. If $\beta \in \mathbf{C}$, the function $\varphi_\beta \in PC$ is defined by

$$\varphi_\beta(\exp(i\zeta)) := \exp(\beta(\zeta - \pi)), \quad \zeta \in [0, 2\pi).$$

It is easily seen that the function φ_β has at most one discontinuity—viz. a jump at the point $z_0 = 1$ so that $\varphi_\beta(1+0) = \exp(-\pi i\beta)$ and $\varphi_\beta(1-0) = \exp(\pi i\beta)$. Therefore, one has

$$\varphi_\beta(\exp(i\zeta)) = \exp(-i\beta\pi)\exp(i\beta\zeta),$$

and the representations $a\widetilde{a}^{-1}(\exp(i\zeta)) = -i\exp(i\zeta/2)$ and $\widetilde{a}^{-1}(\exp(i\zeta)) = -i\exp(i\zeta/4)$ imply that

$$a\widetilde{a}^{-1} = c_1\varphi_{1/2}, \quad \widetilde{a}^{-1} = c_2\varphi_{1/4}, \tag{4.5}$$

where $c_1, c_2 \in \mathbf{C}/\{0\}$.

Consider also the functions ξ_β, η_β defined on $\mathbf{T}/\{0\}$ by

$$\begin{aligned}\xi_\beta(t) &= \left(1 - \frac{1}{t}\right)^\beta := \exp\left\{\beta\log\left|1 - \frac{1}{t}\right| + i\beta\arg\left(1 - \frac{1}{t}\right)\right\},\\ \eta_\beta(t) &= (1-t)^\beta := \exp\left\{\beta\log|1-t| + i\beta\arg(1-t)\right\}.\end{aligned}$$

Note that ξ_β (resp., η_β) is the limit on the unit circle $\mathbf{T}$ of that branch of the function $(1 - 1/z)^\beta$ (resp., $(1-z)^\beta$) which is analytic for $|z| > 1$ (resp., $|z| < 1$) and takes the value 1 at $z = \infty$ (resp., $z = 0$). We also notice that for all $t \in \mathbf{T}/\{1\}$ the relations

$$\xi_\alpha(t)\xi_\beta(t) = \xi_{\alpha+\beta}(t), \quad \eta_\alpha(t)\eta_\beta(t) = \eta_{\alpha+\beta}(t),$$

and

$$\varphi_\beta(t) = \xi_{-\beta}(t)\eta_\beta(t) \tag{4.6}$$

hold. Moreover, taking into account the invertibility of the operators $T(\varphi_{1/4})$ and $T(\varphi_{1/2})$ on the spaces H^p for $1 < p < 2$, one concludes that the identity (4.6) represents a Wiener–Hopf factorization of the function φ_β with respect to H^p, $1 < p < 2$. Thus using (4.6) and the relation $P\xi_{1/2}P(1) = 1$, one obtains

$$P_0 T(\varphi_{1/4})T^{-1}(\varphi_{1/2})(1) = P_0 P\xi_{-1/4}P\eta_{1/4}P\eta_{-1/2}P\xi_{1/2}P(1) = P_0 P\xi_{-1/4}\eta_{-1/4}(1).$$

Now it is easily seen that

$$P_0 P\xi_{-1/4}\eta_{-1/4}(1) = c_0,$$

where c_0 is the Fourier coefficient at t^0 for the function $\xi_{-1/4}\eta_{-1/4} \in L^1(\mathbf{T})$. In view of the expansions

$$\xi_{-1/4} = 1 + \frac{1}{4}z^{-1} + \frac{(1/4)(1+1/4)}{2!}z^{-2} + \frac{(1/4)(1+1/4)(2+1/4)}{3!}z^{-3} + \cdots,$$
$$\eta_{-1/4} = 1 + \frac{1}{4}z + \frac{(1/4)(1+1/4)}{2!}z^{2} + \frac{(1/4)(1+1/4)(2+1/4)}{3!}z^{3} + \cdots,$$

we get

$$c_0 = 1^2 + \sum_{k=1}^{\infty}\left(\frac{(1/4)(1+1/4)\cdots(k-1+1/4)}{k!}\right)^2 \neq 0.$$

Theorem 3.6(ii) now leads to the conclusion that if $1 < p < 2$, then

$$\dim\ker\operatorname{diag}\left(T(a)+H(t^{-1}a), T(a)-H(t^{-1}a)\right)|_{H^p\to H^p} = 0$$

Recalling that the index of this operator is zero, one immediately obtains that on the space H^p, $1 < p < 2$, the operators $T(a)+H(t^{-1}a)$ and $T(a)-H(t^{-1}a)$ are invertible. Moreover, since for any p, $1 < p < \infty$ the operator $T(a)-H(t^{-1}a)\colon H^p \to H^p$ is Fredholm, it is invertible on each space H^p, $1 < p < \infty$.

Remark 4.15. The above approach can be used to obtain complete information about the invertibility of more general operators $T(a) \pm H(at^{-n})$, $n \in \mathbf{Z}$. However, due to lack of space we do not consider this problem here.

Example 4.16. Let a be the following piecewise continuous function

$$a(t) = \begin{cases} 1 & \text{if } t \in \mathring{\mathbf{T}}_+, \\ -1 & \text{if } t \in \mathbf{T}\setminus\mathbf{T}_+. \end{cases}$$

On the space H^p, $1 < p < \infty$, consider the operator $iI+H(a)$. The identity $i\cdot i = a\widetilde{a} = -1$ shows that the matching condition is satisfied with the matching function $c = ia$. Theorem 3.6 entails that for every $p \in (1,\infty)$ the operators $\operatorname{diag}(iI+H(a), iI-H(a))$ and $\operatorname{diag}(-iT(a), iT(a))$ have identical Fredholm properties on the space $H^p \times H^p$. Note that on the space $L^2(\mathbf{T})$ the operator $T(a)$ is not Fredholm, so one of the operators $iI + H(a)$ or $iI - H(a)$ is also not Fredholm on H^2, cf. Theorem 3.6. However, for $p = 2$ the range of the symbol of the operator $iI - H(a)$ is located in the upper half-plane $\Pi^+ := \{z \in \mathbf{C}\colon \operatorname{Im} z > 0\}$, and Theorem 3.6 shows that the operator $iI - H(a)$ is Fredholm on each space H^p. Moreover, $\operatorname{ind}(iI - H(a)) = 0$ for any $p \in (1,\infty)$ whereas $\operatorname{ind}(iI + H(a)) = -2$ for $1 < p < 2$ and $\operatorname{ind}(iI + H(a)) = 2$ for $2 < p < \infty$. Really, this is a consequence of the fact that $T(a)$ is Fredholm with $\operatorname{ind} T(a) = 1$ if $1 < p < 2$ and $\operatorname{ind} T(a) = -1$ if $2 < p < \infty$. Consequently, in view of Theorem 3.6, we have $iI - H(a)$ is invertible for all $p \in (1,\infty)\setminus\{2\}$ but $iI + H(a)$ is right or left invertible for $1 < p < 2$ and $2 < p < \infty$, respectively. The invertibility of the operator $iI - H(a)$ in the space H^2 can be easily shown, so the proof is left to the reader.

Example 4.17. Consider the function $a \in PC$,

$$a(t) = \begin{cases} 1 & \text{if } \operatorname{Re} t \geq 0, \\ -1 & \text{if } \operatorname{Re} t < 0, \end{cases}$$

and the operator $T(a) + H(at)$ acting on the space H^p, $1 < p < \infty$. It is clear that $a = \widetilde{a}$, so (a, at) is the matching pair with matching functions $c(t) = t^{-1}$ and $d = t$.

By Theorem 3.6, one has

$$\operatorname{ind}\operatorname{diag}(T(a)+H(at),T(a)-H(at))=0.$$

Along with Theorem 4.5, this leads to the equation

$$\operatorname{ind}(T(a)+H(at))=\operatorname{ind}(T(a)-H(at))=0$$

for all spaces H^p. Since $1\in\ker(T(a)-H(at))$, one can use Theorem 3.6 and obtain that the operator $T(a)+H(at)$ is invertible on all spaces H^p, whereas $T(a)-H(at)$ is not one-sided invertible.

Remark 4.18. The previous examples show that if an additional information is available, the invertibility problem can be studied completely, even in the case where Theorem 4.10 is not working.

Remark 4.19. Analyzing Example 4.13, one can establish the fact that if $a\in L^\infty$ and the operator $T(a)+H(ta)$ is Fredholm, then it is one-sided invertible.

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