MULTIPLE SOLUTIONS FOR ELLIPTIC EQUATIONS INVOLVING A GENERAL OPERATOR IN DIVERGENCE FORM

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Abstract. In this paper, exploiting variational methods, the existence of three weak solutions for a class of elliptic equations involving a general operator in divergence form and with Dirichlet boundary condition is investigated. Several special cases are analyzed. In conclusion, for completeness, a concrete example of an application is presented by finding the existence of three nontrivial weak solutions for an uniformly elliptic second-order problem on a bounded Euclidean domain.

1. Introduction

In this paper we study the existence of multiple weak solutions for the following Dirichlet boundary-value problem:

$$(D_\lambda^f) \qquad \begin{cases} -\operatorname{div}(a(x, \nabla u)) = \lambda k(x) f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \, \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ ($N \geq 2$) with smooth boundary $\partial\Omega$, $p > 1$, $a \colon \bar{\Omega} \times \mathbf{R}^N \to \mathbf{R}^N$ is a suitable continuous map of gradient type, and λ is a positive real parameter. Further, $f \colon \mathbf{R} \to \mathbf{R}$ and $k \colon \bar{\Omega} \to \mathbf{R}^+$ are two continuous functions.

We first observe that our multiplicity theorems are related to some known results in the current literature on the subject. For instance, we refer the reader to the paper of Brézis and Oswald [4] in which an existence and uniqueness result was obtained via the minimization technique and by using a maximum principle, as well as to the work [12], in which Lin proved the existence, uniqueness and asymptotical properties of the solutions of problem (D_λ^f), when f behaves like the power t^q, for some $q \in]0, 1[$, and sufficiently large t.

Further, Dirichlet problems involving a general operator in divergence form was studied by De Nápoli and Mariani in [7]. In this paper the existence of one weak solution were proved by exploiting the standard mountain pass geometry and requiring, among other assumptions, that the nonlinearity f has a $(p-1)$-superlinear behaviour at infinity. The non-uniform case was successively considered by Duc and Vu in [8], by extending the result of [7] under the key hypothesis that the map a fulfills the growth condition:

$$|a(x, \xi)| \leq c_0(h_0(x) + h_1(x)|\xi|^{p-1}),$$

for every $(x, \xi) \in \Omega \times \mathbf{R}^N$, where $h_0 \in L^{p/(p-1)}(\Omega)$, $h_1 \in L^1_{loc}(\Omega)$ and c_0 is a positive constant.

In both papers [7] and [8] the function f satisfies the celebrated Ambrosetti–Rabinowitz condition:

(AR) *There exist $s_0 > 0$ and $\theta > p$ such that*

$$0 < \theta \int_0^s f(t)\, dt \le s f(s), \quad \forall\, |s| \ge s_0.$$

In [5], Colasuonno, Pucci and Varga studied different and very general classes of elliptic operators in divergence form looking at the existence of multiple weak solutions; see also Remark 3.13. Their contributions represent a nice improvement, in several directions, of the results obtained by Kristály, Lisei and Varga in [10] in which a uniform Dirichlet problem with parameter is investigated.

Yang, Geng and Yan, in [18], proved the existence of three weak solutions for singular p-Laplacian type equations. Further, Papageorgiou, Rocha and Staicu considered a nonsmooth p-Laplacian problem in divergence form, obtaining the existence of at least two nontrivial weak solutions (see [14]).

We also note that Autuori, Pucci and Varga, in [1], studied a quasilinear elliptic eigenvalue problem, for every value of the parameter λ, by using suitable variational techniques (see [15] for related topics).

A similar approach is adopted by Servadei in [16]. More precisely, in this paper, the author consider the following variational inequality involving a self-adjoint, second-order, uniformly elliptic operator A without lower-order terms:

Find $u \in K$ such that

$$\langle Av, v - u \rangle - \lambda \int_\Omega u(x)(v - u)(x)\, dx \ge \int_\Omega p(x) f(u(x))(v - u)(x)\, dx$$

for all $v \in K$,

where λ is a positive parameter, $p \in L^\infty(\Omega)$ vanishes only on a set of measure zero and positive somewhere on Ω, f is a continuous function satisfying certain superlinear and subcritical growth conditions at zero and at infinity, and K is a suitable subset of $H^1_0(\Omega)$.

If A is the Laplacian and if f has a special form, then this problem is well understood, and the behavior of the solution depends on the value of λ compared to the first eigenvalue of the Laplacian. In the cited paper, the author extends these results to a general setting. In particular, if λ_1 denotes the first eigenvalue of A, then there is a nontrivial nonnegative solution if $\lambda < \lambda_1$. The method of proof is based on the Mountain Pass Lemma. Further, the Linking Theorem was applied in the case $\lambda \ge \lambda_1$ (see [13] for related applications).

Motivated by this large interest in elliptic equations with operators in divergence form, the aim of this paper is to establish, by using variational arguments (see Theorem 2.1), a precise interval of values of the parameter λ for which problem (D_λ^f) admits at least three weak solutions, without explicit perturbations of the nonlinear term, even in the higher dimensional setting (see Theorem 3.5).

A direct consequence of Theorem 3.5 is presented in Corollary 3.9. More precisely, in this case the existence of at least two weak nontrivial solutions for problem (D_λ^f)

is obtained for sufficiently large λ. A special case of Corollary 3.9 is Corollary 3.10, where the classical autonomous setting is stated (see Example 3.11).

The plan of the paper is as follows. Section 2 is devoted to our abstract framework, while Section 3 is dedicated to the main results. A concrete example of an application is then presented (see Example 3.12). Finally, we cite a recent monograph by Kristály, Rădulescu and Varga [11] as a general reference on variational methods adopted here.

2. Abstract framework

Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, $p > 1$, and denote by:

- ∘ X the Sobolev space $W_0^{1,p}(\Omega)$ endowed by the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p};$$

- ∘ X^* the topological dual of X;
- ∘ $\langle \cdot, \cdot \rangle$ the duality brackets for the pair (X^*, X).

In addition, as customary, the symbol

$$p^* := \begin{cases} pN/(N-p) & \text{if} \quad 1 < p < N, \\ \infty & \text{if} \quad p \geq N, \end{cases}$$

denotes the critical Sobolev exponent of p. Fixing $q \in [1, p^*]$, by the Sobolev embedding theorem, there exists a positive constant c_q such that

$$(1) \qquad \|u\|_{L^q(\Omega)} \leq c_q \|u\|, \quad u \in X,$$

and, in particular, the embedding $X \hookrightarrow L^q(\Omega)$ is compact for every $q \in [1, p^*[$. Moreover, if $1 < p < N$, the best constant c_{p^*} is given by

$$(2) \qquad c_{p^*} = \frac{1}{N\sqrt{\pi}} \left(\frac{N! \Gamma\left(\dfrac{N}{2}\right)}{2\Gamma\left(\dfrac{N}{p}\right) \Gamma\left(N + 1 - \dfrac{N}{p}\right)} \right)^{1/N} \eta^{1-1/p},$$

where

$$\eta := \frac{N(p-1)}{N-p},$$

see, for instance, [17]. If $p > N$, let

$$\sup\left\{ \frac{\max_{x \in \overline{\Omega}} |u(x)|}{\|u\|} : u \in W_0^{1,p}(\Omega), u \neq 0 \right\} < +\infty.$$

It is well-known [17, formula (6b)] that by putting

$$(3) \qquad m := \frac{N^{-\frac{1}{p}}}{\sqrt{\pi}} \left[\Gamma\left(1 + \frac{N}{2}\right) \right]^{\frac{1}{N}} \left(\frac{p-1}{p-N} \right)^{1-\frac{1}{p}} (\text{meas}(\Omega))^{\frac{1}{N} - \frac{1}{p}},$$

one has

$$(4) \qquad \|u\|_\infty := \max_{x \in \overline{\Omega}} |u(x)| \leq m\|u\|,$$

for every $u \in X$ (equality occurs when Ω is a ball). Here Γ is the Gamma function defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1} e^{-z} \, dz, \quad \forall \, t > 0,$$

and "meas(Ω)" denotes the usual Lebesgue measure of Ω. Moreover, let

$$\tau := \sup_{x \in \Omega} \operatorname{dist}(x, \partial\Omega).$$

Simple calculations show that there is $x_0 \in \Omega$ such that $B(x_0, \tau) \subseteq \Omega$, where $B(x_0, \tau)$ is the open ball of radius τ centered at the point x_0. We also denote by

$$\omega_s := s^N \frac{\pi^{N/2}}{\Gamma\left(1 + \dfrac{N}{2}\right)},$$

the measure of the N-dimensional ball of radius $s > 0$. At this point, for $\delta > 0$, let $u_\delta \in X$ be the following function

$$u_\delta(x) := \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, \tau), \\ \dfrac{2\delta}{\tau} \left(\tau - |x - x_0|\right) & \text{if } x \in B(x_0, \tau) \setminus B(x_0, \tau/2), \\ \delta & \text{if } x \in B(x_0, \tau/2), \end{cases}$$

that will be useful in the sequel in the proof of our theorems. One has that

$$\|u_\delta\|^p = \int_\Omega |\nabla u_\delta(x)|^p \, dx = \frac{2^p \delta^p \omega_\tau}{\tau^p} \left(1 - \frac{1}{2^N}\right).$$

Indeed,

$$\begin{aligned} \int_\Omega |\nabla u_\delta(x)|^p \, dx &= \delta^p \int_{B(x_0,\tau) \setminus B(x_0,\tau/2)} \frac{2^p}{\tau^p} \, dx \\ &= \frac{2^p \delta^p}{\tau^p} (\operatorname{meas}(B(x_0, \tau)) - \operatorname{meas}(B(x_0, \tau/2))) \\ &= \frac{2^p \delta^p \omega_\tau}{\tau^p} \left(1 - \frac{1}{2^N}\right), \end{aligned}$$

where, from now on, "meas($B(x_0, s)$)" for $s > 0$ stands for the Lebesgue measure of the open ball $B(x_0, s)$.

As pointed out before, our approach is variational. More precisely, the main tool will be the following abstract critical point theorem for smooth functions that can be derived from [2, Theorem 3.6].

Theorem 2.1. *Let X be a reflexive real Banach space and let $\Phi, \Psi \colon X \to \mathbf{R}$ be two C^1-functionals with $\Phi(0_X) = \Psi(0_X) = 0$ and such that:*

(a_1) *Φ is coercive and sequentially weakly lower semicontinuous;*
(a_2) *$\Psi \colon X \to \mathbf{R}$ is sequentially weakly upper semicontinuous.*

For every $\lambda > 0$, put $J_\lambda := \Phi - \lambda\Psi$ and assume that the following conditions are satisfied:

(a_3) *There exist $r > 0$ and $\bar{u} \in X$, with $r < \Phi(\bar{u})$, such that*

$$\frac{\displaystyle \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r} < \frac{\Psi(\bar{u})}{\Phi(\bar{u})};$$

(a$_4$) *For each* $\lambda \in \Lambda_r := \left] \dfrac{\Phi(\bar{u})}{\Psi(\bar{u})}, \dfrac{r}{\sup\limits_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)} \right[$ *the functional* J_λ *is bounded from below and fulfills* (PS)$_\mu$, *with* $\mu \in \mathbf{R}$.

Then for each $\lambda \in \Lambda_r$, *the functional* J_λ *has at least three distinct critical points in* X.

For the sake of completeness, we also recall that the C^1-functional $J_\lambda \colon X \to \mathbf{R}$ satisfies the Palais–Smale condition at level $\mu \in \mathbf{R}$ when:

(PS)$_\mu$ *Every sequence* $\{u_n\} \subset X$ *such that*

$$J_\lambda(u_n) \to \mu \quad and \quad \|J_\lambda'(u_n)\|_{X^*} \to 0,$$

as $n \to \infty$, *possesses a convergent subsequence in* X.

3. Main results

In the sequel, let $p > 1$ and let $\Omega \subset \mathbf{R}^N$ be a bounded Euclidean domain, where $N \geq 2$. Further, let $A \colon \bar{\Omega} \times \mathbf{R}^N \to \mathbf{R}$, and let $A = A(x, \xi)$ be a continuous function in $\bar{\Omega} \times \mathbf{R}^N$, with continuous gradient $a(x, \xi) := \nabla_\xi A(x, \xi) \colon \bar{\Omega} \times \mathbf{R}^N \to \mathbf{R}^N$, and assume that the following conditions hold:

(α_1) $A(x, 0) = 0$ *for all* $x \in \Omega$;

(α_2) a *satisfies the growth condition* $|a(x, \xi)| \leq c(1 + |\xi|^{p-1})$ *for all* $x \in \Omega$, $\xi \in \mathbf{R}^N$, $c > 0$;

(α_3) A *is* p-*uniformly convex, that is*

$$A\left(x, \frac{\xi + \eta}{2}\right) \leq \frac{1}{2}A(x, \xi) + \frac{1}{2}A(x, \eta) - k|\xi - \eta|^p,$$

for every $x \in \bar{\Omega}$, $\xi, \eta \in \mathbf{R}^N$ *and some* $k > 0$;

(α_4) A *is* p-*subhomogeneous, i.e.,* $0 \leq a(x, \xi) \cdot \xi \leq pA(x, \xi)$ *for all* $x \in \bar{\Omega}$, $\xi \in \mathbf{R}^N$;

(α_5) A *satisfies* $\Lambda_1|\xi|^p \leq A(x, \xi) \leq \Lambda_2|\xi|^p$ *for all* $x \in \bar{\Omega}$, $\xi \in \mathbf{R}^N$, *where* Λ_1 *and* Λ_2 *are positive constants.*

Example 3.1. Note that by choosing

$$A(x, \xi) := \frac{|\xi|^p}{p},$$

with $p \geq 2$, we have the usual p-Laplacian operator. See also Remark 3.7 for details on this special case.

Remark 3.2. Define $A^{|\vee|} \colon \bar{\Omega} \times \mathbf{R} \to \mathbf{R}$ as follows,

$$A^{|\vee|}(x, t) := \sup_{|\xi| = t} A(x, \xi), \quad \forall\, x \in \bar{\Omega}.$$

For every $\varepsilon, b \in (0, 1)$ and $x \in \bar{\Omega}$, let us put $E_{\varepsilon, b}(x)$ to be the set

$$\left\{ (\xi, \eta) \in \mathbf{R}^N \times \mathbf{R}^N \colon A\left(x, \frac{\xi - \eta}{2}\right) \geq \frac{1}{2}\max\left\{A(x, \varepsilon\xi), A(x, \varepsilon\eta)\right\}, \right.$$

$$\left. A\left(x, \frac{\xi + \eta}{2}\right) > (1 - b)\frac{A(x, \xi) + A(x, \eta)}{2} \right\},$$

and

$$q_{\varepsilon,b}(x) := \sup \left\{ \frac{|\xi - \eta|}{2} : (\xi, \eta) \in E_{\varepsilon,b}(x) \right\}.$$

It is known that the map A is said to be *uniformly convex* if it satisfies

$$\lim_{b \to 0} \int_\Omega A^{|\vee|}(x, q_{\varepsilon,b}(x))\, dx = 0, \quad \text{for every } \varepsilon \in (0, 1).$$

As pointed out in [9, Remark 2.2], we just observe that our hypotheses (α_1)–(α_5) imply that A is an uniformly convex operator.

Remark 3.3. Integrating (α_2) we deduce that the map A satisfies the growth condition:

$$A(x, \xi) \le c(|\xi| + |\xi|^p),$$

for every $x \in \bar\Omega$ and $\xi \in \mathbf{R}^N$. Indeed, we have

$$A(x, \xi) = \int_0^1 \frac{d}{dt} A(x, t\xi)\, dt = \int_0^1 a(x, t\xi) \cdot \xi\, dt.$$

Hence, we deduce that

$$A(x, \xi) \le c \int_0^1 (1 + |\xi|^{p-1} t^{p-1}) |\xi|\, dt \le c|\xi| + \frac{c}{p}|\xi|^p \le c(|\xi| + |\xi|^p),$$

for every $x \in \bar\Omega$ and $\xi \in \mathbf{R}^N$. Moreover, we also note that by condition (α_3), the functional $\Phi \colon X \to \mathbf{R}$ defined by

$$u \mapsto \int_\Omega A(x, \nabla u(x))\, dx,$$

is a locally uniformly convex operator. Taking into account the above facts, its Gâteaux derivative $\Phi' \colon X \to X^*$ satisfies the (S_+) condition; see [7, Proposition 2.1]:

(S_+) *For every sequence $\{u_n\} \subset X$ such that $u_n \rightharpoonup u$ (weakly) in X and*

$$\limsup_{n \to \infty} \int_\Omega a(x, \nabla u_n(x)) \cdot \nabla(u_n - u)(x)\, dx \le 0,$$

it follows that $u_n \to u$ (strongly) in X.

From now on, we say that a continuous function $f \colon \mathbf{R} \to \mathbf{R}$ belongs to the class $\mathfrak{F}_q$ if

$$(5) \qquad\qquad |f(t)| \le a_1 + a_2 |t|^{q-1}, \quad (\forall\, t \in \mathbf{R})$$

for some nonnegative constants a_1, a_2, where $q \in\,]1, pN/(N - p)[$ if $p < N$ and $1 < q < +\infty$ if $p \ge N$.

Let $\Psi \colon X \to \mathbf{R}$ be defined

$$\Psi(u) := \int_\Omega k(x) F(u(x))\, dx,$$

where $k \colon \bar\Omega \to \mathbf{R}$ is a positive and continuous function, and

$$F(s) := \int_0^s f(t)\, dt,$$

for every $s \in \mathbf{R}$.

Fixing $\lambda > 0$, we say that $u \in X$ is a *weak solution* of problem (D_λ^f) if

$$\int_\Omega a(x, \nabla u(x)) \cdot \nabla v(x)\, dx = \lambda \int_\Omega k(x) f(u(x)) v(x)\, dx,$$

for every $v \in X$.

In the sequel we look at the existence of weak solutions of problem (D_λ^f) finding critical points for a suitable associated energy functional J_λ.

Lemma 3.4. *Let us assume that $f \in \mathfrak{F}_q$. Then the functionals Φ and Ψ are respectively sequentially weakly lower and upper semicontinuous. Hence in particular, for every $\lambda \in \mathbf{R}^+$, the functional $J_\lambda \colon X \to \mathbf{R}$ given by*

$$J_\lambda := \Phi - \lambda\Psi,$$

is sequentially weakly lower semicontinuous.

Proof. The functional Φ being locally uniformly convex, is weakly lower semicontionous. On the other hand, since $f \in \mathfrak{F}_q$ and $k \in C^0(\bar{\Omega})$, we have $|k(x)f(t)| \leq \|k\|_\infty (a_1 + a_2|t|^{q-1})$, for every $(x,t) \in \bar{\Omega} \times \mathbf{R}$. Finally, due to the fact that the embedding $X \hookrightarrow L^q(\Omega)$ is compact, we obtain that Ψ is sequentially weakly (upper) semicontinuous in the standard way. $\square$

Set

$$\kappa := \left(\frac{2^{N-p}}{\omega_\tau (2^N - 1)\Lambda_1} \right)^{1/p} \tau,$$

and

$$G_1 := \left(\frac{2^p(2^N - 1)\Lambda_2}{\tau^p \min\limits_{x \in \bar{\Omega}} k(x)} \right) \frac{\|k\|_\infty c_1}{\Lambda_1^{1/p}},$$

as well as

$$G_2 := \left(\frac{2^p(2^N - 1)\Lambda_2}{\tau^p \min\limits_{x \in \bar{\Omega}} k(x)} \right) \frac{\|k\|_\infty c_q^q}{q\Lambda_1^{q/p}}.$$

Our main result is as follows.

Theorem 3.5. *Let $f \in \mathfrak{F}_q$ and assume that*

(h_0) *$F(s) \geq 0$ for every $s \in \mathbf{R}^+$,*

(h_1) $\lim\limits_{|t| \to \infty} \dfrac{f(t)}{|t|^{p-1}} = 0,$

(h_2) *there are positive constants γ and δ, with $\delta > \gamma\kappa$, such that*

$$\frac{F(\delta)}{\delta^p} > a_1 \frac{G_1}{\gamma^{p-1}} + a_2 G_2 \gamma^{q-p}.$$

Then for each parameter λ belonging to

$$\Lambda_{(\gamma,\delta)} := \frac{2^p(2^N-1)\Lambda_2}{\tau^p \min\limits_{x \in \bar{\Omega}} k(x)} \left] \frac{\delta^p}{F(\delta)}, \frac{1}{\left(a_1 \dfrac{G_1}{\gamma^{p-1}} + a_2 G_2 \gamma^{q-p} \right)} \right[,$$

problem (D_λ^f) possesses at least three weak solutions.

Proof. Our aim is to apply Theorem 2.1. Let $X := W_0^{1,p}(\Omega)$ and consider the functionals $\Phi, \Psi \colon X \to \mathbf{R}$ defined before. Clearly, $\Phi \colon X \to \mathbf{R}$ is coercive since, by condition (α_5), it follows that

$$\Phi(u) \geq \Lambda_1 \|u\|^p \to +\infty,$$

when $\|u\| \to \infty$. Moreover, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional (by Lemma 3.4). On the other hand, Ψ is well-defined, continuously Gâteaux differentiable and, again by Lemma 3.4, a weakly upper semicontinuous functional. More precisely, one has

$$\Phi'(u)(v) = \int_\Omega a(x, \nabla u(x)) \cdot \nabla v(x) \, dx,$$

$$\Psi'(u)(v) = \int_\Omega k(x) f(u(x)) v(x) \, dx,$$

for every $u, v \in X$. Hence conditions (a_1) and (a_2) of Theorem 2.1 are satisfied. Now, fix $\lambda > 0$. A critical point of the functional $J_\lambda := \Phi - \lambda \Psi$ is a function $u \in X$ such that

$$\Phi'(u)(v) - \lambda \Psi'(u)(v) = 0,$$

for every $v \in X$. Hence, the critical points of the functional J_λ are weak solutions of problem (D_λ^f). Moreover, $\Phi(0_X) = \Psi(0_X) = 0$. Since $f \in \mathfrak{F}_q$, one has that

$$(6) \qquad\qquad F(s) \leq a_1 |s| + a_2 \frac{|s|^q}{q},$$

for every $s \in \mathbf{R}$.

Let $r \in]0, +\infty[$ and consider the function

$$\chi(r) := \frac{\displaystyle \sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u)}{r}.$$

Taking into account (6), it follows that

$$\Psi(u) = \int_\Omega k(x) F(u(x)) \, dx \leq \|k\|_\infty \left(a_1 \|u\|_{L^1(\Omega)} + \frac{a_2}{q} \|u\|_{L^q(\Omega)}^q \right).$$

Then let $u \in X$ be such that $\Phi(u) \leq r$, that is

$$\int_\Omega A(x, \nabla u(x)) \, dx \leq r.$$

Hence the above relation together with condition (α_5) implies that

$$\|u\| \leq \frac{r^{1/p}}{\Lambda_1^{1/p}}.$$

In other words, one has the following algebraic inclusion

$$\left\{ u \in X \colon \int_\Omega A(x, \nabla u(x)) \, dx \leq r \right\} \subseteq \left\{ u \in X \colon \|u\| \leq \frac{r^{1/p}}{\Lambda_1^{1/p}} \right\}.$$

Consequently, by the Sobolev inequalities (1), one has

$$\Psi(u) \leq \|k\|_\infty \left(\frac{c_1 a_1 r^{1/p}}{\Lambda_1^{1/p}} + \frac{c_q^q a_2 r^{q/p}}{q \Lambda_1^{q/p}} \right),$$

for every $u \in \Phi^{-1}(]-\infty, r])$. Hence,

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) \le \|k\|_\infty \left(\frac{c_1 a_1 r^{1/p}}{\Lambda_1^{1/p}} + \frac{c_q^q a_2 r^{q/p}}{q \Lambda_1^{q/p}} \right).$$

By the above inequality, the following relation holds

$$(7) \qquad\qquad \chi(r) \le \|k\|_\infty L(r),$$

for every $r > 0$, where we set

$$L(r) := \frac{c_1 a_1 r^{1/p-1}}{\Lambda_1^{1/p}} + \frac{c_q^q a_2 r^{q/p-1}}{q \Lambda_1^{q/p}}.$$

Next, let u_δ be the function defined in Section 2. Clearly $u_\delta \in X$ and, since (α_5) holds, we have

$$(8) \qquad\qquad \Phi(u_\delta) = \int_\Omega A(x, \nabla u_\delta(x)) \, dx \le \Lambda_2 \left(\frac{2^p \delta^p \omega_\tau}{\tau^p} \left(1 - \frac{1}{2^N} \right) \right).$$

Taking into account that $\delta > \gamma\kappa$, by a direct computation, one has $\gamma^p < \Phi(u_\delta)$. Moreover from (h_0) and taking into account that the map k is positive and continuous in $\bar\Omega$, we easily derive

$$(9) \qquad\qquad \int_\Omega k(x) F(u_\delta(x)) \, dx \ge F(\delta) \frac{\pi^{N/2}}{\Gamma(1+N/2)} \frac{\tau^N}{2^N} \min_{x \in \bar\Omega} k(x).$$

Hence, by (8) and (9), one has

$$(10) \qquad\qquad \frac{\Psi(u_\delta)}{\Phi(u_\delta)} \ge \left(\frac{\tau^p \min\limits_{x \in \bar\Omega} k(x)}{2^p (2^N - 1) \Lambda_2} \right) \frac{F(\delta)}{\delta^p}.$$

In view of (h_2) and taking into account (7) and (10), we get

$$\chi(\gamma^p) = \frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, \gamma^p])} \Psi(u)}{\gamma^p} \le \|k\|_\infty L(\gamma^p) = \frac{\tau^p \min\limits_{x \in \bar\Omega} k(x)}{2^p (2^N - 1) \Lambda_2} \left(a_1 \frac{G_1}{\gamma^{p-1}} + a_2 G_2 \gamma^{q-p} \right)$$

$$< \left(\frac{\tau^p \min\limits_{x \in \bar\Omega} k(x)}{2^p (2^N - 1) \Lambda_2} \right) \frac{F(\delta)}{\delta^p} \le \frac{\Psi(u_\delta)}{\Phi(u_\delta)}.$$

Therefore, assumption (a_3) of Theorem 2.1 is satisfied.

Let us fix $\lambda > 0$. By condition (h_1), there exists δ_λ such that

$$|f(t)| \le \frac{\Lambda_1}{c_p^p (1 + \lambda) \|k\|_\infty} |t|^{p-1},$$

for every $|t| \ge \delta_\lambda$. By integration we get

$$|F(s)| \le \frac{\Lambda_1}{c_p^p (1 + \lambda) \|k\|_\infty} |s|^p + \max_{|t| \le \delta_\lambda} |f(t)| |s|,$$

for every $s \in \mathbf{R}$. Thus

$$J_\lambda(u) \ge \Phi(u) - \lambda |\Psi(u)|$$

$$\ge \Lambda_1 \|u\|^p - \frac{\Lambda_1 \lambda}{(1 + \lambda)} \|u\|^p - c_1 \|k\|_\infty \lambda \max_{|t| \le \delta_\lambda} |f(t)| \|u\|.$$

Then the functional J_λ is bounded from below and, since $p > 1$, $J_\lambda(u) \to +\infty$ whenever $\|u\| \to +\infty$. Hence J_λ is coercive. Now, let us prove that J_λ satisfies the condition $(PS)_\mu$ for $\mu \in \mathbf{R}$. For our goal, let $\{u_n\} \subset X$ be a Palais-Smale sequence, i.e.

$$J_\lambda(u_n) \to \mu, \quad \text{and} \quad \|J'_\lambda(u_n)\|_{X^*} \to 0.$$

Taking into account the coercivity of J_λ, the sequence $\{u_n\}$ is necessarily bounded in X. Since X is reflexive, we may extract a subsequence, which for simplicity we call again $\{u_n\}$, such that $u_n \rightharpoonup u$ in X. We will prove that $\{u_n\}$ strongly converges to $u \in X$. Exploiting the derivative $J'_\lambda(u_n)(u_n - u)$, we obtain

$$\int_\Omega a(x, \nabla u_n(x)) \cdot \nabla(u_n - u)(x)\,dx$$

$$= \langle J'_\lambda(u_n), u_n - u \rangle + \lambda \int_\Omega k(x) f(u_n(x))(u_n - u)(x)\,dx.$$

Since $\|J'_\lambda(u_n)\|_{X^*} \to 0$ and the sequence $\{u_n - u\}$ is bounded in X, taking into account that $|\langle J'_\lambda(u_n), u_n - u \rangle| \le \|J'_\lambda(u_n)\|_{X^*}\|u_n - u\|$, one has

$$\langle J'_\lambda(u_n), u_n - u \rangle \to 0.$$

Further, by the asymptotic condition (h_1), there exists a real positive constant c such that $|f(t)| \le c(1 + |t|^{p-1})$, for every $t \in \mathbf{R}$. Then

$$\int_\Omega k(x)|f(u_n(x))||u_n(x) - u(x)|\,dx$$

$$\le \sigma \int_\Omega |u_n(x) - u(x)|\,dx + \sigma \int_\Omega |u_n(x)|^{p-1}|u_n(x) - u(x)|\,dx$$

$$\le \sigma((\text{meas}(\Omega))^{1/p'} + \|u_n\|_{L^p(\Omega)}^{p-1})\|u_n - u\|_{L^p(\Omega)},$$

where $\sigma := c\|k\|_\infty$ and p' is the conjugate exponent of p.

Now, the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, hence $u_n \to u$ strongly in $L^p(\Omega)$. So we obtain

$$\int_\Omega k(x)|f(u_n(x))||u_n(x) - u(x)|\,dx \to 0.$$

We may conclude

$$\limsup_{n\to\infty}\langle a(x, u_n), u_n - u \rangle = 0,$$

where $\langle a(x, u_n), u_n - u \rangle$ denotes

$$\int_\Omega a(x, \nabla u_n(x)) \cdot \nabla(u_n - u)(x)\,dx.$$

But as observed in Remark 3.3, the operator Φ' has the (S_+) property. So, in conclusion, $u_n \to u$ strongly in X. Hence, J_λ is bounded from below and fulfills $(PS)_\mu$ (with $\mu \in \mathbf{R}$), for every positive parameter, in particular, for every

$$\lambda \in \Lambda_{(\gamma, \delta)} \subseteq \left] \frac{\Phi(u_\delta)}{\Psi(u_\delta)}, \frac{\gamma^p}{\displaystyle\sup_{u \in \Phi^{-1}(]-\infty, \gamma^p])} \Psi(u)} \right[.$$

Then also condition (a_4) holds. Hence all the assumptions of Theorem 2.1 are satisfied. Consequently, for each $\lambda \in \Lambda_{(\gamma,\delta)}$, the functional J_λ has at least three distinct critical points that are weak solutions of the problem (D_λ^f). $\qquad\square$

Remark 3.6. A careful analysis of the proof of Theorem 3.5 shows that condition (h_0) can be replaced by a more general sign condition on the potential:

(h_0') $F(s) \geq 0$, *for every* $s \in \,]0, \delta[$,

where δ is the constant that appears in hypothesis (h_2).

Remark 3.7. If

$$a(x, \nabla u) := |\nabla u|^{p-2}\nabla u, \quad p \geq 2,$$

for every $x \in \bar{\Omega}$, the geometrical constants in the main result assume simpler expressions

$$G_1 := \left(\frac{2^p(2^N - 1)}{\tau^p \min\limits_{x\in\bar\Omega} k(x)}\right)\frac{\|k\|_\infty c_1}{p^{\frac{p-1}{p}}}, \quad G_2 := \left(\frac{2^p(2^N - 1)}{\tau^p \min\limits_{x\in\bar\Omega} k(x)}\right)\frac{\|k\|_\infty c_q^q}{qp^{\frac{p-q}{p}}}.$$

In this setting, under the same hypotheses of Theorem 3.5, for each parameter λ belonging to

$$\widetilde{\Lambda}_{(\gamma,\delta)} := \frac{2^p(2^N - 1)}{p\tau^p \min\limits_{x\in\bar\Omega} k(x)} \left]\frac{\delta^p}{F(\delta)}, \frac{1}{\left(a_1\dfrac{G_1}{\gamma^{p-1}} + a_2 G_2 \gamma^{q-p}\right)}\right[,$$

the following problem

$$(\widehat{D}_\lambda^f) \qquad\qquad \begin{cases} -\Delta_p u = \lambda k(x) f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_p := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator, possesses at least three weak solutions. The case $p = 2$ has been studied in [3, Theorem 3.1] by using a similar variational approach. See also the work [6] for the Neumann setting.

Remark 3.8. An explicit upper bound for the constants c_q in Theorem 3.5 can be easily obtained. Indeed, let $1 < p < N$ and fix $q \in [1, p^*[$. By formula (2) one has

$$c_q \leq \frac{\operatorname{meas}(\Omega)^{\frac{p^*-q}{p^*q}}}{N\sqrt{\pi}}\left(\frac{N!\,\Gamma\left(\dfrac{N}{2}\right)}{2\Gamma\left(\dfrac{N}{p}\right)\Gamma\left(N + 1 - \dfrac{N}{p}\right)}\right)^{1/N}\eta^{1-1/p},$$

where

$$\eta := \frac{N(p-1)}{N-p}.$$

On the other hand, if $p > N$, due to inequality (4), we also have

$$c_q \leq \frac{N^{-\frac{1}{p}}}{\sqrt{\pi}}\left[\Gamma\left(1 + \frac{N}{2}\right)\right]^{\frac{1}{N}}\left(\frac{p-1}{p-N}\right)^{1-\frac{1}{p}}(\operatorname{meas}(\Omega))^{\frac{1}{N} + \frac{p-q}{qp}}.$$

We also have the following multiplicity result that can be proved by Theorem 3.5.

Corollary 3.9. *Assume that $f: \mathbf{R} \to \mathbf{R}$ is a continuous and nonnegative function such that*

$$F(s) \le a_2 s^{q-p}, \quad (\forall\, s \in \mathbf{R}^+)$$

for some $a_2 > 0$ and $p < q < p^*$. In addition, suppose that condition (h_1) holds. Then for every

$$\lambda > \frac{2^p(2^N - 1)\Lambda_2}{\tau^p \min\limits_{x \in \bar{\Omega}} k(x)} \inf_{\delta > 0} \frac{\delta^p}{F(\delta)},$$

problem (D_λ^f) has at least two distinct, nontrivial weak solutions.

Proof. Since f is a nonnegative function, hypothesis (h_0) clearly holds. Now, fixing $\bar{\lambda}$ as in the conclusion, there exists $\bar{\delta} > 0$ such that

$$\bar{\lambda} > \left(\frac{2^p(2^N - 1)\Lambda_2}{\tau^p \min\limits_{x \in \bar{\Omega}} k(x)} \right) \frac{\bar{\delta}^p}{F(\bar{\delta})}.$$

Moreover, picking a positive constant $\bar{\gamma}$, with

$$\bar{\gamma} < \min \left\{ \frac{\bar{\delta}}{\kappa}, \left(\frac{2^p(2^N - 1)\Lambda_2}{\lambda a_2 \tau^p G_2 \min\limits_{x \in \bar{\Omega}} k(x)} \right)^{\frac{1}{q-p}} \right\},$$

condition (h_2) is verified and, since

$$\bar{\lambda} \in \frac{2^p(2^N - 1)\Lambda_2}{\tau^p \min\limits_{x \in \bar{\Omega}} k(x)} \left] \frac{\bar{\delta}^p}{F(\bar{\delta})}, \frac{1}{a_2 G_2 \bar{\gamma}^{q-p}} \right[,$$

Theorem 3.5 ensures that problem (D_λ^f) admits at least two nontrivial weak solutions. $\square$

The next is a special case of the above result.

Corollary 3.10. *Let $2 \le p < N$ and assume that $f: \mathbf{R} \to \mathbf{R}$ is a continuous and nonnegative function that is $(p-1)$-sublinear at infinity, i.e.*

$$\lim_{|t| \to \infty} \frac{f(t)}{|t|^{p-1}} = 0.$$

Suppose that

$$F(s) \le a_2 s^{q-p}, \quad (\forall\, s \in \mathbf{R}^+)$$

for some $a_2 > 0$ and $p < q < pN/(N-p)$. Then for every

$$\lambda > \frac{2^p(2^N - 1)}{p\tau^p} \inf_{\delta > 0} \frac{\delta^p}{\displaystyle\int_0^\delta f(t)\,dt},$$

the following Dirichlet problem

$$(\widehat{D}_\lambda^f) \qquad\qquad \begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least two distinct, nontrivial weak solutions in $W_0^{1,p}(\Omega)$.

Example 3.11. Let $\Omega \subset \mathbf{R}^5$ be a domain with smooth boundary. Taking $p = 4$, an example of map that satisfies all the assumptions of Corollary 3.10 is the nonnegative function $g\colon \mathbf{R} \to \mathbf{R}$ given by $g(t) := \log(1 + t^4)$. Indeed,

$$\lim_{|t|\to\infty} \frac{\log(1 + t^4)}{|t|^3} = 0,$$

and its potential

$$G(s) = \sqrt{2}\arctan(\sqrt{2}s - 1) + \sqrt{2}\arctan(\sqrt{2}s + 1)$$
$$- \frac{\sqrt{2}}{2}\log\left(\frac{s^2 - \sqrt{2}s + 1}{s^2 + \sqrt{2}s + 1}\right) + s\log(1 + s^4) - 4s,$$

satisfies $G(s) \leq s^2$, for every $s \in \mathbf{R}$. Then, for every

$$\lambda > \frac{124}{\tau^4} \inf_{\delta > 0} \frac{\delta^4}{\displaystyle\int_0^\delta g(t)\,dt} \approx \frac{752}{\tau^4},$$

the following Dirichlet problem

$$(\widehat{D}_\lambda^g) \qquad\qquad \begin{cases} -\Delta_4 u = \lambda g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least two distinct, nontrivial weak solutions in $W_0^{1,4}(\Omega)$.

In conclusion we present an application that is a direct consequence of Theorem 3.5.

Example 3.12. Let $(a^{ij}) \in C^0(\bar{\Omega}; \mathbf{R}^{N \times N})$ be a positive definite matrix such that:

(β_1) $a^{ij}(x) = a^{ji}(x)$, for every $x \in \bar{\Omega}$;
(β_2) There are positive constants Λ_1 and Λ_2 for which

$$\Lambda_1|\xi|^2 \leq \frac{1}{2}\sum_{i,j=1}^N a^{ij}(x)\xi_i\xi_j \leq \Lambda_2|\xi|^2, \quad \forall x \in \bar{\Omega},\ \xi \in \mathbf{R}^N.$$

Put

$$A(x, \xi) := \frac{1}{2}\sum_{i,j=1}^N a^{i,j}(x)\xi_i\xi_j, \quad \forall\,(x, \xi) \in \bar{\Omega} \times \mathbf{R}^N,$$

and consider the elliptic operator in divergent form

$$L[u] = \sum_{i,j=1}^N \frac{\partial}{\partial x_j}\left(a^{ij}(x)\frac{\partial u}{\partial x_i}\right).$$

As observed by De Nápoli and Mariani in [7], the map A satisfies assumptions (α_1)–(α_5). At this point, let Ω be a nonempty bounded open subset of the Euclidean space $(\mathbf{R}^N, |\cdot|)$ with boundary of class C^1, and let $k\colon \bar{\Omega} \to \mathbf{R}$ be a positive continuous

function. Further, let us fix $q \in]2, 2^*[$ and consider the function $h \colon \mathbf{R} \to \mathbf{R}$ defined by

$$h(t) := \begin{cases} 1 + |t|^{q-1} & \text{if } |t| \leq r, \\ \dfrac{(1+r^2)(1+r^{q-1})}{1+t^2} & \text{if } |t| > r, \end{cases}$$

where r is a fixed constant such that

$$(11) \qquad\qquad r > \max\left\{\kappa, q^{\frac{1}{q-2}}(G_1 + G_2)^{\frac{1}{q-2}}\right\}.$$

Clearly $h(0) \neq 0$ and $h(t) \leq (1 + |t|^{q-1})$ for every $t \in \mathbf{R}$. Hence, condition (5) is satisfied for $a_1 = a_2 = 1$. Moreover, also condition (h_1) is verified since $\lim\limits_{|t| \to +\infty} h(t)/|t| = 0$. Finally, owing to (11), it follows that

$$G_1 + G_2 < \frac{r^{q-2}}{q}.$$

Therefore,

$$\frac{H(r)}{r^2} = \frac{r^{q-2}}{q} + \frac{1}{r} > G_1 + G_2,$$

where $H(r) := \displaystyle\int_0^r h(t)dt$, and condition ($h_2$) holds choosing $\delta = r$.

Consequently, owing to Theorem 3.5, for each parameter λ belonging to the interval

$$\Lambda_\delta := \left. \frac{4(2^N - 1)\Lambda_2}{\tau^2 \min\limits_{x \in \overline{\Omega}} k(x)} \right] \frac{r^2}{H(r)}, \frac{1}{G_1 + G_2} \left[, \right.$$

the following elliptic Dirichlet problem

$$(\widetilde{D}_\lambda^h) \qquad\qquad \begin{cases} -L[u] = \lambda k(x)h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

possesses at least three nontrivial weak solutions in $H_0^1(\Omega)$.

Remark 3.13. We observe that, to the contrary to the result of Kristály, Lisei and Varga studied [10, Theorem 2.1], in Theorem 3.5, as for instance Example 3.12 shows, we don't require the following behaviour at zero

$$\lim_{t \to 0} \frac{f(t)}{|t|^{p-1}} = 0,$$

that automatically implies $f(0) = 0$.

Moreover, for completeness, we emphasize that our results can be investigated also for different classes of elliptic operators in divergence form looking at the existence of at least three nontrivial weak solutions. See, for instance, the recent and interesting paper of Colasuonno, Pucci, and Varga [5] for related topics.

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