

GENERALIZED HILBERT OPERATORS

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Abstract. If g is an analytic function in the unit disc \mathbf{D} , we consider the generalized Hilbert operator \mathcal{H}_g defined by

$$\mathcal{H}_g(f)(z) = \int_0^1 f(t)g'(tz) dt.$$

We study these operators acting on classical spaces of analytic functions in \mathbf{D} . More precisely, we address the question of characterizing the functions g for which the operator \mathcal{H}_g is bounded (compact) on the Hardy spaces H^p , on the weighted Bergman spaces A_α^p or on the spaces of Dirichlet type \mathcal{D}_α^p .

1. Introduction

1.1. Generalized Hilbert operators. We denote by \mathbf{D} the unit disc in the complex plane \mathbf{C} , and by $\mathcal{H}ol(\mathbf{D})$ the space of all analytic functions in \mathbf{D} .

The Hilbert matrix

$$\mathcal{H} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdot \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdot \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

can be viewed as an operator on spaces of analytic functions, called the *Hilbert operator*, by its action on the Taylor coefficients:

$$a_n \mapsto \sum_{k=0}^{\infty} \frac{a_k}{n+k+1}, \quad n = 0, 1, 2, \dots,$$

that is, if $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(\mathbf{D})$, we define

$$(1.1) \quad \mathcal{H}(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \right) z^n,$$

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whenever the right hand side makes sense and defines an analytic function in \mathbf{D} .

Hardy’s inequality [5, p. 48] guarantees that the transformed power series in (1.1) converges on \mathbf{D} and defines there an analytic function $\mathcal{H}(f)$ whenever $f \in H^1$. In other words, $\mathcal{H}(f)$ is a well defined analytic function for every $f \in H^1$.

It turns out that $\mathcal{H}(f)$ can be written also in the form,

$$\mathcal{H}(f)(z) = \sum_{n=0}^{\infty} \left(\int_0^1 t^n f(t) dt \right) z^n = \int_0^1 f(t) \frac{1}{1-tz} dt,$$

or, equivalently,

$$\mathcal{H}(f)(z) = \int_0^1 f(t)g'(tz) dt, \quad \text{with } g(z) = \log \frac{1}{1-z}.$$

The resulting Hilbert operator \mathcal{H} is bounded from H^p to H^p , whenever $1 < p < \infty$ but \mathcal{H} is not bounded on H^1 [3, Theorem 1.1]. In [4] the norm of \mathcal{H} acting on Hardy spaces was computed. Concerning the Bergman spaces A^p , the operator $\mathcal{H}: A^p \rightarrow A^p$ is bounded if and only if $2 < p < \infty$, [2]. But \mathcal{H} is not even defined in A^2 , for it was shown in [4] that there exist functions $f \in A^2$ such that the series defining $\mathcal{H}(f)(0)$ is divergent. In a recent paper [14] Lanucha, Nowak, and Pavlović have considered the question of finding subspaces of H^1 and of A^2 which are mapped by \mathcal{H} into H^1 and A^2 , respectively, and also the action of the operator \mathcal{H} on the Bloch space and on Besov spaces.

In this article we shall be dealing with certain generalized Hilbert operators. Given $g \in \mathcal{H}ol(\mathbf{D})$, we consider the generalized Hilbert operator \mathcal{H}_g defined by

$$(1.2) \quad \mathcal{H}_g(f)(z) = \int_0^1 f(t)g'(tz) dt.$$

As noted above, $\mathcal{H} = \mathcal{H}_g$ with $g(z) = \log \frac{1}{1-z}$. We mention [8] for a different generalization of the classical Hilbert operator.

The Fejér–Riesz inequality [5, page 46] guarantees that given any $g \in \mathcal{H}ol(\mathbf{D})$, the integral in (1.2) converges absolutely, and therefore the right hand side of (1.2) defines an analytic function on \mathbf{D} , for every $f \in H^1$.

We note that \mathcal{H}_g has a representation in terms of the Taylor coefficients similar to (1.1). Indeed, a simple computation shows that if $g(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{H}ol(\mathbf{D})$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^1$, then

$$\mathcal{H}_g(f)(z) = \sum_{k=0}^{\infty} \left((k+1)b_{k+1} \int_0^1 t^k f(t) dt \right) z^k = \sum_{k=0}^{\infty} \left((k+1)b_{k+1} \sum_{n=0}^{\infty} \frac{a_n}{n+k+1} \right) z^k.$$

Our main objective in this paper is characterizing those functions g for which \mathcal{H}_g is bounded on the Hardy spaces H^p , the Bergman spaces A^p_α and on the the spaces of Dirichlet type \mathcal{D}^p_α ($0 < p < \infty, \alpha > -1$). These results are stated in Section 2.

1.2. Spaces of analytic functions. If $0 < r < 1$ and $f \in \mathcal{H}ol(\mathbf{D})$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{1/p}, \quad 0 < p < \infty,$$

$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

1.2.1. Hardy and Bergman spaces. If $0 < p \leq \infty$, the Hardy space H^p consists of those $f \in \mathcal{H}ol(\mathbf{D})$ such that $\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0 < r < 1} M_p(r, f) < \infty$. Functions f in Hardy spaces have non-tangential boundary values $f(e^{i\theta})$ almost everywhere on the unit circle \mathbf{T} .

If $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A_α^p consists of those $f \in \mathcal{H}ol(\mathbf{D})$ such that

$$\|f\|_{A_\alpha^p} \stackrel{\text{def}}{=} \left((\alpha + 1) \int_{\mathbf{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) \right)^{1/p} < \infty.$$

The unweighted Bergman space A_0^p is simply denoted by A^p . Here, $dA(z) = \frac{1}{\pi} dx dy$ denotes the normalized Lebesgue area measure in \mathbf{D} . For each $p \in (0, \infty)$ the Hardy space H^p is contained in A^{2p} and the exponent $2p$ cannot be improved. We refer to [5] for the theory of Hardy spaces, and to [6], [13] and [20] for Bergman spaces.

1.2.2. Dirichlet type spaces. If $0 < p < \infty$ and $\alpha > -1$, the space of Dirichlet type \mathcal{D}_α^p consists of all indefinite integrals of functions in A_α^p . Hence, if f is analytic in \mathbf{D} , then $f \in \mathcal{D}_\alpha^p$ if and only if

$$\|f\|_{\mathcal{D}_\alpha^p} \stackrel{\text{def}}{=} |f(0)|^p + \|f'\|_{A_\alpha^p}^p < \infty.$$

The space \mathcal{D}_0^2 is the classical Dirichlet space \mathcal{D} and $\mathcal{D}_1^2 = H^2$. For each p , the range of values of the parameter α for which \mathcal{D}_α^p is most interesting is $p - 2 \leq \alpha \leq p - 1$.

If $\alpha > p - 1$, then it is easy to see that $\mathcal{D}_\alpha^p = A_{\alpha-p}^p$. Indeed, this follows from the well known estimate

$$\int_{\mathbf{D}} |f(z)|^p (1 - |z|)^s dA(z) \asymp |f(0)|^p + \int_{\mathbf{D}} |f'(z)|^p (1 - |z|)^{p+s} dA(z),$$

(see, e. g., [7, Theorem 6]). On the other hand, if $\alpha < p - 2$, then $\frac{\alpha+2-p}{p} < 0$ and then it follows easily that $\mathcal{D}_\alpha^p \subset H^\infty$ in this case. For $\alpha = p - 2$ the space \mathcal{D}_{p-2}^p coincides with the Besov space usually denoted by B^p .

For $\alpha = p - 1$ the space \mathcal{D}_{p-1}^p is the closest to the Hardy space H^p but does not coincide with it for $p \neq 2$. If $0 < p \leq 2$, then $\mathcal{D}_{p-1}^p \subset H^p$ [7], and if $2 \leq p < \infty$, then $H^p \subset \mathcal{D}_{p-1}^p$ [15].

1.2.3. Mean Lipschitz spaces. We shall consider also the mean Lipschitz spaces $\Lambda(p, \alpha)$. For $1 \leq p < \infty$ and $0 < \alpha \leq 1$ the space $\Lambda(p, \alpha)$ consists of those $g \in \mathcal{H}ol(\mathbf{D})$ having a non-tangential limit $g(e^{i\theta})$ almost everywhere and such that

$$\omega_p(g, t) = O(t^\alpha), \quad t \rightarrow 0,$$

where

$$\omega_p(g, t) = \sup_{0 < h \leq t} \left(\int_0^{2\pi} |g(e^{i(\theta+h)}) - g(e^{i\theta})|^p \frac{d\theta}{2\pi} \right)^{1/p}$$

is the integral modulus of continuity of order p . A classical result of Hardy and Littlewood [11] (see also Chapter 5 of [5]) asserts that

$$(1.3) \quad \Lambda(p, \alpha) = \left\{ f \in H^p : M_p(r, f') = O((1 - r)^{\alpha-1}) \right\},$$

for $1 \leq p < \infty$, $0 < \alpha \leq 1$. The corresponding ‘‘little oh’’ spaces are denoted by $\lambda(p, \alpha)$.

Among all the mean Lipschitz spaces, the spaces $\Lambda(p, \frac{1}{p})$, $1 < p < \infty$, will play a fundamental role in our work. They form a nested scale of spaces contained in the space $BMOA$ [1]:

$$\Lambda\left(q, \frac{1}{q}\right) \subset \Lambda\left(p, \frac{1}{p}\right) \subset BMOA, \quad 1 \leq q < p < \infty.$$

Furthermore the function $\log(\frac{1}{1-z})$ belongs to $\Lambda\left(p, \frac{1}{p}\right)$ for each $p > 1$.

2. Main results

Our main results regarding Hardy spaces are contained in Theorem 1 and Theorem 2.

Theorem 1. *Suppose that $1 < p \leq 2$ and $g \in \mathcal{H}ol(\mathbf{D})$. Then \mathcal{H}_g is bounded from H^p to H^p if and only if $g \in \Lambda\left(p, \frac{1}{p}\right)$.*

Theorem 2. *Suppose that $2 < p < \infty$ and $g \in \mathcal{H}ol(\mathbf{D})$. We have:*

- (i) *If \mathcal{H}_g is bounded from H^p to H^p , then $g \in \Lambda\left(p, \frac{1}{p}\right)$.*
- (ii) *If $g \in \Lambda\left(q, \frac{1}{q}\right)$ for some q with $1 < q < p$, then \mathcal{H}_g is bounded from H^p to H^p .*

It is natural to ask whether or not the condition $g \in \Lambda\left(p, \frac{1}{p}\right)$ implies that \mathcal{H}_g is bounded from H^p to H^p , for $2 < p < \infty$. We do not know the answer to this question but we conjecture that it is affirmative. The condition $g \in \Lambda\left(q, \frac{1}{q}\right)$ for some q with $1 < q < p$ which appears in (ii) is slightly stronger than that of g belonging to $\Lambda\left(p, \frac{1}{p}\right)$.

Using (1.3) it follows that if $g \in \mathcal{H}ol(\mathbf{D})$ has power series $g(z) = \sum_{k=0}^{\infty} b_k z^k$ with $\sup_{k \in \mathbf{N}} k|b_k| < \infty$, then $g \in \Lambda\left(2, \frac{1}{2}\right)$. Also, using (1.3) and the Littlewood subordination principle, it follows easily that a function $g \in \mathcal{H}ol(\mathbf{D})$ such that $\Re(g'(z)) \geq 0$, for all $z \in \mathbf{D}$, belongs to $\Lambda\left(q, \frac{1}{q}\right)$ for all $q > 1$, a result which readily implies that the same is true for any $g \in \mathcal{H}ol(\mathbf{D})$ which is the Cauchy transform of a finite, complex, Borel measure μ on the circle \mathbf{T} , that is,

$$g(z) = \int_{\mathbf{T}} \frac{d\mu(\zeta)}{1 - \zeta z}.$$

Consequently, it is clear that we have the following.

Corollary 1. *Let \mathcal{K} be the class of those analytic functions in \mathbf{D} which are the Cauchy transform of a finite, complex, Borel measure on \mathbf{T} and let*

$$\mathcal{C} = \left\{ g(z) = \sum_{k=0}^{\infty} b_k z^k \in \mathcal{H}ol(\mathbf{D}) : \sup_{k \in \mathbf{N}} k|b_k| < \infty \right\}.$$

We have:

- (i) *If $2 \leq p < \infty$ and $g \in \mathcal{C}$, then $\mathcal{H}_g: H^p \rightarrow H^p$ is bounded.*
- (ii) *If $1 < p < \infty$ and $g \in \mathcal{K}$, then $\mathcal{H}_g: H^p \rightarrow H^p$ is bounded.*

We note that \mathcal{K} and \mathcal{C} are subclasses of the mentioned mean Lipschitz spaces containing the function $g(z) = \log \frac{1}{1-z}$. Thus, Corollary 1 generalizes the classical result on the boundedness of the Hilbert operator on H^p .

It turns out that $g \in \Lambda \left(p, \frac{1}{p} \right)$ is equivalent to the boundedness of the operator \mathcal{H}_g on the weighted Bergman spaces A_α^p and on the spaces of Dirichlet type \mathcal{D}_α^p for the admissible values of p and α .

Theorem 3. *Suppose that $1 < p < \infty$, $-1 < \alpha < p - 2$ and $g \in \mathcal{H}ol(\mathbf{D})$. Then $\mathcal{H}_g: A_\alpha^p \rightarrow A_\alpha^p$ is bounded if and only if $g \in \Lambda \left(p, \frac{1}{p} \right)$.*

The condition $-1 < \alpha < p - 2$ is not a real restriction. It is needed to insure that any function $f \in A_\alpha^p$ satisfies that $\int_0^1 |f(t)| dt < \infty$, which is necessary for the operator \mathcal{H}_g being well defined on A_α^p . The result does not remain true for $\alpha \geq p - 2$.

Theorem 4. *Suppose that $1 < p < \infty$, $p - 2 < \alpha \leq p - 1$ and $g \in \mathcal{H}ol(\mathbf{D})$. Then $\mathcal{H}_g: \mathcal{D}_\alpha^p \rightarrow \mathcal{D}_\alpha^p$ is bounded if and only if $g \in \Lambda \left(p, \frac{1}{p} \right)$.*

The paper is organized as follows. In Section 3 we state and prove a number of lemmas which will be used specially in Section 4 where we shall prove the necessity parts of our just mentioned results. Section 5 will be devoted to study the *the sublinear Hilbert operator* $\tilde{\mathcal{H}}$ defined by

$$\tilde{\mathcal{H}}(f)(z) = \int_0^1 \frac{|f(t)|}{1-tz} dt.$$

We shall prove that if $g \in \Lambda \left(p, \frac{1}{p} \right)$ and X is either H^p with $1 < p \leq 2$, or A_α^p with $1 < p < \infty$ and $-1 < \alpha < p - 2$, or \mathcal{D}_α^p with $1 < p < \infty$ and $p - 2 < \alpha \leq p - 1$, then

$$\|\mathcal{H}_g(f)\|_X \leq C \|\tilde{\mathcal{H}}(f)\|_X, \quad f \in X.$$

The sufficiency parts of our Theorems 1-4 will follow using this and the following result which has independent interest.

- Theorem 5.**
- (i) *If $p > 1$, then $\tilde{\mathcal{H}}: H^p \rightarrow H^p$ is bounded.*
 - (ii) *If $p > 1$ and $-1 < \alpha < p - 2$, then $\tilde{\mathcal{H}}: A_\alpha^p \rightarrow A_\alpha^p$ is bounded.*
 - (iii) *If $p > 1$ and $p - 2 < \alpha \leq p - 1$, then $\tilde{\mathcal{H}}: \mathcal{D}_\alpha^p \rightarrow \mathcal{D}_\alpha^p$ is bounded.*

In Section 7 we shall deal with the question of characterizing the functions g for which \mathcal{H}_g is compact on Hardy, Bergman and Dirichlet spaces. We prove the “expected results”, that is, Theorems 1-4 remain true if we change “bounded” to “compact” and the mean Lipschitz space $\Lambda(s, \alpha)$ appearing there to the corresponding “little oh” space $\lambda(s, \alpha)$. We also obtain the characterization of the functions g for which the operator \mathcal{H}_g is Hilbert-Schmidt on the relevant Hilbert spaces.

Theorem 6. *The following are equivalent:*

- (i) \mathcal{H}_g is Hilbert-Schmidt on H^2 .
- (ii) \mathcal{H}_g is Hilbert-Schmidt on A_α^2 for any $-1 < \alpha < 0$.
- (iii) \mathcal{H}_g is Hilbert-Schmidt on \mathcal{D}_α^2 for any $0 < \alpha \leq 1$.
- (iv) $g \in \mathcal{D}$.

Note that the case $\alpha = 1$ of (iii) is assertion (i).

We close this section noticing that, as usual, we shall be using the convention that $C = C(p, \alpha, q, \beta) \dots$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta \dots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Moreover, for two real-valued functions E_1, E_2 we write $E_1 \asymp E_2$, or $E_1 \lesssim E_2$, if there exists a positive constant C independent of the argument such that $\frac{1}{C}E_1 \leq E_2 \leq CE_1$, respectively $E_1 \leq CE_2$.

3. Preliminary results

Throughout the paper we shall use the following notation: If $g(z) = \sum_{k=0}^\infty b_k z^k \in \mathcal{H}ol(\mathbf{D})$ and $n \geq 0$, we set

$$\Delta_n g(z) = \sum_{k \in I(n)} b_k z^k$$

where $I(n) = \{k \in \mathbf{N} : 2^n \leq k \leq 2^{n+1} - 1\}$.

Let us recall several distinct characterizations of $\Lambda(p, \alpha)$ spaces, (see [1], [5], [9] and [17]).

Theorem A. *Suppose that $1 < p < \infty$, $0 < \alpha < 1$ and $g \in \mathcal{H}ol(\mathbf{D})$. The following conditions are equivalent:*

- (i) $g \in \Lambda(p, \alpha)$.
- (ii) $M_p(r, g') = O\left(\frac{1}{(1-r)^{1-\alpha}}\right)$, as $r \rightarrow 1^-$.
- (iii) $\|\Delta_n g\|_{H^p} = O(2^{-n\alpha})$, as $n \rightarrow \infty$.
- (iv) $\|\Delta_n g'\|_{H^p} = O(2^{n(1-\alpha)})$, as $n \rightarrow \infty$.
- (v) $\|\Delta_n g''\|_{H^p} = O(2^{n(2-\alpha)})$, as $n \rightarrow \infty$.

Remark 1. The corresponding results for the little-oh space $\lambda(p, \alpha)$ remain true, and they can be proved following the proofs in the references for Theorem A.

Suppose $W(z) = \sum_{k \in J} b_k z^k$ is a polynomial, so J is a finite subset of \mathbf{N} , and $f(z) = \sum_{k=0}^\infty a_k z^k \in \mathcal{H}ol(\mathbf{D})$. We consider the Hadamard product

$$(W * f)(z) = \sum_{k \in J} b_k a_k z^k,$$

and observe that if $f \in H^1$, then

$$(W * f)(e^{it}) = \frac{1}{2\pi} \int_0^{2\pi} W(e^{i(t-\theta)}) f(e^{i\theta}) d\theta$$

is the usual convolution.

If $\Phi: \mathbf{R} \rightarrow \mathbf{C}$ is a C^∞ -function such that $\text{supp}(\Phi)$ is a compact subset of $(0, \infty)$, we set

$$A_\Phi = \max_{s \in \mathbf{R}} |\Phi(s)| + \max_{s \in \mathbf{R}} |\Phi''(s)|,$$

and for $N = 1, 2, \dots$, we consider the polynomials

$$W_N^\Phi(z) = \sum_{k \in \mathbf{N}} \Phi\left(\frac{k}{N}\right) z^k.$$

Now, we are ready to state the next result on smooth partial sums.

Theorem B. *Assume that $\Phi: \mathbf{R} \rightarrow \mathbf{C}$ is a C^∞ -function with $\text{supp}(\Phi)$ a compact set contained in $(0, \infty)$. Then:*

- (i) There exists an absolute constant $C > 0$ such that if $m \in \{0, 1, 2, \dots\}$ and $N \in \{1, 2, 3, \dots\}$, then

$$|W_N^\Phi(e^{i\theta})| \leq C \min \left\{ N \max_{s \in \mathbf{R}} |\Phi(s)|, N^{1-m} |\theta|^{-m} \max_{s \in \mathbf{R}} |\Phi^{(m)}(s)| \right\},$$

for $0 < |\theta| < \pi$.

- (ii) There exists a positive constant C such that

$$|(W_N^\Phi * f)(e^{i\theta})| \leq C A_\Phi M(|f|)(e^{i\theta}), \quad \text{for all } f \in H^1,$$

where M is the Hardy–Littlewood maximal-operator, that is,

$$M(|f|)(e^{i\theta}) = \sup_{0 < h < \pi} \frac{1}{2h} \int_{\theta-h}^{\theta+h} |f(e^{it})| dt.$$

- (iii) For every $p \in (1, \infty)$ there exists $C_p > 0$ such that

$$\|W_N^\Phi * f\|_{H^p} \leq C_p A_\Phi \|f\|_{H^p}, \quad f \in H^p.$$

- (iv) For every $p \in (1, \infty)$ and $\alpha > -1$ there is $C_p > 0$ such that

$$\|W_N^\Phi * f\|_{A_\alpha^p} \leq C_p A_\Phi \|f\|_{A_\alpha^p}, \quad f \in A_\alpha^p.$$

Theorem B follows from the results and proofs in [18, p. 111 – 113].

The following lemma also plays an essential role in our work.

Lemma 1. Suppose that $1 < p < \infty$ and $\alpha > -1$. For $N = 1, 2, \dots$, let $a_N = 1 - \frac{1}{N}$ and define the functions

$$(3.1) \quad \psi_{N,\alpha}(s) = \frac{1}{N^{3-\frac{2+\alpha}{p}}} \int_0^1 \frac{t^{sN}}{(1 - a_N t)^2} dt, \quad s > 0.$$

and

$$(3.2) \quad \varphi_{N,\alpha}(s) = \frac{1}{\psi_{N,\alpha}(s)} \quad s > 0.$$

Then:

- (i) $\psi_{N,\alpha}, \varphi_{N,\alpha} \in C^\infty((0, \infty))$.
- (ii) Asymptotically, $|\psi_{N,\alpha}(s)| \asymp \frac{1}{N^{2-\frac{2+\alpha}{p}}}$, $\frac{1}{2} < s < 4$, $N \rightarrow \infty$.
- (iii) For each $m \in \mathbf{N}$ there is a constant $C(m) > 0$ (depending on m but not on N) such that

$$|\psi_{N,\alpha}^{(m)}(s)| \leq \frac{C(m)}{N^{2-\frac{2+\alpha}{p}}}, \quad \frac{1}{2} < s < 4, \quad N = 1, 2, \dots$$

- (iv) For each $m \in \mathbf{N}$ there is a constant $C(m) > 0$ (depending on m but not on N) such that

$$(3.3) \quad |\varphi_{N,\alpha}^{(m)}(s)| \leq C(m) N^{2-\frac{2+\alpha}{p}}, \quad \frac{1}{2} < s < 4, \quad N = 1, 2, \dots$$

Proof. (i) is clear.

(ii) We note that

$$\int_0^1 \frac{t^{sN}}{(1 - a_N t)^2} dt \leq \int_0^1 \frac{1}{(1 - a_N t)^2} dt = N,$$

while if $\frac{1}{2} < s < 4$, then

$$\int_0^1 \frac{t^{sN}}{(1 - a_N t)^2} dt \geq \int_{a_N}^1 \frac{t^{sN}}{(1 - a_N t)^2} dt \geq (1 - a_N) \frac{a_N^{4N}}{(1 - a_N^2)^2} = \frac{(1 - \frac{1}{N})^{4N}}{(2 - \frac{1}{N})^2} N \geq CN.$$

So, for $\frac{1}{2} < s < 4$, we have $|\psi_{N,\alpha}(s)| \asymp \frac{1}{N^{2-\frac{2+\alpha}{p}}}$, as $N \rightarrow \infty$.

(iii) Since

$$\sup_{0 < t < 1, \frac{1}{2} < s < 4} \left(\log \frac{1}{t^N}\right)^m t^{sN} \leq \sup_{0 < x < 1} \left(\log \frac{1}{x}\right)^m x^{1/2} = C(m) < \infty,$$

we deduce that

$$\begin{aligned} |\psi_{N,\alpha}^{(m)}(s)| &= \frac{1}{N^{3-\frac{2+\alpha}{p}}} \int_0^1 \frac{\left(\log \frac{1}{t^N}\right)^m t^{sN}}{(1 - a_N t)^2} dt \leq C(m) \frac{1}{N^{3-\frac{2+\alpha}{p}}} \int_0^1 \frac{1}{(1 - a_N t)^2} dt \\ &\leq C(m) \frac{1}{N^{2-\frac{2+\alpha}{p}}}, \quad \frac{1}{2} < s < 4, \quad N = 1, 2, \dots \end{aligned}$$

(iv) For $m = 0$, the assertion follows from part (ii). For $m = 1$, using parts (ii) and (iii), we have

$$|\varphi'_{N,\alpha}(s)| \leq \frac{|\psi'_{N,\alpha}(s)|}{|\psi_{N,\alpha}(s)|^2} \leq C(1)N^{2-\frac{2+\alpha}{p}}, \quad \frac{1}{2} < s < 4.$$

Now we shall proceed by induction. Assume that (3.3) holds for $j = 0, 1, \dots, m-1$. Since $1 = \varphi_{N,\alpha}(s)\psi_{N,\alpha}(s)$, we have

$$0 = (\varphi_{N,\alpha}(s)\psi_{N,\alpha}(s))^{(m)}(s) = \sum_{j=0}^m \binom{m}{j} \psi_{N,\alpha}^{(m-j)}(s)\varphi_{N,\alpha}^{(j)}(s),$$

which implies

$$|\varphi_{N,\alpha}^{(m)}(s)| \leq \frac{\sum_{j=0}^{m-1} \binom{m}{j} |\psi_{N,\alpha}^{(m-j)}(s)\varphi_{N,\alpha}^{(j)}(s)|}{|\psi_{N,\alpha}(s)|}, \quad \frac{1}{2} < s < 4.$$

This together with the induction hypothesis and part (iii) concludes the proof. \square

We shall use also the following lemma which follows easily from results in [17].

Lemma 2. Assume that $0 < p < \infty$, $\alpha > -1$, $N \in \mathbf{N}$, and set

$$h(z) = \sum_{N/2 \leq k \leq 4N} a_k z^k.$$

Then

$$\|h\|_{A_\alpha^p} \asymp N^{-\frac{1+\alpha}{p}} \|h\|_{H^p}.$$

Proof. Assume N is even. (If N is odd the proof can be adjusted by using $[\frac{N}{2}] + 1$ instead of $\frac{N}{2}$). Using [17, Lemma 3.1] we have for each $0 < r < 1$,

$$\|h\|_{H^p}^p r^{p4N} \leq M_p^p(r, h) \leq \|h\|_{H^p}^p r^{p\frac{N}{2}}$$

which gives

$$\|h\|_{H^p}^p \int_0^1 r^{p4N+1} (1 - r^2)^\alpha dr \leq \frac{\|h\|_{A_\alpha^p}^p}{\alpha + 1} \leq \|h\|_{H^p}^p \int_0^1 r^{p\frac{N}{2}+1} (1 - r^2)^\alpha dr.$$

Each of the two integrals appearing above can be expressed in terms of the usual Beta function, and using the Stirling asymptotic series we can see that each of the integrals grows as $N^{-(\alpha+1)}$ as $N \rightarrow \infty$, and the assertion follows. \square

4. Necessary conditions for the boundedness of \mathcal{H}_g

Putting together the conditions stated in Theorem 1 and Theorem 2 as necessary for the boundedness of the operator on H^p for $1 < p \leq 2$ and $2 < p < \infty$, respectively, yields the following statement.

Theorem 7. *Suppose that $1 < p < \infty$ and $g \in \text{Hol}(\mathbf{D})$. If \mathcal{H}_g is bounded from H^p to H^p , then $g \in \Lambda\left(p, \frac{1}{p}\right)$.*

Proof. Let $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be the Taylor expansion of g . We start by considering the functions $\psi_{N,\alpha}$ and $\varphi_{N,\alpha} = \frac{1}{\psi_{N,\alpha}}$ defined in Lemma 1 with $\alpha = p - 1$ and, for simplicity, write $\psi_N = \psi_{N,p-1}$ and $\varphi_N = \varphi_{N,p-1}$.

For each $N = 1, 2, \dots$, we can find a C^∞ -function $\Phi_N: \mathbf{R} \rightarrow \mathbf{C}$ with $\text{supp}(\Phi_N) \subset \left(\frac{1}{2}, 4\right)$, satisfying

$$(4.1) \quad \Phi_N(s) = \varphi_N(s), \quad 1 \leq s \leq 2,$$

and such that, by using part (iv) of Lemma 1, for each $m \in \mathbf{N}$ there exists $C(m)$ (independent of N) with

$$(4.2) \quad |\Phi_N^{(m)}(s)| \leq C(m)N^{1-\frac{1}{p}}, \quad s \in \mathbf{R}, \quad N = 1, 2, \dots$$

In particular, we have

$$(4.3) \quad A_{\Phi_N} = \max_{s \in \mathbf{R}} |\Phi_N(s)| + \max_{s \in \mathbf{R}} |\Phi_N''(s)| \leq CN^{1-\frac{1}{p}}.$$

Let us consider now the family of test functions $\{f_N\}$ given by

$$f_N(z) = \frac{1}{N^{2-\frac{1}{p}}} \frac{1}{(1 - a_N z)^2}, \quad z \in \mathbf{D}, \quad N = 1, 2, \dots$$

An easy calculation using [5, Lemma, p. 65]) shows that the H^p -norms of the functions f_N are uniformly bounded. By the hypothesis

$$\sup_N \|\mathcal{H}_g(f_N)\|_{H^p} = C < \infty.$$

This, together with part (iii) of Theorem B and (4.3), implies

$$(4.4) \quad \|W_N^{\Phi_N} * \mathcal{H}_g(f_N)\|_{H^p} \leq C_p A_{\Phi_N} \|\mathcal{H}_g(f_N)\|_{H^p} \leq C_p N^{1-\frac{1}{p}}.$$

On the other hand,

$$\begin{aligned} (W_N^{\Phi_N} * \mathcal{H}_g(f_N))(z) &= \sum_{\frac{N}{2} \leq k \leq 4N} \left[(k+1)b_{k+1} \left(\int_0^1 t^k f_N(t) dt \right) \Phi_N\left(\frac{k}{N}\right) \right] z^k \\ &= \sum_{\frac{N}{2} \leq k \leq N-1} [\dots] z^k + \sum_{N \leq k \leq 2N-1} [\dots] z^k + \sum_{2N \leq k \leq 4N} [\dots] z^k \\ &= F_1^N(z) + F_2^N(z) + F_3^N(z) \end{aligned}$$

and by (4.1)

$$\begin{aligned}
 F_2^N(z) &= \sum_{N \leq k \leq 2N-1} (k+1)b_{k+1} \left(\int_0^1 t^k f_N(t) dt \right) \Phi_N \left(\frac{k}{N} \right) z^k \\
 (4.5) \quad &= \sum_{N \leq k \leq 2N-1} (k+1)b_{k+1} \psi_N \left(\frac{k}{N} \right) \varphi_N \left(\frac{k}{N} \right) z^k \\
 &= \sum_{N \leq k \leq 2N-1} (k+1)b_{k+1} z^k.
 \end{aligned}$$

Using the M. Riesz projection theorem and (4.4) we have

$$\|F_2^N\|_{H^p} \leq C_p \|W_N^{\Phi_N} * \mathcal{H}_g(f_N)\|_{H^p} \leq C_p N^{1-\frac{1}{p}},$$

valid for each N . Finally observing that for $n \in \mathbf{N}$,

$$\Delta_n g'(z) = \sum_{k=2^n}^{2^{n+1}-1} (k+1)b_{k+1} z^k = F_2^{2^n}(z),$$

we obtain

$$\|\Delta_n g'\|_{H^p} \leq C_p 2^{n(1-\frac{1}{p})},$$

and using part (iv) of Theorem A, we conclude $g \in \Lambda \left(p, \frac{1}{p} \right)$. □

Proof of the necessity statement in Theorem 4. The proof is similar to that of Theorem 7, hence, we shall omit some details. Let p, α and $g(z) = \sum_{k=0}^\infty b_k z^k \in \mathcal{H}ol(\mathbf{D})$ be as in the statement and assume that $\mathcal{H}_g: \mathcal{D}_\alpha^p \rightarrow \mathcal{D}_\alpha^p$ is bounded. We consider the functions $\psi_{N,\alpha}$ and $\varphi_{N,\alpha} = \frac{1}{\psi_{N,\alpha}}$ defined in Lemma 1. By part (iv) of Lemma 1, for each $N = 1, 2, \dots$, there is a C^∞ -function $\Phi_{N,\alpha}: \mathbf{R} \rightarrow \mathbf{C}$ with $\text{supp}(\Phi_{N,\alpha}) \subset \left(\frac{1}{2}, 4\right)$ such that

$$(4.6) \quad \Phi_{N,\alpha}(s) = \varphi_{N,\alpha} \left(s + \frac{1}{N} \right), \quad 1 \leq s \leq 2,$$

and for each $m \in \mathbf{N}$ there exists $C(m)$ (independent of N) such that

$$(4.7) \quad |\Phi_{N,\alpha}^{(m)}(s)| \leq C(m) N^{2-\frac{2+\alpha}{p}}, \quad s \in \mathbf{R}, \quad N = 1, 2, \dots$$

Since $\alpha < 3p - 2$, the family of test functions

$$(4.8) \quad f_N(z) = f_{N,\alpha}(z) = \frac{1}{N^{3-\frac{2+\alpha}{p}}} \frac{1}{(1 - a_N z)^2}, \quad z \in \mathbf{D},$$

forms a bounded set in \mathcal{D}_α^p (see [20, Lemma 3.10]), and the hypothesis implies that

$$\sup_N \|\mathcal{H}_g(f_N)\|_{\mathcal{D}_\alpha^p} < \infty.$$

This, together with the easily checked identity $\mathcal{H}_g(f)' = \mathcal{H}_{g'}(zf)$, gives

$$\sup_N \|\mathcal{H}_{g'}(zf_N)\|_{A_\alpha^p} = C < \infty.$$

Then part (iv) of Theorem B and (4.7) imply that

$$(4.9) \quad \|W_N^{\Phi_{N,\alpha}} * \mathcal{H}_{g'}(zf_N)\|_{A_\alpha^p} \leq C_p A_{\Phi_{N,\alpha}} \|\mathcal{H}_{g'}(zf_N)\|_{A_\alpha^p} \leq C_p N^{2-\frac{2+\alpha}{p}}.$$

Moreover,

$$(W_N^{\Phi_{N,\alpha}} * \mathcal{H}_{g'}(zf_N))(z) = \sum_{\frac{N}{2} \leq k \leq 4N} (k+1)(k+2)b_{k+2} \left(\int_0^1 t^{k+1} f_N(t) dt \right) \Phi_{N,\alpha} \left(\frac{k}{N} \right) z^k$$

and, by (4.6),

$$(4.10) \quad \begin{aligned} & \sum_{k=N}^{2N-1} (k+1)(k+2)b_{k+2} \left(\int_0^1 t^{k+1} f_N(t) dt \right) \Phi_{N,\alpha} \left(\frac{k}{N} \right) z^k \\ &= \sum_{k=N}^{2N-1} (k+1)(k+2)b_{k+2} z^k. \end{aligned}$$

Consequently, using (4.10), the M. Riesz projection theorem, Lemma 2, and (4.9), and setting $N = 2^n, n \in \mathbf{N}$,

$$\begin{aligned} \|\Delta_n g''\|_{H^p} &= \left\| \sum_{k=2^n}^{2^{n+1}-1} (k+1)(k+2)b_{k+2} z^k \right\|_{H^p} \\ &\leq C_p \left\| \sum_{k=2^{n-1}}^{2^{n+2}} (k+1)(k+2)b_{k+2} \left(\int_0^1 t^{k+1} f_{2^n}(t) dt \right) \Phi_{2^n,\alpha} \left(\frac{k}{2^n} \right) z^k \right\|_{H^p} \\ &= C_p \left\| W_{2^n}^{\Phi_{2^n,\alpha}} * \mathcal{H}_{g'}(zf_{2^n}) \right\|_{H^p} \leq C_p 2^{n(\frac{1+\alpha}{p})} \left\| W_{2^n}^{\Phi_{2^n,\alpha}} * \mathcal{H}_{g'}(zf_{2^n}) \right\|_{A_\alpha^p} \\ &\leq C_p 2^{n(2-\frac{1}{p})}, \end{aligned}$$

and by part (v) of Theorem A, we deduce that $g \in \Lambda \left(p, \frac{1}{p} \right)$. □

We note that the proof we have just finished remains valid for $p > 1$ and $\alpha < 3p - 2$.

Proof of the necessity statement in Theorem 3. Let p, α be as in the statement and assume $\mathcal{H}_g: A_\alpha^p \rightarrow A_\alpha^p$ is bounded. Since $\alpha < p - 2$, then $\alpha + p < 3p - 2$. This together with the fact that $A_\alpha^p = \mathcal{D}_{p+\alpha}^p$ gives the assertion as a consequence of the preceding proof. □

5. The sublinear Hilbert operator

Let us consider the following space of analytic functions in \mathbf{D}

$$A_{[0,1]}^1 = \left\{ f \in \mathcal{H}ol(\mathbf{D}) : \int_0^1 |f(t)| dt < \infty \right\}.$$

The well-known Fejér–Riesz inequality [5] implies that $H^1 \subset A_{[0,1]}^1$. We remark also that an application of Hölder’s inequality yields

$$(5.1) \quad A_\alpha^p \subset A_{[0,1]}^1, \quad \text{if } p > 1 \text{ and } -1 < \alpha < p - 2,$$

an inclusion which is not longer true for $\alpha \geq p - 2$.

Condition (5.1) insures that $\tilde{\mathcal{H}}$ is well defined on A_α^p for p and α in that range of values.

Now, we proceed to state some lemmas which will be needed for the proof Theorem 5.

Lemma 3. (i) Assume that $0 < p < \infty$. Then there exists a positive constant $C = C(p)$ such that

$$\int_0^1 M_\infty^p(r, g) dr \leq C \|g\|_{H^p}^p, \quad \text{for all } g \in \mathcal{H}ol(\mathbf{D}).$$

(ii) Assume that $0 < p < \infty$ and $\alpha > -1$. Then there exists a positive constant $C = C(p, \alpha)$ such that

$$\int_0^1 M_\infty^p(r, f)(1-r)^{\alpha+1} dr \leq C \|f\|_{A_\alpha^p}^p, \quad \text{for all } f \in \mathcal{H}ol(\mathbf{D}).$$

Proof. Part (i) follows taking $q = \infty$ and $\lambda = p$ in Theorem 5.11 of [5].

Now we proceed to prove part (ii). Applying (i) to $g(z) = f(sz)$ ($0 < s < 1$) and making a change of variables, we obtain

$$\int_0^s M_\infty^p(r, f) dr \leq CsM_p^p(s, f), \quad 0 \leq r < 1,$$

Multiplying both sides of the last inequality by $(1-s)^\alpha$, integrating the resulting inequality, and applying Fubini's theorem yields

$$\int_0^1 M_\infty^p(r, f)(1-r)^{\alpha+1} dr = C \int_0^1 (1-s)^\alpha \int_0^s M_\infty^p(r, f) dr ds \leq C \|f\|_{A_\alpha^p}^p. \quad \square$$

Lemma 4. Assume that $1 < p < \infty$ and $p-2 < \alpha$. Then there exists a constant $C = C(p, \alpha)$ such that for any $f \in \mathcal{H}ol(\mathbf{D})$

$$\int_0^1 M_\infty^p(r, f)(1-r)^{\alpha-p+1} dr \leq C \|f\|_{\mathcal{D}_\alpha^p}^p.$$

Proof. The identity $f(z) = f(0) + \int_0^z f'(\zeta) d\zeta$, $z \in \mathbf{D}$, gives

$$M_\infty^p(r, f) \leq C \left(|f(0)|^p + \left(\int_0^r M_\infty(t, f') dt \right)^p \right),$$

for some constant C . Since $\alpha - p + 1 > -1$ we have

$$\begin{aligned} \int_0^1 M_\infty^p(r, f)(1-r)^{\alpha-p+1} dr &\leq C |f(0)|^p + C \int_0^1 \left(\int_0^r M_\infty(t, f') dt \right)^p (1-r)^{\alpha-p+1} dr \\ &= C |f(0)|^p + C \int_0^1 \left(\int_{1-r}^1 M_\infty(1-s, f') ds \right)^p (1-r)^{\alpha-p+1} dr. \end{aligned}$$

We now use the following version of the classical Hardy inequality [12, pp. 244–245]: If $k > 0$, $q > 1$ and h is a nonnegative function defined in $(0, \infty)$, then

$$\int_0^\infty \left(\int_x^\infty h(t) dt \right)^q x^{k-1} dx \leq \left(\frac{q}{k} \right)^q \int_0^\infty h(x)^q x^{q+k-1} dx.$$

Taking $h \equiv 0$ in $[1, \infty)$, and making the change of variable $x = 1 - r$ in each side, the inequality takes the form

$$(5.2) \quad \int_0^1 \left(\int_{1-r}^1 h(t) dt \right)^q (1-r)^{k-1} dr \leq \left(\frac{q}{k} \right)^q \int_0^1 (h(1-r))^q (1-r)^{q+k-1} dr.$$

Now apply this inequality to the function $h(s) = M_\infty(1-s, f')$ with $k = \alpha - p + 2 > 0$ to obtain

$$\int_0^1 \left(\int_{1-r}^1 M_\infty(1-s, f') ds \right)^p (1-r)^{\alpha-p+1} dr \leq C \int_0^1 M_\infty^p(r, f') (1-r)^{\alpha+1} dr.$$

Putting together the above and using Lemma 3 we find,

$$\begin{aligned} \int_0^1 M_\infty^p(r, f)(1-r)^{\alpha-p+1} dr &\leq C \left(|f(0)|^p + \int_0^1 M_\infty^p(r, f') (1-r)^{\alpha+1} dr \right) \\ &\leq C \left(|f(0)|^p + \|f'\|_{A_\alpha^p}^p \right) = C \|f\|_{\mathcal{D}_\alpha^p}^p, \end{aligned}$$

and the proof is complete. □

The first part of the following Lemma is a special case of [16, Theorem 2.1], and the second part is an immediate consequence of the first part.

Lemma 5. (i) *If $1 < p < \infty$ and $\alpha > -1$, then the dual of A_α^p can be identified with A_β^q where $\frac{1}{p} + \frac{1}{q} = 1$ and β is any number with $\beta > -1$, under the pairing*

$$(5.3) \quad \langle f, g \rangle_{A_{p,\alpha,\beta}} = \int_{\mathbf{D}} f(z) \overline{g(z)} (1-|z|^2)^{\frac{\alpha}{p} + \frac{\beta}{q}} dA(z).$$

(ii) *If $1 < p < \infty$ and $\alpha > -1$, then the dual of \mathcal{D}_α^p can be identified with \mathcal{D}_β^q where $\frac{1}{p} + \frac{1}{q} = 1$ and β is any number with $\beta > -1$, under the pairing*

$$(5.4) \quad \langle f, g \rangle_{\mathcal{D}_{p,\alpha,\beta}} = f(0) \overline{g(0)} + \int_{\mathbf{D}} f'(z) \overline{g'(z)} (1-|z|^2)^{\frac{\alpha}{p} + \frac{\beta}{q}} dA(z).$$

Proof of Theorem 5. (i) Recall that for $1 < p < \infty$, the dual of H^p can be identified with H^q , $\frac{1}{p} + \frac{1}{q} = 1$, under the H^2 -pairing,

$$\langle f, h \rangle_{H^2} = \lim_{r \rightarrow 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) \overline{h(re^{i\theta})} d\theta,$$

thus it is enough to prove that there exists a constant $C > 0$ such that

$$(5.5) \quad \left| \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} \tilde{\mathcal{H}}(f)(re^{i\theta}) \overline{h(re^{i\theta})} d\theta \right| \leq C \|f\|_{H^p} \|h\|_{H^q}$$

for any $f \in H^p$ and $g \in H^q$. Now, by Fubini's theorem

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \tilde{\mathcal{H}}(f)(re^{i\theta}) \overline{h(re^{i\theta})} d\theta &= \int_0^1 |f(t)| \overline{\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{h(re^{i\theta})}{1-tre^{-i\theta}} d\theta \right)} dt \\ &= \int_0^1 |f(t)| \overline{h(r^2t)} dt. \end{aligned}$$

Using Hölder's inequality and the Fejér–Riesz inequality we have

$$\begin{aligned} \left| \frac{1}{2\pi} \int_0^{2\pi} \tilde{\mathcal{H}}(f)(re^{i\theta}) \overline{h(re^{i\theta})} d\theta \right| &\leq \left(\int_0^1 |f(t)|^p dt \right)^{1/p} \left(\int_0^1 |h(r^2t)|^q dt \right)^{1/q} \\ &\leq C \|f\|_{H^p} M_q(r^2, h) \leq C \|f\|_{H^p} \|h\|_{H^q}, \end{aligned}$$

which implies (5.5) and finishes the proof of (i).

(ii) Using Lemma 5 we can choose $\beta = \frac{-\alpha q}{p} = \frac{-\alpha}{p-1}$ so that the weight in the pairing (5.3) is identically equal to 1, and we have for $f \in A_\alpha^p$ and $h \in A_\beta^q$,

$$\begin{aligned}
 \langle \tilde{\mathcal{H}}(f), h \rangle_{A_{p,\alpha,\beta}} &= \int_{\mathbf{D}} \tilde{\mathcal{H}}(f)(z) \overline{h(z)} dA(z) \\
 &= \int_{\mathbf{D}} \left(\int_0^1 \frac{|f(t)|}{1-tz} dt \right) \overline{h(z)} dA(z) \\
 (5.6) \quad &= \int_0^1 |f(t)| \overline{\left(\int_{\mathbf{D}} \frac{h(z)}{1-t\bar{z}} dA(z) \right)} dt \\
 &= 2 \int_0^1 |f(t)| \left(\int_0^1 \overline{h(r^2t)} r dr \right) dt,
 \end{aligned}$$

so that

$$(5.7) \quad |\langle \tilde{\mathcal{H}}(f), h \rangle_{A_{p,\alpha,\beta}}| \leq 2 \int_0^1 |f(t)| G(t) dt,$$

where $G(t) = \int_0^1 |h(r^2t)| r dr$. Using Hölder's inequality we obtain,

$$\begin{aligned}
 \int_0^1 |f(t)| G(t) dt &= \int_0^1 |f(t)| (1-t)^{\frac{\alpha+1}{p}} G(t) (1-t)^{-\frac{\alpha+1}{p}} dt \\
 &\leq \left(\int_0^1 |f(t)|^p (1-t)^{\alpha+1} dt \right)^{1/p} \left(\int_0^1 |G(t)|^q (1-t)^{-\frac{q(\alpha+1)}{p}} dt \right)^{1/q} \\
 &\leq \left(\int_0^1 M_\infty^p(t, f) (1-t)^{\alpha+1} dt \right)^{1/p} \left(\int_0^1 |G(t)|^q (1-t)^{-\frac{q(\alpha+1)}{p}} dt \right)^{1/q} \\
 &\leq C \|f\|_{A_\alpha^p} \left(\int_0^1 |G(t)|^q (1-t)^{-\frac{q(\alpha+1)}{p}} dt \right)^{1/q},
 \end{aligned}$$

where in the last step we have used Lemma 3. Next we show that

$$(5.8) \quad \int_0^1 |G(t)|^q (1-t)^{-\frac{q(\alpha+1)}{p}} dt \leq C \|h\|_{A_\beta^q}^q.$$

This together with (5.7) will finish the proof. To show (5.8) observe first that if $0 < t < 1/2$ then $|h(r^2t)| \leq M_\infty(\frac{1}{2}, h)$ for each $0 < r < 1$, thus

$$\int_0^1 |h(r^2t)| r dr \leq M_\infty\left(\frac{1}{2}, h\right), \quad 0 < t < 1/2,$$

and we have

$$\begin{aligned}
 \int_0^{1/2} |G(t)|^q (1-t)^{-\frac{q(\alpha+1)}{p}} dt &= \int_0^{1/2} \left(\int_0^1 |h(r^2t)| r dr \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}} dt \\
 &\leq C M_\infty^q\left(\frac{1}{2}, h\right) \leq C \|h\|_{A_\beta^q}^q.
 \end{aligned}$$

On the other hand,

$$-\frac{q(\alpha+1)}{p} = \frac{-\alpha+p-2}{p-1} - 1 > -1,$$

and making a change of variable we obtain $\int_0^1 |h(r^2t)|rdr = \frac{1}{2t} \int_0^t |h(s)|ds$ so,

$$\begin{aligned} \int_{1/2}^1 |G(t)|^q (1-t)^{-\frac{q(\alpha+1)}{p}} dt &= \int_{1/2}^1 \left(\int_0^1 |h(r^2t)|rdr \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}} dt \\ &= \int_{1/2}^1 \frac{1}{(2t)^q} \left(\int_0^t |h(s)|ds \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}} dt \\ &\leq \int_{1/2}^1 \left(\int_0^t M_\infty(s, h) ds \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}} dt \\ &\leq \int_0^1 \left(\int_{1-t}^1 M_\infty(1-s, h) ds \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}} dt \\ &\leq C \int_0^1 M_\infty^q(t, h) (1-t)^{-\frac{q\alpha}{p}+1} dt \quad (\text{by (5.2)}) \\ &\leq C \|h\|_{A_p^q}^q, \end{aligned}$$

where we have used Lemma 3 in the last step. Thus (5.8) is proved and the proof of (ii) is complete.

(iii) Case $\alpha = p - 1$. By Lemma 5 the dual of \mathcal{D}_{p-1}^p can be identified with \mathcal{D}_{q-1}^q , $\frac{1}{p} + \frac{1}{q} = 1$, taking $\alpha = p - 1$ and $\beta = q - 1$ in the relevant pairing in (5.4), so that the weight becomes $(1 - |z|^2)$. Thus for $f \in \mathcal{D}_{p-1}^p$ and $h \in \mathcal{D}_{q-1}^q$ we have by Fubini's theorem

$$\begin{aligned} \langle \tilde{\mathcal{H}}(f), h \rangle_{\mathcal{D}_{p,p-1,q-1}} &= \tilde{\mathcal{H}}(f)(0)\overline{h(0)} + \int_{\mathbf{D}} \tilde{\mathcal{H}}(f)'(z)\overline{h'(z)}(1 - |z|^2) dA(z) \\ &= \tilde{\mathcal{H}}(f)(0)\overline{h(0)} + \int_0^1 |f(t)| \left(\int_{\mathbf{D}} \frac{\overline{th'(z)}}{(1-tz)^2} (1 - |z|^2) dA(z) \right) dt, \end{aligned}$$

and a routine calculation gives

$$\int_{\mathbf{D}} \frac{\overline{th'(z)}}{(1-tz)^2} (1 - |z|^2) dA(z) = \overline{h(t)} - \int_0^1 \overline{h(rt)} dr.$$

Now

$$\left| \overline{h(t)} - \int_0^1 \overline{h(rt)} dr \right| \leq 2M_\infty(t, h),$$

therefore

$$\begin{aligned} \left| \int_{\mathbf{D}} \tilde{\mathcal{H}}(f)'(z)\overline{h'(z)}(1 - |z|^2) dA(z) \right| &\leq \int_0^1 |f(t)| \left| \overline{h(t)} - \int_0^1 \overline{h(rt)} dr \right| dt \\ &\leq 2 \int_0^1 M_\infty(t, f)M_\infty(t, h) dt \leq 2 \left(\int_0^1 M_\infty^p(t, f) dt \right)^{1/p} \left(\int_0^1 M_\infty^q(t, h) dt \right)^{1/q} \\ &\leq C \|f\|_{\mathcal{D}_{p-1}^p} \|h\|_{\mathcal{D}_{q-1}^q}, \end{aligned}$$

where for the last inequality we have used Lemma 4 twice with $\alpha = p - 1$ and $\alpha = q - 1$ in the two integrals respectively. Moreover, $|h(0)| \leq \|h\|_{\mathcal{D}_{q-1}^q}$ and

$$|\tilde{\mathcal{H}}(f)(0)| = \int_0^1 |f(t)| dt \leq \left(\int_0^1 M_\infty^p(t, f) dt \right)^{1/p} \leq C \|f\|_{\mathcal{D}_{p-1}^p},$$

and combining the above we obtain

$$|\langle \tilde{\mathcal{H}}(f), h \rangle_{\mathcal{D}_{p,p-1,q-1}}| \leq C \|f\|_{\mathcal{D}_{p-1}^p} \|h\|_{\mathcal{D}_{q-1}^q}$$

which completes the proof of this case.

Case $p - 2 < \alpha < p - 1$. In this case the dual of \mathcal{D}_α^p can be identified with \mathcal{D}_β^q with $\beta = \frac{-\alpha q}{p}$. The weight in the pairing (5.4) is then identically equal to 1. Thus for $f \in \mathcal{D}_\alpha^p$ and $h \in \mathcal{D}_\beta^q$ we have

$$\langle \tilde{\mathcal{H}}(f), h \rangle_{\mathcal{D}_{p,\alpha,\beta}} = \tilde{\mathcal{H}}(f)(0)\overline{h(0)} + \int_{\mathbf{D}} \tilde{\mathcal{H}}(f)'(z)\overline{h'(z)} dA(z).$$

Now using Fubini's theorem and the reproducing formula

$$h'(a) = \int_{\mathbf{D}} \frac{h'(z)}{(1 - a\bar{z})^2} dA(z), \quad a \in \mathbf{D}, \quad h \in \mathcal{D}_\beta^q,$$

we find

$$\int_{\mathbf{D}} \tilde{\mathcal{H}}(f)'(z)\overline{h'(z)} dA(z) = \int_0^1 t|f(t)| \left(\int_{\mathbf{D}} \frac{\overline{h'(z)}}{(1 - tz)^2} dA(z) \right) dt = \int_0^1 t|f(t)|\overline{h'(t)} dt.$$

We set $s = -1 + \frac{\alpha+1}{p}$ and use Hölder's inequality to obtain

$$\begin{aligned} \left| \int_0^1 t|f(t)|\overline{h'(t)} dt \right| &= \left| \int_0^1 t|f(t)|(1-t)^s \overline{h'(t)}(1-t)^{-s} dt \right| \\ &\leq \left(\int_0^1 |f(t)|^p (1-t)^{ps} dt \right)^{\frac{1}{p}} \left(\int_0^1 |h'(t)|^q (1-t)^{-qs} dt \right)^{\frac{1}{q}}. \end{aligned}$$

By Lemma 4 the first integral above is

$$\begin{aligned} \int_0^1 |f(t)|^p (1-t)^{ps} dt &= \int_0^1 |f(t)|^p (1-t)^{\alpha-p+1} dt \\ &\leq \int_0^1 M_\infty^p(f, t) (1-t)^{\alpha-p+1} dt \leq C \|f\|_{\mathcal{D}_\alpha^p}^p, \end{aligned}$$

while the second integral by Lemma 3 is

$$\begin{aligned} \int_0^1 |h'(t)|^q (1-t)^{-qs} dt &= \int_0^1 |h'(t)|^q (1-t)^{1-\frac{\alpha q}{p}} dt \\ &= \int_0^1 |h'(t)|^q (1-t)^{\beta+1} dt \leq \int_0^1 M_\infty^q(h', t) (1-t)^{\beta+1} dt \\ &\leq C \|h'\|_{A_\beta^q}^q \leq C \|h\|_{\mathcal{D}_\beta^q}^q. \end{aligned}$$

Thus

$$\left| \int_{\mathbf{D}} \tilde{\mathcal{H}}(f)'(z)\overline{g'(z)} dA(z) \right| \leq C \|f\|_{\mathcal{D}_\alpha^p} \|g\|_{\mathcal{D}_\beta^q}.$$

This together with the inequalities $|h(0)| \leq \|h\|_{\mathcal{D}_\beta^q}$ and

$$|\tilde{\mathcal{H}}(f)(0)| = \int_0^1 |f(t)| dt \leq C \|f\|_{\mathcal{D}_\alpha^p}$$

imply that

$$|\langle \tilde{\mathcal{H}}(f), h \rangle_{\mathcal{D}_{p,\alpha,\beta}}| \leq C \|f\|_{\mathcal{D}_\alpha^p} \|h\|_{\mathcal{D}_\beta^q}$$

and the proof is complete. □

6. Sufficient conditions

In this section we will prove the sufficient conditions for Theorems 1, 2(ii), 3, and 4. In order to do that we state first some needed results.

A nice result of Hardy–Littlewood [5, Section 6.2] [18, Theorem 7.5.1] asserts that if $0 < p \leq 2$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H^p$, then

$$(6.1) \quad K_p(f) = \sum_{k=0}^{\infty} (k+1)^{p-2} |a_k|^p \leq C_p \|f\|_{H^p}^p.$$

On the other hand, if $2 \leq p < \infty$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(D)$ satisfies that $K_p(f) < \infty$, then $f \in H^p$ and

$$(6.2) \quad \|f\|_{H^p}^p \leq C_p K_p(f).$$

The converse of each of these two statements is not true for a general power series $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(\mathbf{D})$ and for arbitrary indices $p \neq 2$. If, however, we restrict to the class of power series with non-negative decreasing coefficients, then we have the following result (see [10], [21, Chapter XII, Lemma 6.6], [18, 7.5.9] and [19]).

Theorem C. *Assume that $1 \leq p < \infty$ and $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(\mathbf{D})$, where $\{a_n\}$ is a sequence of positive numbers which decreases to zero. Then the following assertions are equivalent:*

- (i) $f \in H^p$.
- (ii) $f \in \mathcal{D}_{p-1}^p$.
- (iii) $K_p(f) < \infty$.

Furthermore, $\|f\|_{H^p}^p \asymp \|f\|_{\mathcal{D}_{p-1}^p}^p \asymp K_p(f)$.

The following decomposition theorem can be found in [17, Theorem 2.1] and [18, 7.5.8].

Theorem D. (i) *Assume $0 < p < \infty$, $1 < q < \infty$ and $0 < \alpha < \infty$. Then,*

$$\int_0^1 (1-r)^{p\alpha-1} M_q^p(r, f) dr \asymp |f(0)|^p + \sum_{n=0}^{\infty} 2^{-n(p\alpha)} \|\Delta_n f\|_{H^q}^p, \quad \text{for all } f \in \mathcal{H}ol(\mathbf{D}).$$

(ii) *In particular, if $p > 1$ and $\beta > -1$,*

$$\|f\|_{A_{\beta}^p}^p \asymp |f(0)|^p + \sum_{n=0}^{\infty} 2^{-n(\beta+1)} \|\Delta_n f\|_{H^p}^p, \quad \text{for all } f \in \mathcal{H}ol(\mathbf{D}).$$

The following lemma can be found in [18, 7.3.5], in a slightly different form. The proof suggested there can be applied to obtain it in the form we need it.

Lemma 6. *Suppose $0 < p < \infty$ and $\gamma \in \mathbf{R}$. For $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}ol(\mathbf{D})$ let $F(z) = \sum_{k=0}^{\infty} (k+1)^{\gamma} a_k z^k$. Then $\|\Delta_n F\|_{H^p} \asymp 2^{n\gamma} \|\Delta_n f\|_{H^p}$.*

Lemma 7. *Suppose that $1 < p < \infty$. There exists a constant $C = C(p) > 0$ such that if $f \in H^1$, $g(z) = \sum_{k=0}^{\infty} c_k z^k \in \mathcal{H}ol(\mathbf{D})$, and we set*

$$h(z) = \sum_{k=0}^{\infty} c_k \left(\int_0^1 t^{k+1} f(t) dt \right) z^k,$$

then

$$\|\Delta_n h\|_{H^p} \leq C \left(\int_0^1 t^{2^{n-2}+1} |f(t)| dt \right) \|\Delta_n g\|_{H^p}, \quad n \geq 3.$$

Proof. For each $n = 1, 2, \dots$, define

$$\Upsilon_n(s) = \int_0^1 t^{2^n s+1} f(t) dt, \quad s \geq 0.$$

Clearly, Υ_n is a $C^\infty(0, \infty)$ -function and

$$(6.3) \quad |\Upsilon_n(s)| \leq \int_0^1 t^{2^{n-2}+1} |f(t)| dt, \quad s \geq \frac{1}{2}.$$

Furthermore, since $\sup_{0 < x < 1} (\log \frac{1}{x})^2 x^{1/2} = C(2) < \infty$, we have

$$(6.4) \quad \begin{aligned} |\Upsilon_n''(s)| &\leq \int_0^1 \left[\left(\log \frac{1}{t^{2^n}} \right)^2 t^{2^{n-1}} \right] t^{2^n s+1-2^{n-1}} |f(t)| dt \\ &\leq C(2) \int_0^1 t^{2^n s+1-2^{n-1}} |f(t)| dt \leq C(2) \int_0^1 t^{2^{n-2}+1} |f(t)| dt, \quad s \geq \frac{3}{4}. \end{aligned}$$

Then, using (6.3) and (6.4), for each $n = 1, 2, \dots$, we can take a function $\Phi_n \in C^\infty(\mathbf{R})$ with $\text{supp}(\Phi_n) \in (\frac{3}{4}, 4)$, and such that

$$\Phi_n(s) = \Upsilon_n(s), \quad s \in [1, 2],$$

and

$$A_{\Phi_n} = \max_{s \in \mathbf{R}} |\Phi_n(s)| + \max_{s \in \mathbf{R}} |\Phi_n''(s)| \leq C \int_0^1 t^{2^{n-2}+1} |f(t)| dt.$$

We can then write

$$\Delta_n h(z) = \sum_{k \in I(n)} c_k \left(\int_0^1 t^{k+1} f(t) dt \right) z^k = \sum_{k \in I(n)} c_k \Phi_n \left(\frac{k}{2^n} \right) z^k = W_{2^n}^{\Phi_n} * \Delta_n g(z).$$

So by using part (iii) of Theorem B, we have

$$\begin{aligned} \|\Delta_n h\|_{H^p} &= \|W_{2^n}^{\Phi_n} * \Delta_n g\|_{H^p} \leq C_p A_{\Phi_n} \|\Delta_n g\|_{H^p} \\ &\leq C \left(\int_0^1 t^{2^{n-2}+1} |f(t)| dt \right) \|\Delta_n g\|_{H^p}. \end{aligned} \quad \square$$

We shall need several lemmas. The first one can be found in [19, p. 4]

Lemma A. Assume that $1 < p < \infty$ and $\lambda = \{\lambda_n\}_{n=0}^\infty$ is a monotone sequence of non negative numbers. Let $(\lambda g)(z) = \sum_{n=0}^\infty \lambda_n b_n z^n$, where $g(z) = \sum_{n=0}^\infty b_n z^n$. Then:

(a) If $\{\lambda_n\}_{n=0}^\infty$ is nondecreasing, there is $C > 0$ such that

$$C^{-1} \lambda_{2^{n-1}} \|\Delta_n g\|_{H^p}^p \leq \|\Delta_n \lambda g\|_{H^p}^p \leq C \lambda_{2^n} \|\Delta_n g\|_{H^p}^p.$$

(b) If $\{\lambda_n\}_{n=0}^\infty$ is nonincreasing, there is $C > 0$ such that

$$C^{-1} \lambda_{2^n} \|\Delta_n g\|_{H^p}^p \leq \|\Delta_n \lambda g\|_{H^p}^p \leq C \lambda_{2^{n-1}} \|\Delta_n g\|_{H^p}^p.$$

Lemma 8. (i) Assume $1 < p < \infty$, $-1 < \alpha < \infty$, and $f \in \mathcal{D}_\alpha^p$. Then

$$\|\tilde{\mathcal{H}}(f)\|_{\mathcal{D}_\alpha^p}^p \asymp |\tilde{\mathcal{H}}(f)(0)|^p + \sum_{j=1}^\infty (j+1)^{2p-3-\alpha} \left(\int_0^1 t^{j+1} |f(t)| dt \right)^p.$$

(ii) Assume $1 < p < \infty$, $-1 < \alpha < p - 2$, and $f \in A_\alpha^p$. Then

$$\|\tilde{\mathcal{H}}(f)\|_{A_\alpha^p}^p \asymp |\tilde{\mathcal{H}}(f)(0)|^p + \sum_{j=1}^\infty (j+1)^{p-3-\alpha} \left(\int_0^1 t^j |f(t)| dt \right)^p.$$

Proof. Set $r_n = 1 - \frac{1}{2^n}$. Applying [17, Lemma 3.1] to the function $h(z) = \frac{1}{1-z} = \sum_{k=0}^\infty z^k$, we deduce that

$$(6.5) \quad \|\Delta_n h\|_{H^p}^p \asymp \int_{-\pi}^\pi \frac{1}{|1 - r_n e^{it}|^p} dt \asymp \frac{1}{(1 - r_n)^{p-1}} \asymp 2^{n(p-1)}.$$

Now, we shall prove (i). By Theorem D (ii) we have

$$\begin{aligned} \|\tilde{\mathcal{H}}(f)\|_{\mathcal{D}_\alpha^p}^p &= |\tilde{\mathcal{H}}(f)(0)|^p + \|\tilde{\mathcal{H}}(f)'\|_{A_\alpha^p}^p \\ &\asymp |\tilde{\mathcal{H}}(f)(0)|^p + |\tilde{\mathcal{H}}(f)'(0)|^p + \sum_{n=0}^\infty 2^{-n(\alpha+1)} \|\Delta_n \tilde{\mathcal{H}}(f)'\|_{H^p}^p \\ &\asymp |\tilde{\mathcal{H}}(f)(0)|^p + \sum_{n=0}^\infty 2^{-n(\alpha+1)} \|\Delta_n \tilde{\mathcal{H}}(f)'\|_{H^p}^p, \end{aligned}$$

where we have taken into account that

$$\tilde{\mathcal{H}}(f)'(0) = \int_0^1 t|f(t)| dt \asymp \int_0^1 |f(t)| dt = \tilde{\mathcal{H}}(f)(0).$$

We then apply Lemma 6 and subsequently Lemma A (b) with the nonincreasing sequence $\int_0^1 t^{j+1}|f(t)|dt$ and (6.5) to obtain

$$\begin{aligned} \sum_{n=0}^\infty 2^{-n(\alpha+1)} \|\Delta_n \tilde{\mathcal{H}}(f)'\|_{H^p}^p &= \sum_{n=0}^\infty 2^{-n(\alpha+1)} \left\| \sum_{j \in I(n)} (j+1) \left(\int_0^1 t^{j+1}|f(t)| dt \right) z^j \right\|_{H^p}^p \\ &\asymp \sum_{n=0}^\infty 2^{-n(\alpha+1-p)} \left\| \sum_{j \in I(n)} \left(\int_0^1 t^{j+1}|f(t)| dt \right) z^j \right\|_{H^p}^p \\ &\lesssim \sum_{n=0}^\infty 2^{-n(\alpha+1-p)} \left(\int_0^1 t^{2^{n-1}+1}|f(t)| dt \right)^p \left\| \sum_{j \in I(n)} z^j \right\|_{H^p}^p \\ &\asymp \sum_{n=0}^\infty 2^{n(2p-2-\alpha)} \left(\int_0^1 t^{2^{n-1}+1}|f(t)| dt \right)^p \\ &\asymp \sum_{j=1}^\infty (j+1)^{(2p-3-\alpha)} \left(\int_0^1 t^{j+1}|f(t)| dt \right)^p. \end{aligned}$$

Analogously, it can be proved that

$$\sum_{n=0}^\infty 2^{-n(\alpha+1)} \|\Delta_n \tilde{\mathcal{H}}(f)'\|_{H^p}^p \gtrsim \sum_{j=1}^\infty (j+1)^{(2p-3-\alpha)} \left(\int_0^1 t^{j+1}|f(t)| dt \right)^p.$$

and the assertion of (i) follows.

The proof of (ii) is similar and is omitted. □

We are now ready to prove the sufficient conditions.

Proof of the sufficiency statement in Theorem 4. Let p, α be as in the statement and assume $g \in \Lambda\left(p, \frac{1}{p}\right)$. For $f \in \mathcal{D}_\alpha^p$, bearing in mind that $\mathcal{H}_g(f)'(z) = \mathcal{H}_{g'}(zf)$ and using Theorem D (ii), we obtain

$$\begin{aligned} \|\mathcal{H}_g(f)\|_{\mathcal{D}_\alpha^p}^p &= |\mathcal{H}_g(f)(0)|^p + \|\mathcal{H}_{g'}(zf)\|_{A_\alpha^p}^p \\ &= |\mathcal{H}_g(f)(0)|^p + |\mathcal{H}_{g'}(zf)(0)|^p + \sum_{n=0}^\infty 2^{-n(\alpha+1)} \|\Delta_n \mathcal{H}_{g'}(zf)\|_{Hp}^p. \end{aligned}$$

Now,

$$\begin{aligned} |\mathcal{H}_g(f)(0)|^p + |\mathcal{H}_{g'}(zf)(0)|^p &\leq (|g'(0)|^p + |g''(0)|^p) \left(\int_0^1 |f(t)| dt \right)^p \\ &\leq C(g) \int_0^1 |f(t)|^p dt \leq C(g) \int_0^1 M_\infty^p(t, f)(1-t)^{\alpha-p+1} dt \leq C(g, p, \alpha) \|f\|_{\mathcal{D}_\alpha^p}^p, \end{aligned}$$

where in the last step we have used Lemma 4 and the observation that since $p-2 < \alpha \leq p-1$ we have $-1 < \alpha-p+1 \leq 0$.

On the other hand, if we write $g''(z) = \sum_{k=0}^\infty c_k z^k$, then

$$\mathcal{H}_{g'}(zf)(z) = \sum_{k=0}^\infty c_k \left(\int_0^1 t^{k+1} f(t) dt \right) z^k$$

and we can apply Lemma 7 and part (v) of Theorem A to obtain

$$\begin{aligned} \|\Delta_n \mathcal{H}_{g'}(zf)\|_{Hp}^p &\leq C \left(\int_0^1 t^{2^{n-2}+1} |f(t)| dt \right)^p \|\Delta_n g''\|_{Hp}^p \\ &\leq C \left(\int_0^1 t^{2^{n-2}+1} |f(t)| dt \right)^p 2^{pn(2-\frac{1}{p})} \end{aligned}$$

for $n \geq 3$. Thus

$$\begin{aligned} \sum_{n=3}^\infty 2^{-n(\alpha+1)} \|\Delta_n \mathcal{H}_{g'}(zf)\|_{Hp}^p &\leq C \sum_{n=3}^\infty 2^{n(2p-2-\alpha)} \left(\int_0^1 t^{2^{n-2}+1} |f(t)| dt \right)^p \\ &\asymp C \sum_{n=0}^\infty 2^{(n+1)(2p-2-\alpha)} \left(\int_0^1 t^{2^{n+1}+1} |f(t)| dt \right)^p. \end{aligned}$$

Now it is easy to see that

$$2^n \left(\int_0^1 t^{2^{n+1}+1} |f(t)| dt \right)^p \leq \sum_{j \in I(n)} \left(\int_0^1 t^{j+1} |f(t)| dt \right)^p,$$

and we can continue the above estimate as follows

$$\begin{aligned} &\leq C \sum_{n=0}^\infty 2^{(n+1)(2p-3-\alpha)} \sum_{j \in I(n)} \left(\int_0^1 t^{j+1} |f(t)| dt \right)^p \\ &\asymp C \sum_{j=1}^\infty (j+1)^{(2p-3-\alpha)} \left(\int_0^1 t^{j+1} |f(t)| dt \right)^p \\ &\leq C \left(|\tilde{\mathcal{H}}(f)(0)|^p + \sum_{j=1}^\infty (j+1)^{(2p-3-\alpha)} \left(\int_0^1 t^{j+1} |f(t)| dt \right)^p \right) \asymp \|\tilde{\mathcal{H}}(f)\|_{\mathcal{D}_\alpha^p}^p \leq C \|f\|_{\mathcal{D}_\alpha^p}^p, \end{aligned}$$

where we have used Lemma 8 (i) and Theorem 5 (iii). This together with the inequality for $|\mathcal{H}_g(f)(0)|^p + |\mathcal{H}_{g'}(zf)(0)|^p$ finishes the proof. \square

Proof of the sufficiency statement in Theorem 3. Let p, α be as in the statement and let $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be the power series for g . For $f \in A_{\alpha}^p$, from Theorem D we have

$$(6.6) \quad \|\mathcal{H}_g(f)\|_{A_{\alpha}^p}^p \asymp |\mathcal{H}_g(f)(0)|^p + \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \|\Delta_n \mathcal{H}_g(f)\|_{H^p}^p.$$

Now,

$$|\mathcal{H}_g(f)(0)| \leq |g'(0)| \int_0^1 |f(t)| dt = |g'(0)| \int_0^1 |f(t)|(1-t)^{\frac{\alpha+1}{p}} (1-t)^{-\frac{\alpha+1}{p}} dt$$

and by Hölder's inequality,

$$\leq C(g, p, \alpha) \left(\int_0^1 M_{\infty}^p(t, f)(1-t)^{\alpha+1} dt \right)^{1/p} \leq C(g, p, \alpha) \|f\|_{A_{\alpha}^p},$$

where the last inequality is from Lemma 3.

Now write $g'(z) = \sum_{k=0}^{\infty} c_k z^k$, then

$$\mathcal{H}_g(f)(z) = \sum_{k=0}^{\infty} \left(c_k \int_0^1 t^k f(t) dt \right) z^k.$$

Now Lemma 7 remains valid if we replace the power t^{k+1} appearing in the definition of the function h in statement of the Lemma by t^k , and the power $t^{2^{n-2}+1}$ in the conclusion by $t^{2^{n-2}}$. This variation can be proved in the same way as the original version. Applying the Lemma in this new form and using the assumption for g , we find

$$\|\Delta_n \mathcal{H}_g(f)\|_{H^p}^p \leq C \left(\int_0^1 t^{2^{n-2}} |f(t)| dt \right)^p \|\Delta_n g'\|_{H^p}^p \leq C \left(\int_0^1 t^{2^{n-2}} |f(t)| dt \right)^p 2^{pn(1-\frac{1}{p})}.$$

Now, the proof can be completed as the previous one using Theorem D (ii), Lemma 8 (ii) and Theorem 5 (ii). Namely,

$$\begin{aligned} \sum_{n=3}^{\infty} 2^{-n(\alpha+1)} \|\Delta_n \mathcal{H}_g(f)\|_{H^p}^p &\leq C \sum_{n=3}^{\infty} 2^{n(p-2-\alpha)} \left(\int_0^1 t^{2^{n-2}} |f(t)| dt \right)^p \\ &\asymp C \sum_{n=0}^{\infty} 2^{(n+1)(p-2-\alpha)} \left(\int_0^1 t^{2^{n+1}} |f(t)| dt \right)^p \\ &\leq C \sum_{n=0}^{\infty} 2^{(n+1)(p-3-\alpha)} \sum_{j \in I(n)} \left(\int_0^1 t^j |f(t)| dt \right)^p \\ &\asymp C \sum_{j=1}^{\infty} (j+1)^{(p-3-\alpha)} \left(\int_0^1 t^j |f(t)| dt \right)^p \\ &\leq C \left(|\tilde{\mathcal{H}}(f)(0)|^p + \sum_{j=1}^{\infty} (j+1)^{(p-3-\alpha)} \left(\int_0^1 t^j |f(t)| dt \right)^p \right) \\ &\asymp \|\tilde{\mathcal{H}}(f)\|_{A_{\alpha}^p}^p \leq C \|f\|_{A_{\alpha}^p}^p, \end{aligned}$$

and together with the inequality $|\mathcal{H}_g(f)(0)| \leq C(g, p, \alpha)\|f\|_{A_\alpha^p}$ this finishes the proof. \square

If $p > 1$ and $f \in H^p$, then $\tilde{\mathcal{H}}(f)(z) = \sum_{j=0}^\infty \left(\int_0^1 t^j |f(t)| dt\right) z^j$ is analytic in \mathbf{D} and has nonnegative Taylor coefficients decreasing to zero. Thus Theorem C implies that

$$(6.7) \quad \|\tilde{\mathcal{H}}(f)\|_{\mathcal{D}_{p-1}^p}^p \asymp \|\tilde{\mathcal{H}}(f)\|_{H^p}^p \asymp \sum_{j=0}^\infty (j+1)^{p-2} \left(\int_0^1 t^j |f(t)| dt\right)^p.$$

Proof of the sufficiency statement in Theorem 1. Assume that $1 < p \leq 2$ and $g(z) = \sum_{k=0}^\infty b_k z^k \in \Lambda\left(p, \frac{1}{p}\right)$. Take $f \in H^p$. Since $\mathcal{D}_{p-1}^p \subset H^p$ with domination in the norms, by the proof of Theorem 4 with $\alpha = p - 1$, (6.7) and Theorem 5 (i) we obtain

$$\|\mathcal{H}_g(f)\|_{H^p}^p \leq C\|\mathcal{H}_g(f)\|_{\mathcal{D}_{p-1}^p}^p \leq C\|\tilde{\mathcal{H}}(f)\|_{\mathcal{D}_{p-1}^p}^p \asymp \|\tilde{\mathcal{H}}(f)\|_{H^p}^p \leq C\|f\|_{H^p}^p.$$

Hence $\mathcal{H}_g: H^p \rightarrow H^p$ is bounded. This finishes the proof. \square

Proof of Theorem 2 (ii). Let $2 < p < \infty$ and $g \in \Lambda\left(q, \frac{1}{q}\right)$ for some q with $1 < q < p$. Let $f \in H^p$. Applying [17, Corollary 3.1] to the analytic function $\mathcal{H}_g(f)$ we have,

$$\|\mathcal{H}_g(f)\|_{H^p}^p \leq C \left(|\mathcal{H}_g(f)(0)|^p + \int_0^1 (1-r)^{p\left(1-\frac{1}{q}\right)} M_q^p(r, \mathcal{H}_g(f)') dr \right)$$

where $C = C(p, q)$ is an absolute constant. By Theorem D (i), applied here with $\alpha = 1 - \frac{1}{q} + \frac{1}{p}$ we further have

$$(6.8) \quad \begin{aligned} & \int_0^1 (1-r)^{p\left(1-\frac{1}{q}\right)} M_q^p(r, \mathcal{H}_g(f)') dr \\ & \asymp |\mathcal{H}_g(f)'(0)|^p + \sum_{n=0}^\infty 2^{-n\left(p-\frac{p}{q}+1\right)} \|\Delta_n \mathcal{H}_g(f)'\|_{H^q}^p. \end{aligned}$$

Now for the constant terms of the two relations above it is easy to see, using Hölder’s inequality and the Fejér–Riesz inequality that

$$(6.9) \quad |\mathcal{H}_g(f)(0)|^p + |\mathcal{H}_g(f)'(0)|^p \leq C(g, p)\|f\|_{H^p}^p.$$

To estimate the sum in (6.8) write $g''(z) = \sum_{k=0}^\infty c_k z^k$ so that

$$\mathcal{H}_g(f)'(z) = \mathcal{H}_{g'}(zf)(z) = \sum_{k=0}^\infty \left(c_k \int_0^1 t^{k+1} f(t) dt \right) z^k,$$

and use Lemma 7 and Theorem A (v) to obtain

$$\begin{aligned} & \sum_{n=3}^{\infty} 2^{-n(p(1-\frac{1}{q})+1)} \|\Delta_n \mathcal{H}_g(f)'\|_{H^q}^p \leq C \sum_{n=3}^{\infty} 2^{-n(p-\frac{p}{q}+1)} \left(\int_0^1 t^{2^{n-2}+1} |f(t)| dt \right)^p \|\Delta_n g''\|_{H^q}^p \\ & \leq C \sum_{n=3}^{\infty} 2^{n(p-1)} \left(\int_0^1 t^{2^{n-2}+1} |f(t)| dt \right)^p \leq C \sum_{n=0}^{\infty} 2^{(n+1)(p-1)} \left(\int_0^1 t^{2^{n+1}+1} |f(t)| dt \right)^p \\ & \leq C \sum_{n=0}^{\infty} 2^{(n+1)(p-2)} \sum_{j \in I(n)} \left(\int_0^1 t^{j+1} |f(t)| dt \right)^p \asymp C \sum_{j=0}^{\infty} (j+1)^{p-2} \left(\int_0^1 t^{j+1} |f(t)| dt \right)^p \\ & \asymp \|\tilde{\mathcal{H}}(f)\|_{H^p}^p \leq C \|f\|_{H^p}^p, \end{aligned}$$

where in the last two lines we have used (6.7) and Theorem 5 (ii). This and (6.9) finish the proof. \square

7. Compactness

Let us recall that an operator T acting on a Banach space X is compact if any bounded sequence $\{f_k\}$ of elements of X has a subsequence $\{f_{k_i}\}$ such that $T(f_{k_i})$ converges in X . For the generalized Hilbert operator \mathcal{H}_g acting on the appropriate spaces we have.

Theorem 8. *Suppose that $1 < p < \infty$ and $g \in \mathcal{H}ol(\mathbf{D})$, then:*

- (i) *If $\mathcal{H}_g: H^p \rightarrow H^p$ is compact, then $g \in \lambda(p, \frac{1}{p})$.*
- (ii) *If $1 < p \leq 2$ and $g \in \lambda(p, \frac{1}{p})$, then $\mathcal{H}_g: H^p \rightarrow H^p$ is compact.*
- (iii) *If $2 < p < \infty$ and $g \in \lambda(q, \frac{1}{q})$ for some $1 < q < p$, then $\mathcal{H}_g: H^p \rightarrow H^p$ is compact.*

Theorem 9. *Suppose that $1 < p < \infty$, $-1 < \alpha < p - 2$ and $g \in \mathcal{H}ol(\mathbf{D})$. Then $\mathcal{H}_g: A_\alpha^p \rightarrow A_\alpha^p$ is compact if and only if $g \in \lambda(p, \frac{1}{p})$.*

Theorem 10. *Suppose that $1 < p < \infty$, $p - 2 < \alpha \leq p - 1$ and $g \in \mathcal{H}ol(\mathbf{D})$. Then $\mathcal{H}_g: \mathcal{D}_\alpha^p \rightarrow \mathcal{D}_\alpha^p$ is compact if and only if $g \in \lambda(p, \frac{1}{p})$.*

We shall use the following lemma.

Lemma 9. *Suppose that $1 < p < \infty$ and let X be either H^p , or A_α^p for some α with $-1 < \alpha < p - 2$, or \mathcal{D}_α^p for some α with $p - 2 < \alpha \leq p - 1$. Let $\{f_k\}_{k=1}^\infty$ be a sequence in X satisfying $\sup_k \|f_k\|_X = K < \infty$ and $f_k \rightarrow 0$, as $k \rightarrow \infty$, uniformly on compact subsets of \mathbf{D} . Then:*

- (i) $\lim_{k \rightarrow \infty} \int_0^1 |f_k(t)| dt = 0$.
- (ii) *For every $g \in \mathcal{H}ol(\mathbf{D})$ we have*

$$\mathcal{H}_g(f_k) \rightarrow 0, \quad \text{as } k \rightarrow \infty, \text{ uniformly on compact subsets of } \mathbf{D}.$$

Proof. Let's start with the proof of (i). Let q be the exponent conjugate to p , that is, $\frac{1}{p} + \frac{1}{q} = 1$. Take $\varepsilon > 0$.

Suppose first that $X = H^p$. Take $r_0 \in (0, 1)$ such that $(1 - r_0)^{1/q} < \varepsilon$. By the hypothesis there exists $k_0 \in \mathbf{N}$ such that

$$|f_k(z)| < \varepsilon, \quad \text{if } k \geq k_0 \text{ and } |z| \leq r_0.$$

Then, using Hölder’s inequality and part (i) of Lemma 3, we see that for $k \geq k_0$, we have

$$\begin{aligned} \int_0^1 |f_k(t)| dt &\leq \varepsilon + \int_{r_0}^1 M_\infty(t, f_k) dt \leq \varepsilon + \left(\int_{r_0}^1 M_\infty^p(t, f_k) dt \right)^{1/p} (1 - r_0)^{1/q} \\ &\leq \varepsilon + CK\varepsilon = C'\varepsilon. \end{aligned}$$

Thus (i) holds in this case.

Similarly, if $X = A_\alpha^p$ with $-1 < \alpha < p - 2$, take $r_0 \in (0, 1)$ such that $(1 - r_0)^{\frac{p-\alpha-2}{p}} < \varepsilon$. There exists $k_0 \in \mathbf{N}$ such that

$$|f_k(z)| < \varepsilon, \quad \text{if } k \geq k_0 \text{ and } |z| \leq r_0.$$

Then, using Hölder’s inequality and part (ii) of Lemma 3, we obtain, for $k \geq k_0$,

$$\begin{aligned} \int_0^1 |f_k(t)| dt &\leq \varepsilon + \int_{r_0}^1 |f_k(t)| dt \\ &\leq \varepsilon + \left(\int_{r_0}^1 M_\infty^p(t, f_k)(1 - t)^{\alpha+1} dt \right)^{\frac{1}{p}} \left(\int_{r_0}^1 (1 - t)^{-(\alpha+1)\frac{q}{p}} dt \right)^{\frac{1}{q}} \\ &\leq \varepsilon + K \frac{p}{p - \alpha - 2} (1 - r_0)^{\frac{p-\alpha-2}{p}} \leq C'\varepsilon. \end{aligned}$$

So, we see that (i) holds in this case too.

Finally, suppose that $X = \mathcal{D}_\alpha^p$ for a certain α with $p - 2 < \alpha \leq p - 1$. Since $\alpha - p \leq -1$, we have that $\mathcal{D}_\alpha^p \subset A_\beta^p$ for all $\beta > -1$. Take and fix β with $-1 < \beta < p - 2$. We have $X \subset A_\beta^p$ and then, using the hypothesis and the closed graph theorem, we deduce that $\sup_k \|f\|_{A_\beta^p} < \infty$ and then the result in this case follows from the preceding one.

Part (ii) follows easily from part (i). Indeed, if $g \in \mathcal{H}ol(\mathbf{D})$ and $|z| \leq r < 1$, we have

$$|\mathcal{H}_g(f_k)(z)| = \left| \int_0^1 f_k(t)g'(tz) dt \right| \leq M_\infty(r, g') \int_0^1 |f_k(t)| dt.$$

Thus (ii) holds. □

Now the following result follows easily.

Lemma 10. *Suppose that $1 < p < \infty$ and let X be either H^p , or A_α^p for some α with $-1 < \alpha < p - 2$, or \mathcal{D}_α^p for some α with $p - 2 < \alpha \leq p - 1$. For a function $g \in \mathcal{H}ol(\mathbf{D})$ the following conditions are equivalent:*

- (i) $\mathcal{H}_g: X \rightarrow X$ is compact.
- (ii) If $\{f_k\}_{k=1}^\infty$ is a sequence in X such that

$$(7.1) \quad \sup_k \|f_k\|_X = K < \infty$$

and

$$(7.2) \quad f_k \rightarrow 0, \quad \text{as } k \rightarrow \infty, \text{ uniformly on compact subsets of } \mathbf{D},$$

then $\lim_{k \rightarrow \infty} \|\mathcal{H}_g(f_k)\|_X = 0$.

Proof of Theorem 10. Assume first that $\mathcal{H}_g: \mathcal{D}_\alpha^p \rightarrow \mathcal{D}_\alpha^p$ is compact. Since the family of test functions

$$f_{N,\alpha}(z) = \frac{1}{N^{3-\frac{2+\alpha}{p}}} \frac{1}{(1 - a_N z)^2}, \quad z \in \mathbf{D},$$

considered in (4.8) satisfies (7.1) and (7.2), we have

$$\lim_{N \rightarrow \infty} \|\mathcal{H}_g(f_{N,\alpha})\|_{\mathcal{D}_\alpha^p} = 0.$$

Next, scrutinizing the proof of Theorem 4 (necessity part), we see that the quantity $\|\mathcal{H}_g(f_{N,\alpha})\|_{\mathcal{D}_\alpha^p}$ is incorporated in the constant C_p which appears in the final lines of the argument of the proof. In particular,

$$\|\Delta_n g''\|_{H^p} \leq C'_p (\|\mathcal{H}_g(f_{2^n,\alpha})\|_{\mathcal{D}_\alpha^p}) 2^{n(2-\frac{1}{p})},$$

therefore

$$\lim_{n \rightarrow \infty} \frac{\|\Delta_n g''\|_{H^p}}{2^{n(2-\frac{1}{p})}} = 0,$$

so by Remark 1, $g \in \lambda\left(p, \frac{1}{p}\right)$.

Conversely, let $\varepsilon > 0$ and $g \in \lambda\left(p, \frac{1}{p}\right)$. Suppose $\{f_k\}$ is a sequence of analytic functions in \mathbf{D} satisfying (7.1) and (7.2). Then there exists $n_0 \in \mathbf{N}$ such that

$$\frac{\|\Delta_n g''\|_{H^p}}{2^{n(2-\frac{1}{p})}} < \varepsilon \quad \text{for all } n \geq n_0.$$

Then it follows from the proof of Theorem 4 (sufficiency part) that for all k

$$\|\mathcal{H}_g(f_k)\|_{\mathcal{D}_\alpha^p}^p \lesssim |\mathcal{H}_g(f_k)(0)|^p + \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\int_0^1 t^{2^{n-2}+1} |f_k(t)| dt \right)^p \|\Delta_n g''\|_{H^p}^p.$$

Using Lemma 9 we see that

$$|\mathcal{H}_g(f_k)(0)| = \int_0^1 |f_k(t)| dt \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

On the other hand

$$\begin{aligned} & \sum_{n=0}^{\infty} 2^{-n(\alpha+1)} \left(\int_0^1 t^{2^{n-2}+1} |f_k(t)| dt \right)^p \|\Delta_n g''\|_{H^p}^p \\ & \leq C \sum_{n=0}^{n_0-1} 2^{n(2p-2-\alpha)} \left(\int_0^1 t^{2^{n-2}+1} |f_k(t)| dt \right)^p \\ & \quad + C\varepsilon \sum_{n=n_0}^{\infty} 2^{n(2p-2-\alpha)} \left(\int_0^1 t^{2^{n-2}+1} |f_k(t)| dt \right)^p. \end{aligned}$$

The finite sum above tend to 0 as $k \rightarrow \infty$ by appealing to Lemma 9. The second sum is

$$\begin{aligned} & \sum_{n_0}^{\infty} 2^{n(2p-2-\alpha)} \left(\int_0^1 t^{2^{n-2}+1} |f_k(t)| dt \right)^p \\ & \leq C \sum_{j=1}^{\infty} (j+1)^{(2p-3-\alpha)} \left(\int_0^1 t^{j+1} |f_k(t)| dt \right)^p \\ & \leq C \|\tilde{\mathcal{H}}(f_k)\|_{\mathcal{D}_\alpha^p}^p \leq C \sup_k \|f_k\|_{\mathcal{D}_\alpha^p} \leq CK \end{aligned}$$

by (7.1). This gives

$$\lim_{k \rightarrow \infty} \|\mathcal{H}_g(f_k)\|_{\mathcal{D}_\alpha^p} \leq CK\varepsilon,$$

and since ε is arbitrary the proof is complete. □

Theorem 8 and Theorem 9 can be proved with the same technique. We omit the details.

Finally, we shall prove Theorem 6.

Proof of Theorem 6. We recall that an operator T on a separable Hilbert space H is a Hilbert–Schmidt operator if for an orthonormal basis $\{e_n : n = 0, 1, 2, \dots\}$ of H the sum $\sum_{n=0}^{\infty} \|T(e_n)\|^2$ is finite. The finiteness of this sum does not depend on the basis chosen. The class of Hilbert–Schmidt operators on H is denoted by $S^2(H)$.

(i) The set $\{1, z, z^2, \dots\}$ is a basis of H^2 . If $g(z) = \sum_0^\infty b_k z^k \in \text{Hol}(\mathbf{D})$, then

$$\mathcal{H}_g(z^n) = \int_0^1 t^n g'(tz) dt = \sum_{k=0}^{\infty} \frac{(k+1)b_{k+1}}{n+k+1} z^k,$$

thus

$$\|\mathcal{H}_g(z^n)\|_{H^2}^2 = \sum_{k=0}^{\infty} \frac{(k+1)^2 |b_{k+1}|^2}{(n+k+1)^2}$$

and

$$\begin{aligned} \sum_{n=0}^{\infty} \|\mathcal{H}_g(z^n)\|_{H^2}^2 &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(k+1)^2 |b_{k+1}|^2}{(n+k+1)^2} \\ &= \sum_{k=0}^{\infty} (k+1)^2 |b_{k+1}|^2 \sum_{n=0}^{\infty} \frac{1}{(n+k+1)^2} \\ &\sim \sum_{k=0}^{\infty} (k+1)^2 |b_{k+1}|^2 \frac{1}{k+1} \\ &= \sum_{k=0}^{\infty} (k+1) |b_{k+1}|^2 \sim \|g\|_{\mathcal{D}}^2. \end{aligned}$$

Thus $\mathcal{H}_g \in S^2(H^2)$ if and only if $g \in \mathcal{D}$.

(ii) On A_α^2 , $-1 < \alpha < 0$, an orthonormal basis is

$$\{e_n(z) = c_n z^n : n = 0, 1, 2, \dots\},$$

where

$$c_n = \frac{1}{\|z^n\|_{A_\alpha^2}} = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)}}.$$

Now

$$\mathcal{H}_g(e_n)(z) = c_n \mathcal{H}_g(z^n) = c_n \sum_{k=0}^{\infty} \frac{(k+1)b_{k+1}}{n+k+1} z^k$$

and

$$\|\mathcal{H}_g(e_n)\|_{A_\alpha^2}^2 = c_n^2 \sum_{k=0}^{\infty} \frac{k!\Gamma(2+\alpha)}{\Gamma(k+2+\alpha)} \frac{(k+1)^2|b_{k+1}|^2}{(n+k+1)^2}.$$

Thus using the Stirling formula estimate $\frac{\Gamma(n+\beta)}{n!} \sim (n+1)^{\beta-1}$ we have

$$\begin{aligned} \sum_{n=0}^{\infty} \|\mathcal{H}_g(e_n)\|_{A_\alpha^2}^2 &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{n!\Gamma(2+\alpha)} \frac{k!\Gamma(2+\alpha)}{\Gamma(k+2+\alpha)} \frac{(k+1)^2|b_{k+1}|^2}{(n+k+1)^2} \\ &\sim \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n+1)^{\alpha+1}}{(k+1)^{\alpha+1}} \frac{(k+1)^2|b_{k+1}|^2}{(n+k+1)^2} \\ &= \sum_{k=0}^{\infty} (k+1)^{1-\alpha} |b_{k+1}|^2 \sum_{n=0}^{\infty} \frac{(n+1)^{\alpha+1}}{(n+k+1)^2}. \end{aligned}$$

Now a calculation shows that the asymptotic order of the inside series is

$$\sum_{n=0}^{\infty} \frac{(n+1)^{\alpha+1}}{(n+k+1)^2} \sim (k+1)^\alpha,$$

and it follows that

$$\sum_{n=0}^{\infty} \|\mathcal{H}_g(e_n)\|_{A_\alpha^2}^2 \sim \sum_{k=0}^{\infty} (k+1)|b_{k+1}|^2 \sim \|g\|_{\mathcal{D}}^2.$$

(iii) On \mathcal{D}_α^2 , $0 < \alpha \leq 1$, an orthonormal basis is

$$\{e_n\} = \{1, d_1 z, d_2 z^2, \dots\},$$

where

$$d_n = \frac{1}{\|z^n\|_{\mathcal{D}_\alpha^2}} = \frac{1}{n} \sqrt{\frac{\Gamma(n-1+2+\alpha)}{(n-1)!\Gamma(2+\alpha)}}.$$

In this case we find (omitting the details)

$$\begin{aligned} \sum_{n=0}^{\infty} \|\mathcal{H}_g(e_n)\|_{\mathcal{D}_\alpha^2}^2 &\sim \sum_{k=0}^{\infty} (k+1)^{(3-\alpha)} |b_{k+2}|^2 \sum_{n=0}^{\infty} \frac{(n+1)^{\alpha-1}}{(n+k+1)^2} \\ &\sim \sum_{k=0}^{\infty} (k+1)^{(3-\alpha)} |b_{k+2}|^2 (k+1)^{(\alpha-2)} \\ &\sim \sum_{k=0}^{\infty} (k+1) |b_{k+1}|^2 \sim \|g\|_{\mathcal{D}}^2, \end{aligned}$$

and the assertion follows. □

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