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SOBOLEV'S THEOREM AND DUALITY FOR HERZ–MORREY SPACES OF VARIABLE EXPONENT

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Abstract. In this paper, we consider the Herz–Morrey space $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ of variable exponent consisting of all measurable functions f on a bounded open set $G \subset \mathbf{R}^n$ satisfying

$$\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{2d_G} \left(\omega(x_0,r) \|f\|_{L^{p(\cdot)}(B(x_0,r)\setminus B(x_0,r/2))} \right)^q dr/r \right)^{1/q} < \infty,$$

and set $\mathcal{H}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0 \in G} \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$.

Our first aim in this paper is to give the boundedness of the maximal and Riesz potential operators in $\mathcal{H}^{p(\cdot),q,\omega}(G)$ when $q=\infty$.

In connection with $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\mathcal{H}^{p(\cdot),q,\omega}(G)$, let us consider the families $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$, $\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)$, $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\tilde{\mathcal{H}}^{p(\cdot),q,\omega}(G)$. Following Fiorenza–Rakotoson [18], Di Fratta–Fiorenza [17] and Gogatishvili–Mustafayev [19], we next discuss the duality properties among these Herz–Morrey spaces.

1. Introduction

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. We denote by $B(x,r)$ the open ball centered at x of radius r, and by $|E|$ the Lebesgue measure of a measurable set $E \subset \mathbf{R}^n$.

It is well known that the maximal operator is bounded in the Lebesgue space $L^p(\mathbf{R}^n)$ if $p>1$ (see [34]). In [12], the boundedness of the maximal operator is still valid by replacing the Lebesgue space by several Morrey spaces; the original one was introduced by Morrey [30] to estimate solutions of partial differential equations; for Morrey spaces, we also refer to Peetre [32] and Nakai [31].

One of important applications of the boundedness of the maximal operator is Sobolev's inequality; in the classical case,

$$\|I_\alpha * f\|_{L^{p^\sharp}(\mathbf{R}^n)} \le C\|f\|_{L^p(\mathbf{R}^n)}$$

for $f \in L^p(\mathbf{R}^n)$, $0<\alpha<n$ and $1<p<n/\alpha$, where I_α is the Riesz kernel of order α and $1/p^\sharp = 1/p - \alpha/n$ (see, e.g. [2, Theorem 3.1.4]). Sobolev's inequality for Morrey spaces was given by Adams [1] (also [12]). Further, Sobolev's inequality was also studied on generalized Morrey spaces (see [31]). This result was extended to local and global Morrey type spaces by Burenkov, Gogatishvili, Guliyev and Mustafayev

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[8] (see also [7, 9, 10]). The local Morrey type spaces are also called Herz spaces introduced by Herz [23]. In our paper, those Morrey type spaces are referred to as Herz–Morrey spaces.

In [13], Diening showed that the maximal operator is bounded on the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{R}^n)$ if the variable exponent $p(\cdot)$, which is a constant outside a ball, satisfies the locally log-Hölder condition and $\inf p(x) > 1$ (see condition (P2) in Section 2). In the mean time, variable exponent Lebesgue spaces were used to discuss nonlinear partial differential equations with non-standard growth condition. These spaces have attracted more and more attention, in connection with the study of elasticity and fluid mechanics; see [16, 33]. On the other hand, variable exponent Morrey or Herz versions were discussed in [4, 5, 24, 26, 29].

Let G be a bounded open set in $\mathbf{R}^n$, whose diameter is denoted by d_G. Let $\omega(\cdot,\cdot)\colon G\times(0,\infty)\to(0,\infty)$ be a uniformly almost monotone function on $G\times(0,\infty)$ satisfying the uniformly doubling condition. For $x_0\in G$, $0<q\le\infty$ and a variable exponent $p(\cdot)$, we consider the Herz–Morrey space $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ of variable exponent consisting of all measurable functions f on G satisfying

$$\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r)\setminus B(x_0,r/2))}\right)^q dr/r\right)^{1/q} < \infty;$$

when $q=\infty$,

$$\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} = \sup_{0<r<d_G} \omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r)\setminus B(x_0,r/2))} < \infty.$$

Set

$$\mathcal{H}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0\in G} \mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G),$$

whose norm is defined by

$$\|f\|_{\mathcal{H}^{p(\cdot),q,\omega}(G)} = \sup_{x_0\in G} \|f\|_{\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)}.$$

In connection with $\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$, let us consider the families $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ of all functions f on G satisfying

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty$$

and

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)} = \left(\int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{p(\cdot)}(G\setminus B(x_0,r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty,$$

respectively. In the paper by Fiorenza and Rakotoson [18], the Herz–Morrey space $\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$ is referred to as the generalized Lorentz space denoted by $G\Gamma(p,q,\omega)$.

Note here that

$$\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\cup\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)\subset\mathcal{H}^{p(\cdot),q,\omega}_{\{x_0\}}(G).$$

Similarly we consider the space

$$\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0\in G} \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G),$$

whose norm is defined by

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}(G)} = \sup_{x_0 \in G} \|f\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)}.$$

Our first aim in this paper is to establish the boundedness of the maximal operator and the Riesz potential operator in $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$; when $q < \infty$, we refer to [27]. In the borderline case, Trudinger's exponential integrability is discussed.

Next, following Di Fratta–Fiorenza [17] and Gogatishvili–Mustafayev [19] , we study the duality properties among those Herz–Morrey spaces. In particular, we show the associate spaces of $\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$ and $\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$, which give another characterizations of Morrey spaces by Adams–Xiao [3] (see also [20]).

2. Preliminaries

Throughout this paper, let C denote various constants independent of the variables in question. The symbol $g \sim h$ means that $C^{-1}h \le g \le Ch$ for some constant $C > 1$. Set $A(x, r) = B(x, r) \setminus B(x, r/2)$.

Consider a function $p(\cdot)$ on G such that

(P1) $1 < p^- := \inf_{x\in G} p(x) \le \sup_{x\in G} p(x) =: p^+ < \infty$, and

(P2) $p(\cdot)$ is log-Hölder continuous, namely

$$|p(x) - p(y)| \le \frac{c_p}{\log(2d_G/|x-y|)} \quad \text{for } x, y \in G$$

with a constant $c_p \ge 0$; $p(\cdot)$ is referred to as a variable exponent.

We also consider the family $\Omega(G)$ of all positive functions $\omega(\cdot,\cdot)\colon G \times (0,\infty) \to (0,\infty)$ satisfying the following conditions:

(ω0) $\omega(x, 0) = \lim_{r\to+0} \omega(x, r) = 0$ for all $x \in G$ or $\omega(x, 0) = \infty$ for all $x \in G$;

(ω1) $\omega(x,\cdot)$ is uniformly almost monotone on $(0,\infty)$, that is, there exists a constant $Q_1 > 0$ such that $\omega(x,\cdot)$ is uniformly almost increasing on $(0,\infty)$, that is,

$$\omega(x, r) \le Q_1\omega(x, s) \quad \text{for all } x \in G \text{ and } 0 < r < s$$

or $\omega(x,\cdot)$ is uniformly almost decreasing on $(0,\infty)$, that is,

$$\omega(x, s) \le Q_1\omega(x, r) \quad \text{for all } x \in G \text{ and } 0 < r < s;$$

(ω2) $\omega(x,\cdot)$ is uniformly doubling on $(0,\infty)$, that is, there exists a constant $Q_2 > 0$ such that

$$Q_2^{-1}\omega(x, r) \le \omega(x, 2r) \le Q_2\omega(x, r) \quad \text{for all } x \in G \text{ and } r > 0; \text{ and}$$

(ω3) there exists a constant $Q_3 > 0$ such that

$$Q_3^{-1} \le \omega(x, 1) \le Q_3 \quad \text{for all } x \in G.$$

Then one can find constants $a, b > 0$ and $C > 1$ such that

$$C^{-1}r^a \le \omega(x, r) \le Cr^{-b} \tag{2.1}$$

for all $x \in G$ and $0 < r \le d_G$.

For later use, it is convenient to note the following result, which is proved by (P1), (P2) and (2.1).

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$\omega(x,r)^{p(x)} \le C\omega(x,r)^{p(y)}$$

whenever $|x-y| < r \le d_G$.

For a locally integrable function f on G, set

$$\|f\|_{L^{p(\cdot)}(G)} = \inf\left\{\lambda > 0 \colon \int_G \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} dy \le 1\right\};$$

in what follows, set $f = 0$ outside G. We denote by $L^{p(\cdot)}(G)$ the family of locally integrable functions f on G satisfying $\|f\|_{L^{p(\cdot)}(G)} < \infty$.

Lemma 2.2. *Let $0 < q < \infty$. Then*

(1) $\displaystyle\int_0^{2d_G} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r \sim \sum_{j=1}^{\infty} \left(\omega(x,2^{-j+1}d_G)\|f\|_{L^{p(\cdot)}(A(x,2^{-j+1}d_G))}\right)^q;$

(2) $\displaystyle\int_0^{2d_G} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(B(x,r))}\right)^q dr/r \sim \sum_{j=1}^{\infty} \left(\omega(x,2^{-j+1}d_G)\|f\|_{L^{p(\cdot)}(B(x,2^{-j+1}d_G))}\right)^q;$ and

(3) $\displaystyle\int_0^{2d_G} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(G\setminus B(x,r))}\right)^q dr/r \sim \sum_{j=1}^{\infty} \left(\omega(x,2^{-j}d_G)\|f\|_{L^{p(\cdot)}(G\setminus B(x,2^{-j}d_G))}\right)^q$

for all $x \in G$ and measurable functions f on G.

Proof. We only prove (1), since the remaining assertions can be proved similarly. Since $A(x,r) \supset B(x,3t/2) \setminus B(x,t)$ when $3t/2 < r < 2t \le 2d_G$, we have by (ω1) and (ω2) that

$$\int_{3t/2}^{2t} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r \ge C\left(\omega(x,t)\|f\|_{L^{p(\cdot)}(B(x,3t/2)\setminus B(x,t))}\right)^q$$

and similarly, we have

$$\int_{t}^{3t/2} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r \ge C\left(\omega(x,t)\|f\|_{L^{p(\cdot)}(B(x,t)\setminus B(x,3t/4))}\right)^q.$$

Thus

$$\int_{t}^{2t} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r \ge C\left(\omega(x,t)\|f\|_{L^{p(\cdot)}(B(x,3t/2)\setminus B(x,3t/4))}\right)^q.$$

Therefore, letting $3t/2 = 2^{-j+1}d_G$ for a positive integer j, we see that

$$\int_{2^{-j}d_G}^{2^{-j+2}d_G} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r \ge C\left(\omega(x,2^{-j+1}d_G)\|f\|_{L^{p(\cdot)}(A(x,2^{-j+1}d_G))}\right)^q,$$

so that

$$\begin{aligned}\int_0^{2d_G} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r &\ge \frac{1}{2}\sum_{j=1}^{\infty}\int_{2^{-j}d_G}^{2^{-j+2}d_G} \left(\omega(x,r)\|f\|_{L^{p(\cdot)}(A(x,r))}\right)^q dr/r \\ &\ge C\sum_{j=1}^{\infty}\left(\omega(x,2^{-j+1}d_G)\|f\|_{L^{p(\cdot)}(A(x,2^{-j+1}d_G))}\right)^q.\end{aligned}$$

The converse inequality is easily obtained. □

Further, we obtain the next result.

Lemma 2.3. *Suppose* $0 < q \le \infty$. *If* $\|f\|_{h^{p(\cdot),q,\omega}(G)} \le 1$, *then there exists a constant* $C > 0$ *such that* $\|f\|_{h^{p(\cdot),\infty,\omega}(G)} \le C$, *for* $h = \mathcal{H}_{\{x_0\}}, \underline{\mathcal{H}}_{\{x_0\}}, \overline{\mathcal{H}}_{\{x_0\}}, \mathcal{H}, \underline{\mathcal{H}}$.

By Lemma 2.1, we have the following result.

Lemma 2.4. *There is a constant* $C > 0$ *such that*

$$\int_{B(x_0,r)} |f(y)|^{p(y)}\, dy \le C\omega(x_0,r)^{-p(x_0)}$$

when $x_0 \in G$, $0 < r < d_G$ *and* $\omega(x_0,r)\|f\|_{L^{p(\cdot)}(B(x_0,r))} \le 1$.

Lemma 2.5. *There is a constant* $C > 0$ *such that*

$$\frac{1}{|A(x_0,r)|}\int_{A(x_0,r)} |f(y)|\, dy \le Cr^{-n/p(x_0)}\omega(x_0,r)^{-1}$$

when $x_0 \in G$, $0 < r < d_G$ *and* $\omega(x_0,r)\|f\|_{L^{p(\cdot)}(A(x_0,r))} \le 1$.

Proof. Fix $x_0 \in G$ and $0 < r < d_G$. Let f be a nonnegative measurable function on G satisfying $\omega(x_0,r)\|f\|_{L^{p(\cdot)}(A(x_0,r))} \le 1$. Then we have by (P2) and Lemmas 2.1 and 2.4,

$$\begin{aligned}
&\frac{1}{|A(x_0,r)|}\int_{A(x_0,r)} f(y)\, dy \\
&\le r^{-n/p(x_0)}\omega(x_0,r)^{-1} + \frac{1}{|A(x_0,r)|}\int_{A(x_0,r)} f(y)\left(\frac{f(y)}{r^{-n/p(x_0)}\omega(x_0,r)^{-1}}\right)^{p(y)-1} dy \\
&\le r^{-n/p(x_0)}\omega(x_0,r)^{-1} + C\left(r^{-n/p(x_0)}\omega(x_0,r)^{-1}\right)^{1-p(x_0)} \frac{1}{|A(x_0,r)|}\int_{A(x_0,r)} f(y)^{p(y)}\, dy \\
&\le Cr^{-n/p(x_0)}\omega(x_0,r)^{-1},
\end{aligned}$$

as required. □

3. Boundedness of the maximal operator for $q = \infty$

Let us consider the following conditions: let $\eta \in \Omega(G)$ and $x_0 \in G$.

(ω3.1) There exists a constant $Q > 0$ such that

$$\int_0^r t^{n-n/p(x_0)}\omega(x_0,t)^{-1}\frac{dt}{t} \le Qr^{n-n/p(x_0)}\eta(x_0,r)^{-1}$$

for all $0 < r \le d_G$; and

(ω3.2) there exists a constant $Q > 0$ such that

$$\int_r^{2d_G} t^{-n/p(x_0)}\omega(x_0,t)^{-1}\frac{dt}{t} \le Qr^{-n/p(x_0)}\eta(x_0,r)^{-1}$$

for all $0 < r \le d_G$.

By the doubling condition on ω, one notes from (ω3.1) or (ω3.2) that

$$\omega(x_0,r)^{-1} \le C\eta(x_0,r)^{-1}.$$

Lemma 3.1. *If* $(\omega 3.1)$ *and* $(\omega 3.2)$ *hold for all* $x_0 \in G$ *with the same constant* Q, *then there is a constant* $C > 0$ *such that*

$$\int_{B(x,r)} |f(y)| dy \le C r^{n-n/p(x)} \eta(x,r)^{-1}$$

and

$$\int_{G \setminus B(x,r)} |f(y)||x-y|^{-n} dy \le C r^{-n/p(x)} \eta(x,r)^{-1}$$

for all $x \in G$, $0 < r \le d_G$ *and* f *with* $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$.

Proof. Let f be a nonnegative measurable function on G satisfying $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$. By Lemma 2.5 and $(\omega 3.1)$, we have

$$\begin{aligned}\int_{B(x,r)} f(y)\,dy &= \sum_{j=1}^{\infty} \int_{A(x,2^{-j+1}r)} f(y)\,dy \le C \sum_{j=1}^{\infty} (2^{-j}r)^{n-n/p(x)} \omega(x, 2^{-j}r)^{-1} \\ &\le C r^{n-n/p(x)} \eta(x,r)^{-1}.\end{aligned}$$

Similarly, we obtain by use of Lemma 2.5 and $(\omega 3.2)$

$$\begin{aligned}\int_{G\setminus B(x,r)} |f(y)||x-y|^{-n}\,dy &\le C \sum_{j \ge 1, 2^{j-1} r \le d_G} (2^j r)^{-n} \int_{A(x,2^j r)} f(y)\,dy \\ &\le C \sum_{j \ge 1, 2^{j-1} r \le d_G} (2^j r)^{-n/p(x)} \omega(x, 2^j r)^{-1} \\ &\le C r^{-n/p(x)} \eta(x,r)^{-1},\end{aligned}$$

as required. □

For a locally integrable function f on G, the Hardy–Littlewood maximal operator $\mathcal{M}$ is defined by

$$\mathcal{M} f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,dy;$$

recall that $f = 0$ outside G. Now we state the celebrated result by Diening [13].

Lemma 3.2. *The maximal operator* $\mathcal{M}$ *is bounded in* $L^{p(\cdot)}(G)$, *that is, there exists a constant* $C > 0$ *such that*

$$\|\mathcal{M} f\|_{L^{p(\cdot)}(G)} \le C \|f\|_{L^{p(\cdot)}(G)}.$$

Theorem 3.3. *If* $(\omega 3.1)$ *and* $(\omega 3.2)$ *hold for all* $x_0 \in G$ *with the same constant* Q, *then the maximal operator* $\mathcal{M}$ *is bounded from* $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$ *to* $\mathcal{H}^{p(\cdot),\infty,\eta}(G)$.

Guliyev, Hasanov and Samko [21, 22] proved that if $(\omega 3.2)$ holds for all $x_0 \in G$ with the same constant Q, then the maximal operator $\mathcal{M}$ is bounded from $\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)$ to $\underline{\mathcal{H}}^{p(\cdot),\infty,\eta}(G)$ and if $(\omega 3.1)$ holds for $x_0 \in G$, then the maximal operator $\mathcal{M}$ is bounded from $\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$ to $\overline{\mathcal{H}}^{p(\cdot),\infty,\eta}_{\{x_0\}}(G)$.

Proof of Theorem 3.3. Let f be a nonnegative measurable function on G such that $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$. For $x \in G$ and $0 < r < d_G$, it suffices to show that

$$\|\mathcal{M} f\|_{L^{p(\cdot)}(A(x,r))} \le C \eta(x,r)^{-1}.$$

For this purpose, set

$$f = f\chi_{G\setminus B(x,2r)} + f\chi_{B(x,2r)\setminus B(x,r/4)} + f\chi_{B(x,r/4)} = f_1 + f_2 + f_3,$$

where χ_E denotes the characteristic function of E. We note from Lemma 3.2 that

$$\begin{aligned}\|\mathcal{M}f_2\|_{L^{p(\cdot)}(A(x,r))} &\le C\|f_2\|_{L^{p(\cdot)}(G)} \le C\|f_2\|_{L^{p(\cdot)}(B(x,2r)\setminus B(x,r/4))}\\ &\le C\{\|f_2\|_{L^{p(\cdot)}(B(x,2r)\setminus B(x,r))} + \|f_2\|_{L^{p(\cdot)}(B(x,r)\setminus B(x,r/2))}\\ &\quad + \|f_2\|_{L^{p(\cdot)}(B(x,r/2)\setminus B(x,r/4))}\}\\ &\le C\omega(x,r)^{-1} \le C\eta(x,r)^{-1}.\end{aligned}$$

For $z \in A(x,r)$, Lemma 3.1 gives

$$\mathcal{M}f_3(z) \le Cr^{-n}\int_{B(x,r/4)} f(y)\,dy \le Cr^{-n/p(x)}\eta(x,r)^{-1},$$

so that

$$\|\mathcal{M}f_3\|_{L^{p(\cdot)}(A(x,r))} \le Cr^{-n/p(x)}\eta(x,r)^{-1}\|1\|_{L^{p(\cdot)}(A(x,r))} \le C\eta(x,r)^{-1}.$$

Moreover, Lemma 3.1 again gives

$$\mathcal{M}f_1(z) \le C\int_{G\setminus B(x,2r)} f(y)|x-y|^{-n}\,dy \le Cr^{-n/p(x)}\eta(x,r)^{-1}$$

and hence

$$\|\mathcal{M}f_1\|_{L^{p(\cdot)}(A(x,r))} \le Cr^{-n/p(x)}\eta(x,r)^{-1}\|1\|_{L^{p(\cdot)}(A(x,r))} \le C\eta(x,r)^{-1},$$

as required. □

Remark 3.4. If the conditions on ω hold at $x_0 \in G$ only, then one can see that $\mathcal{M}$ is bounded from $\mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$ to $\mathcal{H}^{p(\cdot),\infty,\eta}_{\{x_0\}}(G)$.

Corollary 3.5. *For bounded functions $\nu(\cdot)\colon G \to (-\infty,\infty)$ and $\beta(\cdot)\colon G \to (-\infty,\infty)$, set $\omega(x,r) = r^{\nu(x)}(\log(2d_G/r))^{\beta(x)}$. If $-n/p^+ < \nu^- \le \nu^+ < n\,(1-1/p^-)$, then the maximal operator $\mathcal{M}$ is bounded in $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$.*

Define

$$\omega_*(x,r) = \left(\int_0^r \omega(x,t)^{-1}\,\frac{dt}{t}\right)^{-1}$$

and

$$\omega^*(x,r) = \left(\int_r^{2d_G} \omega(x,t)^{-1}\,\frac{dt}{t}\right)^{-1}$$

for $x \in G$ and $0 < r \le d_G$.

Theorem 3.6. (1) *If $\omega_*(\cdot,d_G)$ is bounded in G, then $\mathcal{H}^{p(\cdot),\infty,\omega}(G) \subset \underline{\mathcal{H}}^{p(\cdot)\infty,\omega_*}(G)$.*

(2) *For each $x_0 \in G$, $\mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G) \subset \overline{\mathcal{H}}^{p(\cdot)\infty,\omega^*}_{\{x_0\}}(G)$.*

Proof. Let f be a measurable function on G such that $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$. We show only (1), because (2) can be proved similarly.

For (1), we see that

$$\|f\|_{L^{p(\cdot)}(B(x,r))} \le \sum_{j=1}^{\infty}\|f\|_{L^{p(\cdot)}(A(x,2^{-j+1}r))} \le \sum_{j=1}^{\infty}\omega(x,2^{-j}r)^{-1} \le C\omega_*(x,r)^{-1}$$

for all $x \in G$ and $0 < r \le d_G$, as required. □

Remark 3.7. Let $\omega(x,r) = (\log(2d_G/r))^{\beta(x)+1}$ for a bounded function $\beta(\cdot)\colon G \to (-\infty,\infty)$.

(1) If $\operatorname{ess\,inf}_{x\in G} \beta(x) > 0$, then

$$\omega_*(x,r) \sim \left(\log \frac{2d_G}{r}\right)^{\beta(x)}$$

for all $x \in G$ and $0 < r < d_G$; and

(2) if $\beta(x_0) < 0$ for $x_0 \in G$, then

$$\omega^*(x_0,r) \sim \left(\log \frac{2d_G}{r}\right)^{\beta(x_0)}$$

for all $0 < r < d_G$.

Remark 3.8. Let $\omega(x,r) = r^{\nu(x)}$ for a bounded function $\nu(\cdot)\colon G \to (-\infty,\infty)$.

(1) If $\operatorname{ess\,sup}_{x\in G} \nu(x) < 0$, then

$$\omega_*(x,r) \sim \omega(x,r)$$

for all $x \in G$ and $0 < r < d_G$; and

(2) if $\nu(x_0) > 0$ for $x_0 \in G$, then

$$\omega^*(x_0,r) \sim \omega(x_0,r)$$

for all $0 < r < d_G$.

Corollary 3.9. (1) *Suppose* (ω3.1) *and* (ω3.2) *hold for all* $x_0 \in G$ *with the same constant* Q. *If* $\omega_*(\cdot, d_G)$ *is bounded in* G, *then the maximal operator* $\mathcal{M}$ *is bounded from* $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$ *to* $\underline{\mathcal{H}}^{p(\cdot),\infty,\omega_*}(G)$.

(2) *If* (ω3.1) *and* (ω3.2) *hold for* $x_0 \in G$, *then the maximal operator* $\mathcal{M}$ *is bounded from* $\mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$ *to* $\overline{\mathcal{H}}^{p(\cdot),\infty,\omega^*}_{\{x_0\}}(G)$.

Remark 3.10. Let us consider a singular integral operator T associated with a standard kernel $k(x,y)$ in [15, Section 6.3] such that

$$|k(x,y)| \le K_1|x-y|^{-n}$$

for all $x, y \in \mathbf{R}^n$ and

$$\|Tf\|_{L^{p(\cdot)}(\mathbf{R}^n)} \le K_2\|f\|_{L^{p(\cdot)}(\mathbf{R}^n)}$$

for all $f \in L^{p(\cdot)}(\mathbf{R}^n)$.

If (ω3.1) and (ω3.2) hold for all $x_0 \in G$ with the same constant Q, then every singular integral operator T is bounded from $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$ to $\mathcal{H}^{p(\cdot),\infty,\eta}(G)$.

4. Sobolev's inequality for $q = \infty$

We consider the following condition: let $\eta \in \Omega(G)$ and $x_0 \in G$.

(ω4.1) For $0 < \alpha < n$, there exists a constant $Q > 0$ such that

$$\int_r^{2d_G} t^{\alpha-n/p(x)}\omega(x,t)^{-1}\frac{dt}{t} \le Q r^{\alpha-n/p(x)}\eta(x,r)^{-1}$$

for all $0 < r < d_G$.

As in the proof of Lemma 3.1, we have the following result.

Lemma 4.1. *If* (ω4.1) *holds for all* $x_0 \in G$ *with the same constant* Q, *then there is a constant* $C > 0$ *such that*

$$\int_{G\setminus B(x,r)} |x-y|^{\alpha-n}|f(y)|\,dy \le Cr^{\alpha-n/p(x)}\eta(x,r)^{-1}$$

for all $x \in G$, $0 < r < d_G$ *and* f *with* $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$.

For $0 < \alpha < n$, the Riesz potential $I_\alpha f$ is defined by

$$I_\alpha f(x) = I_\alpha * f(x) = \int_G |x-y|^{\alpha-n} f(y)\,dy$$

for measurable functions f on G; and define

$$\frac{1}{p^\sharp(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}.$$

Let us begin with Sobolev's inequality proved by Diening [14, Theorem 5.2]:

Lemma 4.2. *If* $0 < \alpha < n/p^+$, *then there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{L^{p^\sharp(\cdot)}(G)} \le C\|f\|_{L^{p(\cdot)}(G)}$$

for all $f \in L^{p(\cdot)}(G)$.

Our result is stated in the following:

Theorem 4.3. *Let* $0 < \alpha < n/p^+$. *If* (ω3.1) *and* (ω4.1) *hold for all* $x_0 \in G$ *with the same constant* Q, *then there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{\mathcal{H}^{p^\sharp(\cdot),\infty,\eta}(G)} \le C\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \mathcal{H}^{p(\cdot),\infty,\omega}(G)$.

In view of Guliyev, Hasanov and Samko [21, 22], if (ω4.1) holds for all $x_0 \in G$ with the same constant Q, then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{\underline{\mathcal{H}}^{p^\sharp(\cdot),\infty,\eta}(G)} \le C\|f\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)$ and if (ω3.1) holds for $x_0 \in G$, then there exists a constant $C > 0$ (which may depend on x_0) such that

$$\|I_\alpha f\|_{\overline{\mathcal{H}}^{p^\sharp(\cdot),\infty,\eta}_{\{x_0\}}(G)} \le C\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all $f \in \overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$.

Proof of Theorem 4.3. Let f be a nonnegative measurable function on G such that $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$. For $x \in G$ and $0 < r < d_G$, we have only to show the inequality

$$\|I_\alpha f\|_{L^{p^\sharp(\cdot)}(A(x,r))} \le C\eta(x,r)^{-1}.$$

Set

$$f = f\chi_{G\setminus B(x,2r)} + f\chi_{B(x,2r)\setminus B(x,r/4)} + f\chi_{B(x,r/4)} = f_1 + f_2 + f_3,$$

as before. We note from Lemma 4.2 that

$$\|I_\alpha f_2\|_{L^{p^\sharp(\cdot)}(A(x,r))} \le C\|f_2\|_{L^{p(\cdot)}(G)} \le C\|f_2\|_{L^{p(\cdot)}(B(x,2r)\setminus B(x,r/4))} \le C\omega(x,r)^{-1} \le C\eta(x,r)^{-1}.$$

If $z \in A(x,r)$, then Lemma 3.1 gives

$$I_\alpha f_3(z) \le C r^{\alpha-n} \int_{B(x,r/4)} f(y)\,dy \le C r^{\alpha-n/p(x)} \eta(x,r)^{-1},$$

so that

$$\|I_\alpha f_3\|_{L^{p^\sharp(\cdot)}(A(x,r))} \le C r^{\alpha-n/p(x)} \eta(x,r)^{-1} \|1\|_{L^{p^\sharp(\cdot)}(A(x,r))} \le C\eta(x,r)^{-1}.$$

Moreover, Lemma 4.1 gives

$$I_\alpha f_1(z) \le \int_{G\setminus B(x,2r)} |x-y|^{\alpha-n} f(y)\,dy \le C r^{\alpha-n/p(x)} \eta(x,r)^{-1},$$

so that

$$\|I_\alpha f_1\|_{L^{p^\sharp(\cdot)}(A(x,r))} \le C r^{\alpha-n/p(x)} \eta(x,r)^{-1} \|1\|_{L^{p^\sharp(\cdot)}(A(x,r))} \le C\eta(x,r)^{-1},$$

as required. □

Corollary 4.4. *Let $0<\alpha<n/p^+$ and let ν, β and ω be as in Corollary 3.5. If $\alpha - n/p^+ < \nu^- \le \nu^+ < n(1-1/p^-)$, then there exists a constant $C>0$ such that*

$$\|I_\alpha f\|_{\mathcal{H}^{p^\sharp(\cdot),\infty,\omega}(G)} \le C\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \mathcal{H}^{p(\cdot),\infty,\omega}(G)$.

Corollary 4.5. *Assume that $0<\alpha<n/p^+$.*

(1) *Suppose (ω3.1) and (ω4.1) hold for all $x_0 \in G$ with the same constant Q. If $\omega_*(\cdot, d_G)$ is bounded in G, then the operator I_α is bounded from $\mathcal{H}^{p(\cdot),\infty,\omega}(G)$ to $\overline{\mathcal{H}}^{p^\sharp(\cdot),\infty,\omega_*}(G)$.*

(2) *If (ω3.1) and (ω4.1) hold for $x_0 \in G$, then the operator I_α is bounded from $\mathcal{H}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$ to $\overline{\mathcal{H}}^{p^\sharp(\cdot),\infty,\omega^*}_{\{x_0\}}(G)$.*

5. Exponential integrability for $q=\infty$

Set

$$E_1(x,t) = \exp\left(t^{q(x)}\right) - 1,$$

where $1/p(x) + 1/q(x) = 1$. For a locally integrable function f on G, set

$$\|f\|_{L^{E_1}(G)} = \inf\left\{\lambda>0\colon \int_G E_1\left(x, \frac{|f(y)|}{\lambda}\right) dy \le 1\right\}.$$

We denote by $L^{E_1}(G)$ the class of locally integrable functions f on G satisfying $\|f\|_{L^{E_1}(G)} < \infty$.

In connection with $\mathcal{H}^{p(\cdot),q,\omega}(G)$, let us consider $\mathcal{H}^{E_1,q,\omega}(G)$ of all functions f satisfying

$$\|f\|_{\mathcal{H}^{E_1,q,\omega}(G)} \;=\; \sup_{x_0\in G}\left(\int_0^{2d_G} \left(\omega(x_0,r)\|f\|_{L^{E_1}(A(x_0,r))}\right)^q \frac{dr}{r}\right)^{1/q} < \infty.$$

Similarly, we define $\underline{\mathcal{H}}^{E_1,q,\omega}(G)$ and $\overline{\mathcal{H}}^{E_1,q,\omega}_{\{x_0\}}(G)$.

Lemma 5.1.

$$\|1\|_{L^{E_1}(B(x,r))} \sim (\log(1+1/r))^{-1/q(x)}$$

for all $x \in G$ and $0<r<d_G$.

Lemma 5.2. [28, Theorem 4.1, Corollary 4.2] *If $\alpha \ge n/p^-$, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L^{E_1}(G)} \le C\|f\|_{L^{p(\cdot)}(G)}$$

for all $f \in L^{p(\cdot)}(G)$.

Our result is stated in the following:

Theorem 5.3. *Let $\alpha \ge n/p^-$.*

(1) *If (ω3.1) and (ω4.1) hold for all $x_0 \in G$ with the same constant Q, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\mathcal{H}^{E_1,\infty,\eta}(G)} \le C\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \mathcal{H}^{p(\cdot),\infty,\omega}(G)$.

(2) *If (ω4.1) holds for all $x_0 \in G$ with the same constant Q, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\underline{\mathcal{H}}^{E_1,\infty,\eta}(G)} \le C\|f\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)$.

(3) *If (ω3.1) holds for $x_0 \in G$, then there exists a constant $C > 0$ (which may depend on x_0) such that*

$$\|I_\alpha f\|_{\overline{\mathcal{H}}^{E_1,\infty,\eta}_{\{x_0\}}(G)} \le C\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all $f \in \overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$.

Proof. We give only a proof of assertion (1). Let f be a nonnegative measurable function on G such that $\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)} \le 1$. We have only to show the inequality

$$\|I_\alpha f\|_{L^{E_1}(A(x,r))} \le C\eta(x,r)^{-1}$$

for all $x \in G$ and $0 < r < d_G$. Set

$$f = f\chi_{G\setminus B(x,2r)} + f\chi_{B(x,2r)\setminus B(x,r/4)} + f\chi_{B(x,r/4)} = f_1 + f_2 + f_3,$$

as before. We note from Lemma 5.2 that

$$\|I_\alpha f_2\|_{L^{E_1}(A(x,r))} \le C\|f_2\|_{L^{p(\cdot)}(B(x,2r)\setminus B(x,r/4))} \le C\eta(x,r)^{-1}.$$

If $z \in A(x,r)$, then Lemma 3.1 gives

$$I_\alpha f_3(z) \le Cr^{\alpha-n}\int_{B(x,r/4)} f(y)\,dy \le C\eta(x,r)^{-1}$$

since $\alpha \ge n/p^-$, so that

$$\|I_\alpha f_3\|_{L^{E_1}(A(x,r))} \le C\eta(x,r)^{-1}\|1\|_{L^{E_1}(A(x,r))} \le C\eta(x,r)^{-1}$$

by Lemma 5.1. Moreover, Lemma 4.1 gives

$$I_\alpha f_1(z) \le C\int_{G\setminus B(x,2r)} |x-y|^{\alpha-n} f(y)\,dy \le C\eta(x,r)^{-1}$$

since $\alpha \ge n/p^-$, so that

$$\|I_\alpha f_1\|_{L^{E_1}(A(x,r))} \le C\eta(x,r)^{-1}\|1\|_{L^{E_1}(A(x,r))} \le C\eta(x,r)^{-1},$$

as required. □

Corollary 5.4. *Let $\alpha \ge n/p^-$ and let ν, β and ω be as in Corollary 3.5.*

(1) *When $\alpha - n/p^+ < \nu^- \le \nu^+ < n(1-1/p^-)$, there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\mathcal{H}^{E_1,\infty,\omega}(G)} \le C\|f\|_{\mathcal{H}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \mathcal{H}^{p(\cdot),\infty,\omega}(G)$.

(2) *When $\alpha - n/p^+ < \nu^-$, there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{\underline{\mathcal{H}}^{E_1,\infty,\omega}(G)} \le C\|f\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)}$$

for all $f \in \underline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)$.

(3) *When $\nu(x_0) < n(1-1/p(x_0))$ for $x_0 \in G$, there exists a constant $C > 0$ (which may depend on x_0) such that*

$$\|I_\alpha f\|_{\overline{\mathcal{H}}^{E_1,\infty,\omega}_{\{x_0\}}(G)} \le C\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all $f \in \overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$.

6. Associate spaces of $\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$

Recall that for $x_0 \in G$ and measurable functions f on G,

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} = \sup_{0<t<d_G} \omega(x_0,t)\|f\|_{L^{p(\cdot)}(G\setminus B(x_0,t))}$$

and

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)} = \int_0^{2d_G} \omega(x_0,t)\|f\|_{L^{p(\cdot)}(B(x_0,t))}\frac{dt}{t}.$$

Remark 6.1. Let $x_0 \in G$. Note here that if $\omega(x_0,0) = \infty$, then $\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} < \infty$ if and only if $f = 0$ a.e. Hence we may assume that $\omega(x_0,0) = 0$ and then $\omega(x_0,\cdot)$ is uniformly almost increasing on $(0,\infty)$ when $\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} < \infty$.

By the above remark, in this section, suppose

$$\omega(x,0) = 0 \quad \text{for all } x \in G.$$

For $x \in G$ and $0 < t < d_G$, we set

$$p^+(B(x,t)) = \sup_{y\in B(x,t)} p(y),$$

as before. We define $1/q(x) = 1 - 1/p(x)$.

Following Di Fratta and Fiorenza [17], we have the following Hölder type inequality for log-type weights.

Theorem 6.2. *For $x_0 \in G$, suppose*

(ω6.1) *there exist constants $b, Q > 0$ such that*

$$\int_0^t \left(\log\frac{2d_G}{r}\right)^{-bp(x_0)-1} \omega(x_0,r)^{-p^+(B(x_0,t))}\frac{dr}{r} \le Q\left(\log\frac{2d_G}{t}\right)^{-bp(x_0)} \omega(x_0,t)^{-p(x_0)}$$

for all $0 < t < d_G$.

Then there exists a constant $C > 0$ such that

$$\int_G |f(x)g(x)|\,dx \le C\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)}\|g\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all measurable functions f and g on G, where

$$\eta(x_0,r) = \left(\log\frac{2d_G}{r}\right)^{-1}\omega(x_0,r)^{-1}.$$

Proof. Let $x_0 \in G$. Let f and g be nonnegative measurable functions on G such that $\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)} \le 1$ and $\|g\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} \le 1$. We have by Fubini's theorem and Hölder's inequality

$$\begin{aligned}
&\int_G f(x)g(x)\,dx\\
&= \int_G f(x)g(x)\left(b\left(\log\frac{2d_G}{|x-x_0|}\right)^{-b}\int_{|x-x_0|}^{2d_G}\left(\log\frac{2d_G}{t}\right)^{b-1}\frac{dt}{t}\right)dx\\
&= b\int_0^{2d_G}\left(\int_{B(x_0,t)} f(x)g(x)\left(\log\frac{2d_G}{|x-x_0|}\right)^{-b}dx\right)\left(\log\frac{2d_G}{t}\right)^{b-1}\frac{dt}{t}\\
&\le C\int_0^{2d_G}\|f\|_{L^{q(\cdot)}(B(x_0,t))}\left\|g\left(\log\frac{2d_G}{|\cdot-x_0|}\right)^{-b}\right\|_{L^{p(\cdot)}(B(x_0,t))}\left(\log\frac{2d_G}{t}\right)^{b-1}\frac{dt}{t}.
\end{aligned}$$

Here it suffices to show

$$\begin{aligned}
\left\|g\left(\log\frac{2d_G}{|\cdot-x_0|}\right)^{-b}\right\|_{L^{p(\cdot)}(B(x_0,t))} &\le C\left(\log\frac{2d_G}{t}\right)^{-b}\omega(x_0,t)^{-1}\\
&= C\left(\log\frac{2d_G}{t}\right)^{-b+1}\eta(x_0,t)
\end{aligned}$$

for $0 < t < d_G$. In fact, we obtain

$$\begin{aligned}
&\int_{B(x_0,t)}\left(\frac{g(x)}{(\log(2d_G/t))^{-b}\,\omega(x_0,t)^{-1}}\right)^{p(x)}\left(\log\frac{2d_G}{|x-x_0|}\right)^{-bp(x)}dx\\
&\le C\int_{B(x_0,t)}\left(\frac{g(x)}{(\log(2d_G/t))^{-b}\,\omega(x_0,t)^{-1}}\right)^{p(x)}\left(\log\frac{2d_G}{|x-x_0|}\right)^{-bp(x_0)}dx\\
&\le C\int_{B(x_0,t)}\left(\frac{g(x)}{(\log(2d_G/t))^{-b}\,\omega(x_0,t)^{-1}}\right)^{p(x)}\left(\int_0^{|x-x_0|}\left(\log\frac{2d_G}{r}\right)^{-bp(x_0)-1}\frac{dr}{r}\right)dx\\
&\le C\int_0^t\left(\int_{B(x_0,t)\setminus B(x_0,r)} g(x)^{p(x)}\left(\log\frac{2d_G}{t}\right)^{bp(x)}\omega(x_0,t)^{p(x)}\left(\log\frac{2d_G}{r}\right)^{-bp(x_0)-1}dx\right)\frac{dr}{r}
\end{aligned}$$

$$
\begin{aligned}
&\le C\left(\log\frac{2d_G}{t}\right)^{bp(x_0)}\omega(x_0,t)^{p(x_0)}\int_0^t\left(\log\frac{2d_G}{r}\right)^{-bp(x_0)-1}\\
&\cdot\left(\int_{B(x_0,t)\setminus B(x_0,r)}\left(\frac{g(x)}{\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,r))}}\right)^{p(x)}\|g\|^{p(x)}_{L^{p(\cdot)}(G\setminus B(x_0,r))}\,dx\right)\frac{dr}{r}\\
&\le C\left(\log\frac{2d_G}{t}\right)^{bp(x_0)}\omega(x_0,t)^{p(x_0)}\int_0^t\left(\log\frac{2d_G}{r}\right)^{-bp(x_0)-1}\omega(x_0,r)^{-p^+(B(x_0,t))}\frac{dr}{r}\le C
\end{aligned}
$$

by (P2), Lemma 2.1 and (ω6.1). □

Power weights can be treated simpler than Theorem 6.2 in the following manner.

Theorem 6.3. *For $x_0\in G$, suppose*

(ω6.2) *there exist constants $b,Q>0$ such that*

$$\int_0^t r^b\omega(x_0,r)^{-1}\frac{dr}{r}\le Qt^b\omega(x_0,t)^{-1}$$

for all $0<t<d_G$.

Then there exists a constant $C>0$ such that

$$\int_G|f(x)g(x)|\,dx\le C\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)}\|g\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all measurable functions f and g on G, where $\eta(x_0,r)=\omega(x_0,r)^{-1}$.

Proof. Let $x_0\in G$. Let f and g be nonnegative measurable functions on G such that $\|f\|_{\overline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)}\le 1$ and $\|g\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}\le 1$. For $b>0$, we have by Fubini's theorem and Hölder's inequality

$$
\begin{aligned}
\int_G f(x)g(x)\,dx &\le C\int_0^{2d_G}\left(\int_{B(x_0,t)}f(x)g(x)|x-x_0|^b\,dx\right)t^{-b}\frac{dt}{t}\\
&\le C\int_0^{2d_G}\|f\|_{L^{q(\cdot)}(B(x_0,t))}\left\|g|\cdot-x_0|^b\right\|_{L^{p(\cdot)}(B(x_0,t))}t^{-b}\frac{dt}{t}.
\end{aligned}
$$

First, we show that

$$\left\|g|\cdot-x_0|^b\right\|_{L^{p(\cdot)}(B(x_0,2s)\setminus B(x_0,s))}\le Cs^b\omega(x_0,s)^{-1}\le Cs^b\eta(x_0,s)$$

for all $0<s<d_G$. In fact, we obtain

$$
\begin{aligned}
&\int_{B(x_0,2s)\setminus B(x_0,s)}\left(\frac{g(x)}{s^b\omega(x_0,s)^{-1}}\right)^{p(x)}|x-x_0|^{bp(x)}\,dx\\
&\le C\int_{B(x_0,2s)\setminus B(x_0,s)}\left(\frac{g(x)}{\|g\|_{L^{p(\cdot)}(B(x_0,2s)\setminus B(x_0,s))}}\right)^{p(x)}\\
&\quad\cdot\left(\omega(x_0,s)\|g\|_{L^{p(\cdot)}(B(x_0,2s)\setminus B(x_0,s))}\right)^{p(x)}\,dx\le C
\end{aligned}
$$

by (P2) and Lemma 2.1, which gives

$$\left\| g|\cdot - x_0|^b \right\|_{L^{p(\cdot)}(B(x_0,t))} \le \sum_{j=1}^{\infty} \left\| g|\cdot - x_0|^b \right\|_{L^{p(\cdot)}(B(x_0,2^{-j+1}t)\setminus B(x_0,2^{-j}t))}$$
$$\le C\int_0^t r^b \omega(x_0,r)^{-1}\frac{dr}{r} \le Ct^b\omega(x_0,t)^{-1}$$

by (ω6.2). Thus we obtain the required result. □

Theorem 6.4. *Let $\eta(\cdot,\cdot) \in \Omega(G)$. For $x_0 \in G$, suppose*
(ω6.3) *there exists a constant $Q > 0$ such that*

$$\int_t^{2d_G} \eta(x_0,r)\frac{dr}{r} \le Q\omega(x_0,t)^{-1}$$

for all $0 < t < d_G$.
Then there exists a constant $C > 0$ such that

$$\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)} \le C\sup_g \int_G |f(x)g(x)|\,dx$$

for all measurable functions f on G, where the supremum is taken over all measurable functions g on G such that $\|g\|_X \le 1$ with $X = \overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$.

Proof. Let $x_0 \in G$. Let f be a nonnegative measurable function on G. To show the claim, we may assume that

$$\sup_g \int_G |f(x)g(x)|\,dx \le 1,$$

where the supremum is taken over all measurable functions g on G such that $\|g\|_X \le 1$. Take a compact set $K \subset G \setminus \{x_0\}$. Since $L^{p(\cdot)}(K) = \{g\chi_K : g \in L^{p(\cdot)}(G)\} \subset X$, $f\chi_K \in L^{q(\cdot)}(G)$, in view of [25] or [16, Theorem 3.2.13]. By (ω6.3), we find

$$\|f\chi_K\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)} < \infty$$

and, moreover, we have by Lemma 2.2

$$\sum_{j\in N_0} \eta(x_0, 2^{-j+1}d_G)F_j \sim \|f\chi_K\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)},$$

where $F_j = \|f_j\|_{L^{q(\cdot)}(G)}$, $f_j = f\chi_{K\cap B(x_0,2^{-j+1}d_G)}$ and N_0 is the set of positive integers j such that $F_j > 0$. Set

$$g(x) = \sum_{j\in N_0} \eta(x_0,2^{-j+1}d_G)|f_j(x)/F_j|^{q(x)-2}f_j(x)/F_j.$$

Then we see that

$$\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,r))} \le \sum_{j\in N_0,\,2^{-j+1}d_G>r} \eta(x_0,2^{-j+1}d_G)\||f_j/F_j|^{q(\cdot)-2}f_j/F_j\|_{L^{p(\cdot)}(G)}$$
$$\le \sum_{j\ge 1,\,2^{-j+1}d_G>r} \eta(x_0,2^{-j+1}d_G) \le C\omega(x_0,r)^{-1}$$

for all $0 < r < d_G$ by (ω6.3) and hence

$$\|g\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} \le C.$$

Consequently it follows that

$$\int_G f(x)g(x)\,dx = \sum_{j\in N_0} \eta(x_0, 2^{-j+1}d_G) \int_G f(x)|f_j(x)/F_j|^{q(x)-2} f_j(x)/F_j\,dx$$
$$= \sum_{j\in N_0} \eta(x_0, 2^{-j+1}d_G)F_j \ge C\|f\chi_K\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)}.$$

Hence, by the monotone convergence theorem, we have

$$\sup_g \int_G f(x)g(x)\,dx \ge C\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)},$$

which gives the required inequality. □

Let X be a family of measurable functions on G with a norm $\|\cdot\|_X$. Then the associate space X' of X is defined as the family of all measurable functions f on G such that

$$\|f\|_{X'} = \sup_{g\in X:\,\|g\|_X\le 1} \int_G |f(x)g(x)|\,dx < \infty.$$

Theorems 6.2, 6.3 and 6.4 give the following result.

Corollary 6.5. *For $x_0 \in G$, suppose (ω6.1) and (ω6.3) hold. Then*

$$\left(\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)\right)' = \underline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G),$$

where $\eta(x_0, r) = \left(\log \frac{2d_G}{r}\right)^{-1} \omega(x_0, r)^{-1}$. If ($\omega$6.2) and ($\omega$6.3) hold, then the same conclusion is fulfilled with $\eta(x_0, r) = \omega(x_0, r)^{-1}$.

For $0 < q \le \infty$, set

$$\widetilde{\mathcal{H}}^{p(\cdot),q,\omega}(G) = \sum_{x_0\in G} \overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G),$$

whose quasi-norm is defined by

$$\|f\|_{\widetilde{\mathcal{H}}^{p(\cdot),q,\omega}(G)} = \inf_{|f|=\sum_j |f_j|, \{x_j\}\subset G} \sum_j \|f_j\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_j\}}(G)}.$$

The Hölder type inequality in Theorem 6.2 or 6.3, under the same assumptions, implies

$$\int_G |f(x)g(x)|\,dx = \sum_j \int_G |f(x)g_j(x)|\,dx \le C\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}(G)} \sum_j \|g_j\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_j\}}(G)},$$

so that

$$\int_G |f(x)g(x)|\,dx \le C\|f\|_{\underline{\mathcal{H}}^{q(\cdot),1,\eta}(G)} \|g\|_{\widetilde{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)}.$$

Theorem 6.4 gives the converse inequality.

Theorems 6.2, 6.3 and 6.4 give the following result.

Corollary 6.6. *If (ω6.1) and (ω6.3) hold for all $x_0 \in G$ with the same constant Q, then*

$$\left(\widetilde{\mathcal{H}}^{p(\cdot),\infty,\omega}(G)\right)' = \underline{\mathcal{H}}^{q(\cdot),1,\eta}(G),$$

where $\eta(x_0, r) = \left(\log \frac{2d_G}{r}\right)^{-1} \omega(x_0, r)^{-1}$. If ($\omega$6.2) and ($\omega$6.3) hold for all $x_0 \in G$ with the same constant Q, then the same conclusion is fulfilled with $\eta(x_0, r) = \omega(x_0, r)^{-1}$.

Remark 6.7. For $0 < q \le \infty$, set

$$\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G) = \bigcap_{x_0 \in G} \overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)$$

and define the norm

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}(G)} = \sup_{x_0 \in G} \|f\|_{\overline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G)},$$

as usual. Then note that

$$\overline{\mathcal{H}}^{p(\cdot),\infty,\omega}(G) = \begin{cases} L^{p(\cdot)}(G), & \omega(x,0) = 0 \text{ for all } x \in G; \\ \{0\}, & \omega(x,0) = \infty \text{ for all } x \in G. \end{cases}$$

For related results, we refer the reader to the paper by Di Fratta and Fiorenza [17] with logarithmic weights, and the paper by Gagatishvili and Mustafayev [19] with general weights.

Remark 6.8. If $\omega(t) = (\log(2d_G/t))^{-a}$ with $a > 0$, then (ω6.1) and (ω6.3) hold for $\eta(t) = (\log(2d_G/t))^{a-1}$; and if $\omega(t) = r^a$ with $a > 0$, then (ω6.2) and (ω6.3) hold for $\eta(t) = t^{-a}$.

7. Associate spaces of $\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$

Recall that for $x_0 \in G$ and measurable functions f on G,

$$\|f\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} = \sup_{0<t<d_G} \omega(x_0,t)\|f\|_{L^{p(\cdot)}(B(x_0,t))}$$

and

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)} = \int_0^{2d_G} \omega(x_0,t)\|f\|_{L^{p(\cdot)}(G\setminus B(x_0,t))}\frac{dt}{t}.$$

We have the Hölder type inequality for log type weights ω.

Theorem 7.1. *For $x_0 \in G$, suppose*

(ω7.1) *there exist constants $r_0, b, Q > 0$ such that*

$$\int_t^{2r_0} \left(\left(\log\frac{2d_G}{t}\right)^b \omega(x_0,t)^{-1}\right)^{c_p/\log(2d_G/r)} \left(\left(\log\frac{2d_G}{r}\right)^b \omega(x_0,r)^{-1}\right)^{p(x_0)} \cdot \left(\log\frac{2d_G}{r}\right)^{-1}\frac{dr}{r} \le Q\left(\left(\log\frac{2d_G}{t}\right)^b \omega(x_0,t)^{-1}\right)^{p(x_0)}$$

for all $0 < t < r_0$.

Then there exists a constant $C > 0$ such that

$$\int_G |f(x)g(x)|\,dx \le C\|f\|_{\overline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)}\|g\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all measurable functions f, g on G, where $\eta(x_0,r) = \left(\log\frac{2d_G}{r}\right)^{-1}\omega(x_0,r)^{-1}$.

Proof. Let $x_0 \in G$. Let f and g be nonnegative measurable functions on G such that $\|f\|_{\overline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)} \le 1$ and $\|g\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)} \le 1$. For $b > 0$ we have by Fubini's theorem and Hölder's inequality

$$\int_G f(x)g(x)\,dx$$
$$\le C\int_0^{d_G} \|f\|_{L^{q(\cdot)}(G\setminus B(x_0,t))} \left\| g\left(\log\frac{2d_G}{|\cdot - x_0|}\right)^b \right\|_{L^{p(\cdot)}(G\setminus B(x_0,t))} \left(\log\frac{2d_G}{t}\right)^{-b-1}\frac{dt}{t},$$

as in the proof of Theorem 6.2. It suffices to show

$$\left\| g\left(\log\frac{2d_G}{|\cdot - x_0|}\right)^b \right\|_{L^{p(\cdot)}(G\setminus B(x_0,t))} \le C\left(\log\frac{2d_G}{t}\right)^b \omega(x_0,t)^{-1}$$
$$= C\left(\log\frac{2d_G}{t}\right)^{b+1} \eta(x_0,t)$$

for all $0 < t < d_G$. In fact, we obtain for $0 < r_0 < d_G$

$$\int_{B(x_0,r_0)\setminus B(x_0,t)} \left(\frac{g(x)}{(\log(2d_G/t))^b\,\omega(x_0,t)^{-1}}\right)^{p(x)} \left(\log\frac{2d_G}{|x-x_0|}\right)^{bp(x)} dx$$
$$\le C\int_{B(x_0,r_0)\setminus B(x_0,t)} \left(\frac{g(x)}{(\log(2d_G/t))^b\,\omega(x_0,t)^{-1}}\right)^{p(x)} \left(\log\frac{2d_G}{|x-x_0|}\right)^{bp(x_0)} dx$$
$$\le C\int_{B(x_0,r_0)\setminus B(x_0,t)} \left(\frac{g(x)}{(\log(2d_G/t))^b\,\omega(x_0,t)^{-1}}\right)^{p(x)} \left(\int_{|x-x_0|}^{2r_0}\left(\log\frac{2d_G}{r}\right)^{bp(x_0)-1}\frac{dr}{r}\right) dx$$
$$\le C\int_t^{2r_0}\left(\int_{B(x_0,r)\setminus B(x_0,t)} g(x)^{p(x)} \left(\left(\log\frac{2d_G}{t}\right)^{-b}\omega(x_0,t)\right)^{p(x)}\right.$$
$$\left.\cdot\left(\log\frac{2d_G}{r}\right)^{bp(x_0)-1} dx\right)\frac{dr}{r}$$
$$\le C\left(\log\frac{2d_G}{t}\right)^{-bp(x_0)} \omega(x_0,t)^{p(x_0)} \int_t^{2r_0}\left(\left(\log\frac{2d_G}{t}\right)^b\omega(x_0,t)^{-1}\right)^{c_p/\log(2d_G/r)}$$
$$\cdot\left(\log\frac{2d_G}{r}\right)^{bp(x_0)-1}\left(\int_{B(x_0,r)} g(x)^{p(x)}\,dx\right)\frac{dr}{r}$$
$$\le C\left(\log\frac{2d_G}{t}\right)^{-bp(x_0)} \omega(x_0,t)^{p(x_0)} \int_t^{2r_0}\left(\left(\log\frac{2d_G}{t}\right)^b\omega(x_0,t)^{-1}\right)^{c_p/\log(2d_G/r)}$$
$$\cdot\left(\left(\log\frac{2d_G}{r}\right)^b\omega(x_0,r)^{-1}\right)^{p(x_0)}\left(\log\frac{2d_G}{r}\right)^{-1}\frac{dr}{r}$$
$$\le C\left(\log\frac{2d_G}{t}\right)^{-bp(x_0)} \omega(x_0,t)^{p(x_0)} \left(\left(\log\frac{2d_G}{t}\right)^b\omega(x_0,t)^{-1}\right)^{p(x_0)} \le C$$

by (P2), condition (ω7.1) and Lemmas 2.1 and 2.4, which gives

$$\left\| g \left(\log \frac{2d_G}{|\cdot - x_0|} \right)^b \right\|_{L^{p(\cdot)}(B(x_0,r_0)\setminus B(x_0,t))} \le C \left(\log \frac{2d_G}{t} \right)^b \omega(x_0,t)^{-1}$$

for all $0 < t < r_0$. Moreover,

$$\left\| g \left(\log \frac{2d_G}{|\cdot - x_0|} \right)^b \right\|_{L^{p(\cdot)}(G\setminus B(x_0,r_0))} \le C \, \|g\|_{L^{p(\cdot)}(G\setminus B(x_0,r_0))} \le C,$$

which completes the proof. □

Remark 7.2. We show that $\omega(t) = (\log(2d_G/t))^a$ with $a > 0$ satisfies (ω7.1). To show this, for $b, c > 0$ one can find constants $r_0, Q > 0$ such that

$$\int_t^{2r_0} \left(\log \frac{2d_G}{t} \right)^{c/\log(2d_G/r)} \left(\log \frac{2d_G}{r} \right)^{b-1} \frac{dr}{r} \le Q \left(\log \frac{2d_G}{t} \right)^b$$

for all $0 < t < r_0$ and $x_0 \in G$. In fact, first find $0 < r_0 < d_G/e$ such that $\varepsilon = 1/\log(d_G/r_0) < b/2c$, and note for $\tilde{t} = 2d_G e^{-(\log(2d_G/t))^{1/2}}$ that

$$\begin{aligned} \int_t^{\tilde{t}} \left(\log \frac{2d_G}{t} \right)^{c/\log(2d_G/r)} \left(\log \frac{2d_G}{r} \right)^{b-1} \frac{dr}{r} &\le C \int_t^{2d_G} \left(\log \frac{2d_G}{r} \right)^{b-1} \frac{dr}{r} \\ &\le Q \left(\log \frac{2d_G}{t} \right)^b \end{aligned}$$

since $(\log(2d_G/t))^{c/\log(2d_G/r)} \le C$ for all $t < r < \tilde{t}$ and

$$\begin{aligned} \int_{\tilde{t}}^{2r_0} \left(\log \frac{2d_G}{t} \right)^{c/\log(2d_G/r)} \left(\log \frac{2d_G}{r} \right)^{b-1} \frac{dr}{r} &\le C \left(\log \frac{2d_G}{t} \right)^{c\varepsilon} \int_{\tilde{t}}^{2r_0} \left(\log \frac{2d_G}{r} \right)^{b-1} \frac{dr}{r} \\ &\le Q \left(\log \frac{2d_G}{t} \right)^{c\varepsilon + b/2} \le Q \left(\log \frac{2d_G}{t} \right)^b, \end{aligned}$$

as required.

For power weights ω, we obtain the following result.

Theorem 7.3. *For $x_0 \in G$, suppose*

(ω7.2) *there exist constants $b, Q > 0$ such that*

$$\int_t^{2d_G} r^{-b} \omega(x_0,r)^{-1} \frac{dr}{r} \le Q r^{-b} \omega(x_0,t)^{-1}$$

for all $0 < t < d_G$.

Then there exists a constant $C > 0$ such that

$$\int_G |f(x)g(x)| \, dx \le C \|f\|_{\overline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G)} \|g\|_{\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)}$$

for all measurable functions f, g on G, where $\eta(x_0,r) = \omega(x_0,r)^{-1}$.

As in the proof of Theorem 6.4, we have the following result.

Theorem 7.4. *Let $\eta(\cdot,\cdot) \in \Omega(G)$. For $x_0 \in G$, suppose*

(ω7.3) *there exists a constant* $Q > 0$ *such that*

$$\int_0^t \eta(x_0, r) \frac{dr}{r} \le Q\omega(x_0, t)^{-1}$$

for all $0 < t < d_G$.

Then there exists a constant $C > 0$ *such that*

$$\|f\|_{\overline{\mathcal{H}}^{\{x_0\}}_{q(\cdot),1,\eta}(G)} \le C \sup_g \int_G |f(x)g(x)|\,dx$$

for all measurable functions f *on* G, *where the supremum is taken over all measurable functions* g *on* G *such that* $\|g\|_X \le 1$ *with* $X = \underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)$.

Theorems 7.1, 7.3 and 7.4 give the following result.

Corollary 7.5. *If* (ω7.1) *and* (ω7.3) *hold for* $x_0 \in G$, *then*

$$\left(\underline{\mathcal{H}}^{p(\cdot),\infty,\omega}_{\{x_0\}}(G)\right)' = \overline{\mathcal{H}}^{q(\cdot),1,\eta}_{\{x_0\}}(G),$$

where $\eta(x_0, r) = \left(\log \frac{2d_G}{r}\right)^{-1} \omega(x_0, r)^{-1}$. *If* ($\omega$7.2) *and* ($\omega$7.3) *hold for* $x_0 \in G$, *then the same conclusion is fulfilled with* $\eta(x_0, r) = \omega(x_0, r)^{-1}$.

Remark 7.6. If $\omega(t) = (\log(2d_G/t))^a$ with $a > 0$, then (ω7.1) and (ω7.3) hold for $\eta(t) = (\log(2d_G/t))^{-a-1}$; and if $\omega(t) = t^{-a}$ with $a > 0$, then (ω7.2) and (ω7.3) hold for $\eta(t) = t^a$.

For $0 < q \le \infty$, we may consider

$$\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G) = \sum_{x_0 \in G} \underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_0\}}(G),$$

whose quasi-norm is defined by

$$\|f\|_{\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G)} = \inf_{|f| = \sum_j |f_j|, \{x_j\} \subset G} \sum_j \|f_j\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_j\}}(G)}.$$

One can show that

$$\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G) = L^{p(\cdot)}(G).$$

For this, we only show the inclusion $L^{p(\cdot)}(G) \subset \underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G)$. Take $f \in L^{p(\cdot)}(G)$ and $x_1, x_2 \in G$ ($x_1 \ne x_2$). Write

$$f = f\chi_{B(x_2, |x_1 - x_2|/2)} + f\chi_{G \setminus B(x_2, |x_1 - x_2|/2)} = f_1 + f_2.$$

Then

$$\begin{aligned}\|f_1\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_1\}}(G)} &\le \left(\int_{|x_1 - x_2|/2}^{2d_G} \left(\omega(x_1, r)\|f_1\|_{L^{p(\cdot)}(B(x_1,r))}\right)^q dr/r\right)^{1/q} \\ &\le \|f_1\|_{L^{p(\cdot)}(G)} \left(\int_{|x_1 - x_2|/2}^{2d_G} \omega(x_1, r)^q dr/r\right)^{1/q} = A\|f_1\|_{L^{p(\cdot)}(G)}\end{aligned}$$

and

$$\|f_2\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_2\}}(G)} \le \left(\int_{|x_1-x_2|/2}^{2d_G} \left(\omega(x_2,r)\|f_2\|_{L^{p(\cdot)}(B(x_2,r))}\right)^q dr/r\right)^{1/q}$$
$$\le \|f_2\|_{L^{p(\cdot)}(G)} \left(\int_{|x_1-x_2|/2}^{2d_G} \omega(x_2,r)^q dr/r\right)^{1/q} = B\|f_2\|_{L^{p(\cdot)}(G)}.$$

Hence

$$\|f\|_{\underset{\sim}{\mathcal{H}}^{p(\cdot),q,\omega}(G)} \le \|f_1\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_1\}}(G)} + \|f_2\|_{\underline{\mathcal{H}}^{p(\cdot),q,\omega}_{\{x_2\}}(G)} \le A\|f_1\|_{L^{p(\cdot)}(G)} + B\|f_2\|_{L^{p(\cdot)}(G)}$$
$$\le (A+B)\|f\|_{L^{p(\cdot)}(G)} < \infty,$$

as required.

8. Associate spaces of $\underline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)$

Theorem 8.1. *Let $\eta(\cdot,\cdot) \in \Omega(G)$, $x_0 \in G$ and $X = \underline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)$. Suppose*

(ω8.1) *there exists a constant $Q > 0$ such that*

$$\int_t^{2d_G} \omega(x_0,r)\frac{dr}{r} \le Q\eta(x_0,t)^{-1}$$

for all $0 < t < d_G$.

Then there exists a constant $C > 0$ such that

$$\|f\|_{\overline{\mathcal{H}}^{q(\cdot),\infty,\eta}_{\{x_0\}}(G)} \le C\|f\|_{X'}$$

for all measurable functions f on G.

Proof. Let $x_0 \in G$. First we show

$$\int_{G\setminus B(x_0,R)} f(x)g(x)\,dx \le C\eta(x_0,R)^{-1}\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,R))}\|f\|_{X'} \tag{8.1}$$

for $0 < R < d_G$ and nonnegative measurable functions f, g on G. To show this, we consider

$$h = \eta(x_0,R)g\chi_{G\setminus B(x_0,R)}/\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,R))}$$

when $0 < \|g\|_{L^{p(\cdot)}(G\setminus B(x_0,R))} < \infty$. Then we have by ($\omega$8.1)

$$\int_0^{2d_G} \omega(x_0,t)\|h\|_{L^{p(\cdot)}(B(x_0,t))}\frac{dt}{t} \le \eta(x_0,R)\int_R^{2d_G} \omega(x_0,t)\frac{dt}{t} \le C,$$

and hence

$$\int_{G\setminus B(x_0,R)} f(x)h(x)\,dx \le C\|f\|_{X'}.$$

Now we obtain

$$\int_{G\setminus B(x_0,R)} f(x)g(x)\,dx \le C\eta(x_0,R)^{-1}\|g\|_{L^{p(\cdot)}(G\setminus B(x_0,R))}\|f\|_{X'}.$$

If we take $g(x) = |f(x)/\|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))}|^{q(x)-1}\chi_{G\setminus B(x_0,R)}$ when $0 < \|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))} < \infty$, then we have by (8.1) that

$$
\begin{aligned}
1 &= \int_{G\setminus B(x_0,R)} \{f(x)/\|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))}\}^{q(x)}dx \\
&\le C\eta(x_0,R)^{-1}\|\{f/\|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))}\}^{q(\cdot)-1}\|_{L^{p(\cdot)}(G\setminus B(x_0,R))}\|f/\|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))}\|_{X'} \\
&\le C\eta(x_0,R)^{-1}\left\{\|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))}\right\}^{-1}\|f\|_{X'},
\end{aligned}
$$

which shows

$$\eta(x_0,R)\|f\|_{L^{q(\cdot)}(G\setminus B(x_0,R))} \le C\|f\|_{X'}.$$

Thus it follows that

$$\|f\|_{\overline{\mathcal{H}}^{q(\cdot),\infty,\eta}_{\{x_0\}}(G)} \le C\|f\|_{X'},$$

as required. □

Corollary 8.2. *If* (ω8.1) *holds for* $x_0 \in G$ *and* (ω6.1) *holds for* $x_0 \in G$, η *and* $q(\cdot)$, *then*

$$\left(\underline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)\right)' = \overline{\mathcal{H}}^{q(\cdot),\infty,\eta}_{\{x_0\}}(G),$$

where $\eta(x_0,r) = \left(\log\frac{2d_G}{r}\right)^{-1}\omega(x_0,r)^{-1}$. *If* ($\omega$8.1) *holds for* $x_0 \in G$ *and* (ω6.2) *holds for* $x_0 \in G$, η *and* $q(\cdot)$, *then the same conclusion is fulfilled with* $\eta(x_0,r) = \omega(x_0,r)^{-1}$.

As in Fiorenza–Rakotoson [18, Corollary 1], we see that the associate and dual spaces of $\underline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)$ coincides with each other.

Remark 8.3. If $\omega(t) = (\log(2d_G/t))^{-1/a}$ with $a > 1$, then (ω8.1) holds for $\eta(t) = (\log(2d_G/t))^{-1/a'}$; and if $\omega(t) = t^{-a}$ with $a > 0$, then (ω8.1) holds for $\eta(t) = t^a$.

9. Associate space of $\overline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)$

As in the proof of Theorem 8.1, we have the following result.

Theorem 9.1. *Let* $\eta(\cdot,\cdot) \in \Omega(G)$, $x_0 \in G$ *and* $X = \overline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)$. *Suppose*

(ω9.1) *there exists a constant* $Q > 0$ *such that*

$$\int_0^t \omega(x_0,r)\frac{dr}{r} \le Q\eta(x_0,t)^{-1}$$

for all $0 < t < d_G$.

Then there exists a constant $C > 0$ *such that*

$$\|f\|_{\underline{\mathcal{H}}^{q(\cdot),\infty,\eta}_{\{x_0\}}(G)} \le C\|f\|_{X'}$$

for all measurable functions f *on* G.

Corollary 9.2. *If* (ω9.1) *holds for* $x_0 \in G$ *and* (ω7.1) *holds for* $x_0 \in G$, η *and* $q(\cdot)$, *then*

$$\left(\overline{\mathcal{H}}^{p(\cdot),1,\omega}_{\{x_0\}}(G)\right)' = \underline{\mathcal{H}}^{q(\cdot),\infty,\eta}_{\{x_0\}}(G),$$

where $\eta(x_0,r) = \left(\log\frac{2d_G}{r}\right)^{-1}\omega(x_0,r)^{-1}$. *If* ($\omega$9.1) *holds for* $x_0 \in G$ *and* (ω7.2) *holds for* $x_0 \in G$, η *and* $q(\cdot)$, *then the same conclusion is fulfilled with* $\eta(x_0,r) = \omega(x_0,r)^{-1}$.

Corollary 9.3. *If* (ω9.1) *holds for all* $x_0 \in G$ *with the same constant* Q *and* (ω7.1) *holds for* η, $q(\cdot)$ *and all* $x_0 \in G$ *with the same constant* Q, *then*

$$\left(\widetilde{\mathcal{H}}^{p(\cdot),1,\omega}(G)\right)' = \underline{\mathcal{H}}^{q(\cdot),\infty,\eta}(G),$$

where $\eta(x_0,r) = \left(\log \frac{2d_G}{r}\right)^{-1} \omega(x_0,r)^{-1}$. *If* ($\omega$9.1) *holds for all* $x_0 \in G$ *with the same constant* Q *and* (ω7.2) *holds for* η, $q(\cdot)$ *and all* $x_0 \in G$ *with the same constant* Q, *then the same conclusion is fulfilled with* $\eta(x_0,r) = \omega(x_0,r)^{-1}$.

This corollary gives a characterization of Morrey spaces of variable exponents; see also the paper by Gogatishvili and Mustafayev [19] for constant exponents.

Remark 9.4. If $\omega(t) = (\log(2d_G/t))^{-a-1}$ with $a > 0$, then (ω9.1) holds for $\eta(t) = (\log(2d_G/t))^{a}$; and if $\omega(t) = t^a$ with $a > 0$, then (ω9.1) holds for $\eta(t) = t^{-a}$.

10. Grand and small Lebesgue spaces

Following Capone–Fiorenza [11], for $0 < \theta < 1$ and measurable functions f on the unit ball $\mathbf{B} = B(0,1)$, we define the norm

$$\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\theta}_{\{0\}}(\mathbf{B})} = \sup_{0<t<1} \left(\log \frac{2}{t}\right)^{-\theta/p(0)} \|f\|_{L^{p(\cdot)}(\mathbf{B}\setminus B(0,t))}$$

and

$$\|f\|_{L^{p(\cdot)-0,\theta}(\mathbf{B})} = \sup_{0<\varepsilon<p^- -1} \varepsilon^{\theta/p(0)} \|f\|_{L^{p(\cdot)-\varepsilon}(\mathbf{B})}.$$

Theorem 10.1. *There exists a constant* $C > 0$ *such that*

$$\|f\|_{L^{p(\cdot)-0,\theta}(\mathbf{B})} \le C \|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\theta}_{\{0\}}(\mathbf{B})}$$

for all measurable functions f *on* $\mathbf{B}$.

Proof. Let f be a nonnegative measurable function on $\mathbf{B}$ such that $\|f\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\theta}_{\{0\}}(\mathbf{B})} \le 1$ or

$$\int_{\mathbf{B}\setminus B(0,t)} \left(\left(\log \frac{2}{t}\right)^{-\theta/p(0)} f(x)\right)^{p(x)} dx \le 1 \tag{10.1}$$

for all $0 < t < 1$. For $0 < \varepsilon < p^- - 1$, we take $0 < s < 1$ such that $\varepsilon = (p^- - 1)(\log 2)/\log(2/s)$. We have

$$\int_{\mathbf{B}\setminus B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)-\varepsilon} dx \le \int_{\mathbf{B}\setminus B(0,s)} 1\, dx + \int_{\mathbf{B}\setminus B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)} dx \le C.$$

By multiplying (10.1) by $(\log(2/t))^{-b-1}$ for (large) $b > 1$, integration gives

$$\int_0^r \left(\log \frac{2}{t}\right)^{-b-1} \frac{dt}{t}$$
$$\ge \int_0^r \left(\log \frac{2}{t}\right)^{-b-1} \left(\int_{B(0,r)\setminus B(0,t)} \left(\left(\log \frac{2}{t}\right)^{-\theta/p(0)} f(x)\right)^{p(x)} dx\right) \frac{dt}{t}$$

$$
\begin{aligned}
&\geq \int_0^r \left(\log\frac{2}{t}\right)^{-b-1}\left(\int_{B(0,r)\setminus B(0,t)}\left(\log\frac{2}{t}\right)^{-\theta-c_p/\log(2/|x|)} f(x)^{p(x)}\,dx\right)\frac{dt}{t}\\
&= \int_{B(0,r)} f(x)^{p(x)}\left(\int_0^{|x|}\left(\log\frac{2}{t}\right)^{-b-1-\theta-c_p/\log(2/|x|)}\frac{dt}{t}\right)dx\\
&\geq C\int_{B(0,r)} f(x)^{p(x)}\left(\log\frac{2}{|x|}\right)^{-b-\theta} dx,
\end{aligned}
$$

or

$$
\int_{B(0,r)} f(x)^{p(x)}\left(\log\frac{2}{|x|}\right)^{-b-\theta} dx \leq C\left(\log\frac{2}{r}\right)^{-b}
$$

for $0<r<1$.

First consider the case when

$$
A=\int_{B(0,s)}\left(\log\frac{2}{|x|}\right)^{(p(0)-\varepsilon)(\theta+b)/\varepsilon} dx \geq 1.
$$

For $k>1$, we obtain

$$
\begin{aligned}
&\int_{B(0,s)}\left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)-\varepsilon} dx\\
&\leq \int_{B(0,s)}\left(\varepsilon^k A^{-1/p(0)}\left(\log\frac{2}{|x|}\right)^{(\theta+b)/\varepsilon}\right)^{p(x)-\varepsilon} dx\\
&\quad + \int_{B(0,s)}\left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)-\varepsilon}\left(\frac{\varepsilon^{\theta/p(0)} f(x)}{\varepsilon^k A^{-1/p(0)}\left(\log(2/|x|)\right)^{(\theta+b)/\varepsilon}}\right)^{\varepsilon} dx\\
&\leq C\Bigg\{\varepsilon^{kp(0)}\int_{B(0,s)} A^{-(p(x)-\varepsilon)/p(0)}\left(\log\frac{2}{|x|}\right)^{(p(0)-\varepsilon)(\theta+b)/\varepsilon} dx\\
&\quad + \varepsilon^{\theta} A^{\varepsilon/p(0)}\int_{B(0,s)} f(x)^{p(x)}\left(\log\frac{2}{|x|}\right)^{-(\theta+b)} dx\Bigg\}
\end{aligned}
$$

since $\varepsilon^{p(x)-\varepsilon}\leq C\varepsilon^{p(0)}$ by (P2) for all $x\in B(0,s)$. Since $\log(2/t)\leq (2^a/a)t^{-a}$ for $0<t<1$ and $a=\varepsilon/\{2(p(0)-\varepsilon)(\theta+b)\}$, we find

$$
A\leq \int_{B(0,s)}\left(\frac{2^a}{a}|x|^{-a}\right)^{1/(2a)} dx \leq \left(\frac{2^a}{a}\right)^{1/(2a)}\int_{\mathbf{B}}|x|^{-1/2}\,dx \leq Ca^{-1/(2a)},
$$

so that we have by (P2)

$$
A^{-p(x)/p(0)}\leq CA^{-1+c\varepsilon/p(0)} \quad \text{for } x\in B(0,s) \text{ and some constant } c>0
$$

and

$$
A^{\varepsilon/p(0)}\leq C\varepsilon^{-(b+\theta)}.
$$

Hence we have

$$
\begin{aligned}
&\int_{B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)-\varepsilon} dx \\
&\le C\Bigg\{\varepsilon^{kp(0)} A^{-(p(0)-(1+c)\varepsilon)/p(0)} \int_{B(0,s)} \left(\log\frac{2}{|x|}\right)^{(p(0)-\varepsilon)(\theta+b)/\varepsilon} dx \\
&\quad + \varepsilon^{\theta} A^{\varepsilon/p(0)} \int_{B(0,s)} f(x)^{p(x)} \left(\log\frac{2}{|x|}\right)^{-(\theta+b)} dx\Bigg\} \\
&\le C\Bigg\{\varepsilon^{kp(0)} A^{\varepsilon(1+c)/p(0)} + \varepsilon^{\theta} A^{\varepsilon/p(0)} \int_{B(0,s)} f(x)^{p(x)} \left(\log\frac{2}{|x|}\right)^{-(\theta+b)} dx\Bigg\} \\
&\le C\left\{\varepsilon^{kp(0)} A^{\varepsilon(1+c)/p(0)} + \varepsilon^{\theta+b} A^{\varepsilon/p(0)}\right\} \le C\left\{\varepsilon^{kp(0)-(b+\theta)(1+c)} + 1\right\}.
\end{aligned}
$$

If we take b and k such that $kp(0) - (b+\theta)(1+c) \ge 0$, then the present case is obtained.

If $A \le 1$, then we obtain by (P2)

$$
\begin{aligned}
&\int_{B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)-\varepsilon} dx \\
&\le \int_{B(0,s)} \left(\log\frac{2}{|x|}\right)^{(\theta+b)(p(x)-\varepsilon)/\varepsilon} dx + \int_{B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)-\varepsilon} \left(\frac{\varepsilon^{\theta/p(0)} f(x)}{(\log(2/|x|))^{(\theta+b)/\varepsilon}}\right)^{\varepsilon} dx \\
&\le C + \int_{B(0,s)} \left(\varepsilon^{\theta/p(0)} f(x)\right)^{p(x)} \left(\log\frac{2}{|x|}\right)^{-(\theta+b)} dx \\
&\le C\left\{1 + \varepsilon^{\theta} \int_{B(0,s)} f(x)^{p(x)} \left(\log\frac{2}{|x|}\right)^{-(\theta+b)} dx\right\} \le C\left\{1 + \varepsilon^{\theta}\left(\log\frac{2}{s}\right)^{-b}\right\} \le C,
\end{aligned}
$$

which completes the proof. □

Given f on $\mathbf{R}^n$, recall the definition of the symmetric decreasing rearrangement of f by

$$
f^{\star}(x) = \int_0^{\infty} \chi_{E_f(t)^{\star}}(x)\, dt,
$$

where $E^{\star} = \{x\colon |B(0,|x|)| < |E|\}$ and $E_f(t) = \{y\colon |f(y)| > t\}$; see Burchard [6].

Theorem 10.2. *There exists a constant $C > 0$ such that*

$$
\|f^{\star}\|_{\overline{\mathcal{H}}^{p(\cdot),\infty,\theta}_{\{0\}}(\mathbf{B})} \le C\|f^{\star}\|_{L^{p(\cdot)-0,\theta}(\mathbf{B})}
$$

for all measurable functions f on $\mathbf{B}$.

Proof. Let f be a nonnegative measurable function on $\mathbf{B}$ such that $\|f^{\star}\|_{L^{p(\cdot)-0,\theta}(\mathbf{B})} \le 1$. Note that

$$
\int_{\mathbf{B}\setminus B(0,t/2)} \left(\varepsilon^{\theta/p(0)} f^{\star}(x)\right)^{p(x)-\varepsilon} dx \le 1 \tag{10.2}
$$

for all $0 < t < 1$ and $\varepsilon = (p^- - 1)(\log 2)/\log(2/t)$. We have

$$\int_{\mathbf{B}\setminus B(0,t)} \left(\varepsilon^{\theta/p(0)} f^\star(x)\right)^{p(x)} dx$$
$$\le C \left(\frac{1}{|B(0,t)\setminus B(0,t/2)|} \int_{B(0,t)\setminus B(0,t/2)} f^\star(x)\, dx \right)^{\varepsilon} \int_{\mathbf{B}\setminus B(0,t)} \varepsilon^{\theta p(x)/p(0)} f^\star(x)^{p(x)-\varepsilon}\, dx$$

since $f^\star$ is radially decreasing. Set

$$I = \frac{1}{|B(0,t)\setminus B(0,t/2)|} \int_{B(0,t)\setminus B(0,t/2)} f^\star(x)\, dx$$

and

$$J = \left(\frac{1}{|B(0,t)\setminus B(0,t/2)|} \int_{B(0,t)\setminus B(0,t/2)} f^\star(x)^{p(x)-\varepsilon}\, dx \right)^{1/(p(0)-\varepsilon)}.$$

If $J \ge 1$, then we have by (10.2)

$$\begin{aligned} I &\le J + C \frac{1}{|B(0,t)\setminus B(0,t/2)|} \int_{B(0,t)\setminus B(0,t/2)} f^\star(x) \left(\frac{f^\star(x)}{J} \right)^{p(x)-\varepsilon-1} dx \\ &\le J + C J^{-p(0)+\varepsilon+1} \frac{1}{|B(0,t)\setminus B(0,t/2)|} \int_{B(0,t)\setminus B(0,t/2)} f^\star(x)^{p(x)-\varepsilon}\, dx \le CJ \end{aligned}$$

by (P2) since $J \le C t^{-n/p(0)} (\log(2/t))^{\theta/p(0)}$ for all $0 < t < 1$ and if $J \le 1$, then

$$I \le 1 + \frac{1}{|B(0,t)\setminus B(0,t/2)|} \int_{B(0,t)\setminus B(0,t/2)} f^\star(x)^{p(x)-\varepsilon}\, dx \le C.$$

Hence

$$I^\varepsilon \le C \left(t^{-n\varepsilon/p(0)} (\log(2/t))^{\theta\varepsilon/p(0)} + 1 \right) \le C,$$

so that

$$\int_{\mathbf{B}\setminus B(0,t)} \left(\varepsilon^{\theta/p(0)} f^\star(x)\right)^{p(x)} dx \le C \int_{\mathbf{B}\setminus B(0,t/2)} \left(\varepsilon^{\theta/p(0)} f^\star(x)\right)^{p(x)-\varepsilon} dx \le C,$$

which completes the proof. □

References

[1] Adams, D. R.: A note on Riesz potentials. - Duke Math. J. 42, 1975, 765–778.

[2] Adams, D. R., and L. I. Hedberg: Function spaces and potential theory. - Springer, 1996.

[3] Adams, D. R., and J. Xiao: Morrey spaces in harmonic analysis. - Ark. Mat. 50:2, 2012, 201–230.

[4] Almeida, A., and D. Drihem: Maximal, potential and singular type operators on Herz spaces with variable exponents. - J. Math. Anal. Appl. 394:2, 2012, 781–795.

[5] Almeida, A., J. Hasanov, and S. Samko: Maximal and potential operators in variable exponent Morrey spaces. - Georgian Math. J. 15, 2008, 195–208.

[6] Burchard, A.: Rearrangement inequalities. - Lecture notes, June 2009.

[7] Burenkov, V. I., A. Gogatishvili, V. S. Guliyev, and R. Ch. Mustafayev: Boundedness of the fractional maximal operator in local Morrey-type spaces. - Complex Var. Elliptic Equ. 55:8-10, 2010, 739–758.

[8] Burenkov, V. I., A. Gogatishvili, V. S. Guliyev, and R. Ch. Mustafayev: Boundedness of the Riesz potential in local Morrey-type spaces. - Potential Anal. 35:1, 2011, 67–87.

[9] Burenkov, V. I., and H. V. Guliyev: Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces. - Studia Math. 163:2, 2004, 157–176.

[10] Burenkov, V. I., H. V. Guliyev, and V. S. Guliyev: Necessary and sufficient conditions for the boundedness of fractional maximal operators in local Morrey-type spaces. - J. Comput. Appl. Math. 208:1, 2007, 280–301.

[11] Capone, C., and A. Fiorenza: On small Lebesgue spaces. - J. Funct. Spaces Appl. 3:1, 2005, 73–89.

[12] Chiarenza, F., and M. Frasca: Morrey spaces and Hardy–Littlewood maximal function. - Rend. Mat. 7, 1987, 273–279.

[13] Diening, L.: Maximal functions in generalized $L^{p(\cdot)}$ spaces. - Math. Inequal. Appl. 7:2, 2004, 245–254.

[14] Diening, L.: Riesz potentials and Sobolev embeddings on generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$. - Math. Nachr. 263:1, 2004, 31–43.

[15] Diening, L., P. Harjulehto, P. Hästö, Y. Mizuta, and T. Shimomura: Maximal functions in variable exponent spaces: limiting cases of the exponent. - Ann. Acad. Sci. Fenn. Math. 34, 2009, 503–522.

[16] Diening, L., P. Harjulehto, P. Hästö, and M. Růžička: Lebesgue and Sobolev spaces with variable exponents. - Lecture Notes in Math. 2017, Springer, Heidelberg, 2011.

[17] Di Fratta, G., and A. Fiorenza: A direct approach to the duality of grand and small Lebesgue spaces. - Nonlinear Anal. 70:7, 2009, 2582–2592.

[18] Fiorenza, A., and J. M. Rakotoson: Some estimates in $G\Gamma(p,m,w)$ spaces. - J. Math. Anal. Appl. 340:2, 2008, 793–805.

[19] Gogatishvili, A., and R. Ch. Mustafayev: Dual spaces of local Morrey-type spaces. - Czechoslovak Math. J. 61:3, 2011, 609–622.

[20] Gogatishvili, A., and R. Ch. Mustafayev: New pre-dual space of Morrey space. - J. Math. Anal. Appl. 397:2, 2013, 678–692.

[21] Guliyev, V. S., J. J. Hasanov, and S. G. Samko: Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces. - Math. Scand. 107:2, 2010, 285–304.

[22] Guliyev, V. S., J. J. Hasanov, and S. G. Samko: Maximal, potential and singular operators in the local "complementary" variable exponent Morrey type spaces. - J. Math. Anal. Appl. 401:1, 2013, 72–84.

[23] Herz, C.: Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms. - J. Math. Mech. 18, 1968, 283–324.

[24] Izuki, M.: Fractional integrals on Herz–Morrey spaces with variable exponent. - Hiroshima Math. J. 40:3, 2010, 343–355.

[25] Kováčik, O., and J. Rákosník: On spaces $L^{p(x)}$ and $W^{k,p(x)}$. - Czechoslovak Math. J. 41, 1991, 592–618.

[26] Mizuta, Y., E. Nakai, T. Ohno, and T. Shimomura: Riesz potentials and Sobolev embeddings on Morrey spaces of variable exponent. - Complex Var. Elliptic Equ. 56, 2011, 671–695.

[27] Mizuta, Y., and T. Ohno: Duality of Herz–Morrey spaces of variable exponent. - In: The Proceedings on Potential Theory in Sapporo, Japan, 2013 (to appear).

[28] Mizuta, Y., T. Ohno, and T. Shimomura: Sobolev embeddings for Riesz potential spaces of variable exponents near 1 and Sobolev's exponent. - Bull. Sci. Math. 134, 2010, 12–36.

[29] Mizuta, Y., and T. Shimomura: Sobolev embeddings for Riesz potentials of functions in Morrey spaces of variable exponent. - J. Math. Soc. Japan 60, 2008, 583–602.

[30] Morrey, C. B.: On the solutions of quasi-linear elliptic partial differential equations. - Trans. Amer. Math. Soc. 43, 1938, 126–166.

[31] Nakai, E.: Hardy–Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces. - Math. Nachr. 166, 1994, 95–103.

[32] Peetre, J.: On the theory of $L_{p,\lambda}$ spaces. - J. Funct. Anal. 4, 1969, 71–87.

[33] Růžička, M.: Electrorheological fluids: Modeling and mathematical theory. - Springer-Verlag, Berlin, 2000.

[34] Stein, E. M.: Singular integrals and differentiability properties of functions. - Princeton Univ. Press, Princeton, 1970.

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