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JOINTLY MAXIMAL PRODUCTS IN WEIGHTED GROWTH SPACES

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Abstract. It is shown that for any positive, non-decreasing, continuous and unbounded doubling function ω on $[0,1)$, there exist two analytic infinite products f_0 and f_1 such that the asymptotic relation $|f_0(z)|+|f_1(z)| \asymp \omega(|z|)$ is satisfied for all z in the unit disc. It is also shown that both functions f_j for $j=0,1$ satisfy $T(r,f_j) \asymp \log\omega(r)$, as $r \to 1^-$, and hence give examples of analytic functions for which the Nevanlinna characteristic admits the regular slow growth induced by ω.

1. Introduction and results

Let $\mathcal{H}(\mathbf{D})$ denote the algebra of all analytic functions in the unit disc $\mathbf{D}$ of the complex plane $\mathbf{C}$. To consider the growth and the zero distribution of functions in $\mathcal{H}(\mathbf{D})$, we use the following classical notation. The *non-integrated counting function* $n(r,f,0)$ counts the zeros of f in $\{z \in \mathbf{C} \colon |z| \le r\}$ according to multiplicities. Quantities $M_\infty(r,f)$, $M_p(r,f)$, where $0<p<\infty$, $N(r,f,a)$, where $a \in \mathbf{C}$, and $T(r,f)$ denote the *maximum modulus* of f, the *L^p-mean* of f, the *integrated counting function of a-points* of f and the *Nevanlinna characteristic* of f, respectively. We also employ the notation $a \asymp b$, which is equivalent to the conditions $a \lesssim b$ and $b \lesssim a$, where the former means that there exists a constant $C>0$ such that $a \le Cb$, and the latter is defined analogously.

Let $\omega \colon [0,1) \to (0,\infty)$ be non-decreasing, continuous and unbounded. Such a function ω is said to be *doubling*, if there exists a constant $B>1$ such that

$$\omega(1-r/2) \le B\,\omega(1-r), \quad 0<r\le 1. \tag{1}$$

The following result shows that for any doubling function ω there exist two jointly maximal products in the sense that the sum of their moduli behaves asymptotically as $\omega(|z|)$ in $\mathbf{D}$.

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Theorem 1. *Let $\omega\colon [0,1) \to (0,\infty)$ be doubling. Then, there exist $f_0, f_1 \in \mathcal{H}(\mathbf{D})$ such that*

$$|f_0(z)| + |f_1(z)| \asymp \omega(|z|), \quad z \in \mathbf{D}, \tag{2}$$

where both functions f_j for $j = 0, 1$ satisfy $n(r, f_j, 0) = \mathrm{O}\left((1-r)^{-1}\right)$, as $r \to 1^-$. Moreover,

$$M_p(r, f_j) \asymp \omega(r), \quad r \to 1^-, \tag{3}$$

for all $0 < p \le \infty$, and

$$T(r, f_j) \asymp N(r, f_j, a) \asymp \log \omega(r), \quad r \to 1^-, \tag{4}$$

for all $a \in \mathbf{C}$.

The main advantage of our self-contained and constructive proof of Theorem 1 compared to the existing literature is that the zero distribution of the products f_0 and f_1 is explicit. These products are similar to those applied to study the zero distribution of functions in weighted Bergman spaces [7, Section 3]. Note also that our argument gives an alternative way to prove [7, Theorem 3.15], whose original proof is based on certain lacunary series. In fact, Theorem 1 generalizes [7, Theorem 3.15] to doubling functions.

The existence of $f_0, f_1 \in \mathcal{H}(\mathbf{D})$ such that the asymptotic relation (2) is satisfied for a given doubling function ω was recently proved in [1, Lemma 1]. To see that this result is equivalent to [6, Theorem 1.1], that was published essentially at the same time as [1], we may argue as follows. If ω is doubling, then $\psi(x) = \omega(1-1/x)$ is almost subnormal; see [6] for the definitions. Conversely, if ψ is almost subnormal, then by means of [6, Lemma 2.1] there exists a function $\phi \asymp \psi$ such that $\omega(r) = \phi(1/(1-r))$ is doubling. Proofs of [1, Lemma 1] and [6, Theorem 1.1] rest upon the use of lacunary series, which have been the key tool to solve similar problems in the existing literature. The pioneering result [8, Proposition 5.4], which concerns (2) for $\omega(r) = 1/(1-r)$, have been a source of inspiration for several authors. For example, [4, Theorem 1.2] proves (2) for $\omega(r) = -\log(1-r)$. It is well known that (2) has many applications in the operator theory; for details, we refer to [1, 6].

Construction of a product whose Nevanlinna characteristic admits a pregiven asymptotic growth has been studied by several authors. In particular, it is known that whenever $\Lambda(r)$ exceeds the growth of $-\log(1-r)$ as $r \to 1^-$, then there exists a product f whose Nevanlinna characteristic behaves asymptotically as $\Lambda(r)$ [9, Theorem 1]. The asymptotic formula (4) shows that we can find an infinite analytic product in $\mathbf{D}$ such that its Nevanlinna characteristic grows asymptotically as the logarithm of a pregiven doubling ω, and hence we can prescribe characteristics growing slower than $-\log(1-r)$ as $r \to 1^-$. The method and the construction in the proof of Theorem 1 are different from those employed in [9].

The current state of the development concerning complex linear differential equation $f'' + A(z)f = 0$ in $\mathbf{D}$ allows us to deduce a significant amount of information on solutions f, whenever we can analyze the coefficient A in detail. If we take A to be one of the functions f_0 and f_1 in the Theorem 1, then we get an important and intriguing family of examples of such differential equations. These particular equations are way too complicated to be solved explicitly, but the growth and the oscillation of their solutions are well understood due to the asymptotic properties satisfied by the coefficient A. To be brief with regards to this matter, we settle to mention two cases in the recent literature of which the first one concerns polynomial regular functions. This class of regularly growing analytic functions in $\mathbf{D}$ arises naturally in the theory of ODEs [3]. In the sense of linear differential equations, polynomial regular functions play a similar role in the unit disc as polynomials do in the complex plane. For a more general example, see [2].

2. Proof of Theorem 1

The proof of Theorem 1 is divided in several steps. The point of departure is the construction of the infinite products $f_0, f_1 \in \mathcal{H}(\mathbf{D})$, which is followed by a discussion of their growth. Finally, we consider the asserted asymptotic properties of the products f_0 and f_1.

2.1. Construction of the products. Before going into the details of the construction, we note the following lemmas on doubling functions.

Lemma 2. *Let $\omega\colon [0,1) \to (0,\infty)$ be doubling. If $B > 1$ is the constant in (1), then*

$$\omega(t) \le C\left(\frac{1-r}{1-t}\right)^{\alpha}\omega(r), \quad 0 \le r \le t < 1, \tag{5}$$

where $C = \max\left\{B\,\omega(1/2)/\omega(0), B^2\right\}$ and $\alpha = \log_2 B$.

Conversely, it is obvious that, if $\omega : [0,1) \to (0,\infty)$ is non-decreasing, continuous, unbounded and it satisfies (5) for some $C > 1$ and $\alpha > 0$, then ω must be doubling.

Proof of Lemma 2. Since ω is doubling, (1) implies $\omega(t) \le B\,\omega(2t-1)$ for all $t \in [2^{-1}, 1)$. Moreover, if $0 \le r \le t < 1$, then there exist unique constants $j, k \in \mathbf{N} \cup \{0\}$, $j \ge k$, such that $t \in \left[1-2^{-j}, 1-2^{-j-1}\right)$ and $r \in \left[1-2^{-k}, 1-2^{-k-1}\right)$. If $k = 0$, then

$$\omega(t) \le B^j\omega\big(2^j(t-1)+1\big) \le \frac{B^j\omega(1/2)}{\omega(0)}\omega(r) \le \frac{B\,\omega(1/2)}{\omega(0)}\left(\frac{1-r}{1-t}\right)^{\log_2 B}\omega(r),$$

while if $k > 0$, then

$$\omega(t) \le B^{j-k+1}\omega\big(2^{j-k+1}(t-1)+1\big) \le B^{j-k+1}\omega(r) \le B^2\left(\frac{1-r}{1-t}\right)^{\log_2 B}\omega(r).$$

The assertion follows. □

The second lemma introduces a sequence of natural numbers depending on the growth of the doubling function ω. This sequence is the foundation of our construction.

Lemma 3. *Let $\omega\colon [0,1) \to (0,\infty)$ be doubling. Then, there exist a sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers, real constants λ and μ, and a constant $d \in (0,1)$ such that the sequence $\{a_k\}_{k=1}^{\infty}$, defined by*

$$a_k = \frac{\omega\big(1-1/n_{k+2}\big)}{\omega\big(1-1/n_k\big)}, \quad k \in \mathbf{N},$$

satisfies

$$1 < \lambda \le a_k \le \mu < \infty, \quad \frac{\log a_{k+1}}{\log a_k} < d\,\frac{n_{k+1}}{n_k}, \quad k \in \mathbf{N}. \tag{6}$$

Proof. Let $\alpha > 0$ and $C > 1$ be the constants ensured by Lemma 2. Now, let γ be a sufficiently large real constant such that

$$2^{\gamma-\alpha} C^{-1} > 1, \quad \frac{2\gamma + \alpha + \log_2 C}{2\gamma - \alpha - \log_2 C} < \frac{1}{2^{1/\alpha}} \left(\frac{2^{\gamma/\alpha}}{C^{1/\alpha}} - 1 \right). \tag{7}$$

Take $t_1 = 1/2$, and define the sequence $\{t_k\}_{k=1}^{\infty}$ inductively by $\omega(t_{k+1})/\omega(t_k) = 2^{\gamma}$ for $k \in \mathbf{N}$. Let $n_k = \operatorname{floor}\big((1-t_k)^{-1}\big)$, where $\operatorname{floor}(x) = \max\{n \in \mathbf{N} : n \le x\}$. By means of Lemma 2, and the estimates $2^{-1} \le n_k(1-t_k) \le 1$, we may define $1 < \lambda < \mu < \infty$ by

$$a_k \le \frac{C\,\omega(t_{k+2})}{\big(n_k(1-t_k)\big)^{\alpha}\omega(t_k)} \le 2^{\alpha} C \, \frac{\omega(t_{k+2})}{\omega(t_{k+1})} \, \frac{\omega(t_{k+1})}{\omega(t_k)} = 2^{2\gamma+\alpha} C = \mu,$$

and

$$a_k \ge \frac{\big(n_{k+2}(1-t_{k+2})\big)^{\alpha}\omega(t_{k+2})}{C\,\omega(t_k)} \ge \frac{1}{2^{\alpha} C} \, \frac{\omega(t_{k+2})}{\omega(t_{k+1})} \, \frac{\omega(t_{k+1})}{\omega(t_k)} = 2^{2\gamma-\alpha} C^{-1} = \lambda,$$

since these inequalities hold for all $k \in \mathbf{N}$. By Lemma 2 we conclude

$$\begin{aligned} \frac{n_{k+1}}{n_k} &> \left(\frac{1}{1-t_{k+1}} - 1 \right)(1-t_k) > \frac{1-t_k}{1-t_{k+1}} - 1 \\ &\ge \frac{1}{C^{1/\alpha}} \left(\frac{\omega(t_{k+1})}{\omega(t_k)} \right)^{1/\alpha} - 1 = \frac{2^{\gamma/\alpha}}{C^{1/\alpha}} - 1, \quad k \in \mathbf{N}, \end{aligned} \tag{8}$$

and further by (7), we have

$$\frac{\log a_{k+1}}{\log a_k} \le \frac{\log \mu}{\log \lambda} = \frac{2\gamma + \alpha + \log_2 C}{2\gamma - \alpha - \log_2 C} < \frac{1}{2^{1/\alpha}} \left(\frac{2^{\gamma/\alpha}}{C^{1/\alpha}} - 1 \right) < \frac{1}{2^{1/\alpha}} \, \frac{n_{k+1}}{n_k}.$$

This confirms the last inequality in (6) for $d = 2^{-1/\alpha}$. □

Let $\{n_k\}_{k=1}^{\infty}$ be the sequence ensured by Lemma 3, and define

$$f_j(z) = \prod_{k=1}^{\infty} \frac{1 + a_{2k+j} z^{n_{2k+j}}}{1 + a_{2k+j}^{-1} z^{n_{2k+j}}}, \quad z \in \mathbf{D}, \quad j = 0, 1.$$

Evidently both functions f_j belong to $\mathcal{H}(\mathbf{D})$, since all factors are bounded functions in $\mathbf{D}$, and according to (6) the sum

$$\sum_{k=1}^{\infty} \left| \frac{1 + a_{2k+j} z^{n_{2k+j}}}{1 + a_{2k+j}^{-1} z^{n_{2k+j}}} - 1 \right| \le \sum_{k=1}^{\infty} \frac{a_{2k+j} - a_{2k+j}^{-1}}{1 - a_{2k+j}^{-1}} \, |z|^{n_{2k+j}} \le (1+\mu) \sum_{k=1}^{\infty} |z|^{n_{2k+j}}$$

converges uniformly on compact subsets of $\mathbf{D}$.

2.2. Growth estimates for the maximum modulus of the products. To estimate the growth of f_j for $j = 0, 1$ we define $r_{2m+j} = e^{-1/n_{2m+j}}$ for $m \in \mathbf{N}$, and write

$$|f_j(z)| = \left| \prod_{k=1}^{m} a_{2k+j} \frac{a_{2k+j}^{-1} + z^{n_{2k+j}}}{1 + a_{2k+j}^{-1} z^{n_{2k+j}}} \right| \left| \prod_{k=1}^{\infty} \frac{1 + a_{2(m+k)+j} z^{n_{2(m+k)+j}}}{1 + a_{2(m+k)+j}^{-1} z^{n_{2(m+k)+j}}} \right|. \tag{9}$$

First, we prove that the infinite subproduct in (9) is bounded in $\mathbf{D}$. To this end, let $\tau = 2^{\gamma/\alpha} C^{-1/\alpha} - 1$ be the lower bound in (8). According to (7) we know that $\tau > 1$, and

$$\frac{n_{2(m+k)+j}}{n_{2m+j}} = \frac{n_{2(m+k)+j}}{n_{2(m+k)+j-1}} \cdots \frac{n_{2m+j+1}}{n_{2m+j}} \geq \tau^{2k}, \quad k, m \in \mathbf{N}. \tag{10}$$

Since $h_1(x) = (y+x)/(1+yx)$ is increasing on $[0,1)$ for each $y \in [0,1)$, we obtain

$$\begin{aligned} \left| \frac{1 + a_{2(m+k)+j} z^{n_{2(m+k)+j}}}{1 + a^{-1}_{2(m+k)+j} z^{n_{2(m+k)+j}}} \right| &= a_{2(m+k)+j} \left| \frac{a^{-1}_{2(m+k)+j} + z^{n_{2(m+k)+j}}}{1 + a^{-1}_{2(m+k)+j} z^{n_{2(m+k)+j}}} \right| \\ &\leq a_{2(m+k)+j} \frac{a^{-1}_{2(m+k)+j} + |z|^{n_{2(m+k)+j}}}{1 + a^{-1}_{2(m+k)+j} |z|^{n_{2(m+k)+j}}} \\ &< \frac{1 + a_{2(m+k)+j} \left(\frac{1}{e}\right)^{\frac{n_{2(m+k)+j}}{n_{2m+j}}}}{1 + a^{-1}_{2(m+k)+j} \left(\frac{1}{e}\right)^{\frac{n_{2(m+k)+j}}{n_{2m+j}}}} \end{aligned} \tag{11}$$

for $|z| < r_{2m+j}$ and $k, m \in \mathbf{N}$. Moreover, since $h_2(x,y) = (1+xy)/(1+x^{-1}y)$ is increasing in both variables, provided that $x > 1$ and $0 \leq y < 1$, estimates (6), (10) and (11) imply

$$\left| \prod_{k=1}^{\infty} \frac{1 + a_{2(m+k)+j} z^{n_{2(m+k)+j}}}{1 + a^{-1}_{2(m+k)+j} z^{n_{2(m+k)+j}}} \right| < \prod_{k=1}^{\infty} \frac{1 + \mu \left(\frac{1}{e}\right)^{\tau^{2k}}}{1 + \mu^{-1} \left(\frac{1}{e}\right)^{\tau^{2k}}} \leq C^{\star} < \infty, \tag{12}$$

for $|z| < r_{2m+j}$ and $m \in \mathbf{N}$, where $C^{\star} > 0$ is a constant independent of $m \in \mathbf{N}$. Second, we proceed to derive an upper estimate for the maximum modulus of f_j. By means of (6), (9), (12) and the inequality $1 - x \leq e^{-x}$ for $x \geq 0$, we get

$$\begin{aligned} |f_j(z)| &< C^{\star} \prod_{k=1}^{m} a_{2k+j} = C^{\star} \frac{\omega\big(1 - 1/n_{2(m+1)+j}\big)}{\omega\big(1 - 1/n_{2+j}\big)} \leq C^{\star} \mu \frac{\omega\big(1 - 1/n_{2m+j}\big)}{\omega\big(1 - 1/n_{2+j}\big)} \\ &\leq C^{\star} \mu \frac{\omega(r_{2m+j})}{\omega\big(1 - 1/n_{2+j}\big)}, \quad |z| < r_{2m+j}, \quad m \in \mathbf{N}. \end{aligned} \tag{13}$$

If $|z| \geq r_{2+j}$, then $r_{2(m-1)+j} \leq |z| < r_{2m+j}$ for some $m \in \mathbf{N} \setminus \{1\}$. Note that by (10) there exists $t \in \mathbf{N}$ such that $n_{2(m+t)+j} > 2\, n_{2m+j}$ for all $m \in \mathbf{N}$. Since $e^{-x} \leq 1 - x/2$ for $0 \leq x \leq 1$, we conclude

$$r_{2m+j} \leq 1 - (2\, n_{2m+j})^{-1} < 1 - 1/n_{2(m+t)+j}, \quad m \in \mathbf{N}.$$

Then (6), (13) and the inequality $1 - x \leq e^{-x}$ for $0 \leq x \leq 1$, give

$$\begin{aligned} |f_j(z)| &< C^{\star} \mu \frac{\omega\big(1 - 1/n_{2(m+t)+j}\big)}{\omega\big(1 - 1/n_{2+j}\big)} \leq C^{\star} \mu^{2+t} \frac{\omega\big(1 - 1/n_{2(m-1)+j}\big)}{\omega\big(1 - 1/n_{2+j}\big)} \\ &\leq C^{\star} \mu^{2+t} \frac{\omega\big(r_{2(m-1)+j}\big)}{\omega\big(1 - 1/n_{2+j}\big)} \leq C^{\star} \mu^{2+t} \frac{\omega\big(|z|\big)}{\omega\big(1 - 1/n_{2+j}\big)}. \end{aligned}$$

Consequently, the maximum modulus of f_j satisfies

$$M_{\infty}(r, f_j) = \max_{|z|=r} \big| f_j(z) \big| \lesssim \omega(r), \quad 0 \leq r < 1. \tag{14}$$

2.3. Growth estimates for the minimum modulus of the products. The following discussion shows that the difference between the maximum modulus and the minimum modulus of f_j for $j = 0, 1$ is small in a large subset of the unit disc. Define $E_j = \bigcup_{m=1}^{\infty} I_{2m+j}$, where I_{2m+j} is the closed interval whose endpoints are

$$\min I_{2m+j} = \left(a_{2m+j}^{-n_{2m+j}^{-1}}\right)^{1-\delta} \left(a_{2(m+1)+j}^{-n_{2(m+1)+j}^{-1}}\right)^{\delta}, \quad m \in \mathbf{N},$$

and

$$\max I_{2m+j} = \left(a_{2m+j}^{-n_{2m+j}^{-1}}\right)^{\delta \frac{n_{2m+j}}{n_{2m+1+j}}} \left(a_{2(m+1)+j}^{-n_{2(m+1)+j}^{-1}}\right)^{1-\delta \frac{n_{2m+j}}{n_{2m+1+j}}}, \quad m \in \mathbf{N}.$$

Here $0 < \delta < 1$ is a sufficiently small constant, which is to be determined later. According to (6) all elements in the sequence $\{a_m^{-1/n_m}\}_{m=1}^{\infty}$ belong to the interval $(0, 1)$, this sequence is strictly increasing, and it converges to 1, as $m \to \infty$. Moreover, $I_{2m+j} \subset \left(a_{2m+j}^{-1/n_{2m+j}}, a_{2(m+1)+j}^{-1/n_{2(m+1)+j}}\right)$ for all $m \in \mathbf{N}$. First, we prove that the infinite subproduct in (9) is uniformly bounded away from zero for $|z| \in E_j$. If $|z| \in I_{2m+j}$, then $|z|^{n_{2(m+k)+j}} < a_{2(m+k)+j}^{-1}$ for all $k \in \mathbf{N}$, and therefore

$$(15) \qquad \begin{aligned} \left| \frac{1 + a_{2(m+k)+j} z^{n_{2(m+k)+j}}}{1 + a_{2(m+k)+j}^{-1} z^{n_{2(m+k)+j}}} \right| &\geq a_{2(m+k)+j} \frac{a_{2(m+k)+j}^{-1} - |z|^{n_{2(m+k)+j}}}{1 - a_{2(m+k)+j}^{-1} |z|^{n_{2(m+k)+j}}} \\ &= \frac{1 - a_{2(m+k)+j} |z|^{n_{2(m+k)+j}}}{1 - a_{2(m+k)+j}^{-1} |z|^{n_{2(m+k)+j}}} \end{aligned}$$

for $|z| \in I_{2m+j}$ and k, $m \in \mathbf{N}$. Since $h_3(x, y) = (1 - xy)/(1 - x^{-1}y)$ is decreasing in both variables, when $x > 1$ and $0 \leq y < 1$, estimates (6), (10) and (15) imply that there exists a constant $C^* > 0$, independent of $m \in \mathbf{N}$, such that

$$(16) \qquad \begin{aligned} &\left| \prod_{k=1}^{\infty} \frac{1 + a_{2(m+k)+j} z^{n_{2(m+k)+j}}}{1 + a_{2(m+k)+j}^{-1} z^{n_{2(m+k)+j}}} \right| \\ &\geq \prod_{k=1}^{\infty} \frac{1 - a_{2(m+k)+j} \left(a_{2m+j}^{-\delta} \, a_{2(m+1)+j}^{-\frac{n_{2m+1+j}}{n_{2(m+1)+j}} \left(1 - \delta \frac{n_{2m+j}}{n_{2m+1+j}}\right)} \right)^{\frac{n_{2(m+k)+j}}{n_{2m+1+j}}}}{1 - a_{2(m+k)+j}^{-1} \left(a_{2m+j}^{-\delta} \, a_{2(m+1)+j}^{-\frac{n_{2m+1+j}}{n_{2(m+1)+j}} \left(1 - \delta \frac{n_{2m+j}}{n_{2m+1+j}}\right)} \right)^{\frac{n_{2(m+k)+j}}{n_{2m+1+j}}}} \\ &\geq \prod_{k=1}^{\infty} \frac{1 - \mu \left(\lambda^{-\delta}\right)^{\tau^{2k-1}}}{1 - \mu^{-1} \left(\lambda^{-\delta}\right)^{\tau^{2k-1}}} \geq C^*, \quad |z| \in I_{2m+j}, \quad m \in \mathbf{N}. \end{aligned}$$

Second, we proceed to estimate the minimum modulus of f_j on E_j. Note that the last inequality in (6) implies

$$(17) \qquad \begin{aligned} a_{2k+j} |z|^{n_{2k+j}} &\geq a_{2k+j} \left(a_{2m+j}^{-\frac{1}{n_{2m+j}}(1-\delta)} \, a_{2(m+1)+j}^{-\frac{1}{n_{2(m+1)+j}}\delta} \right)^{n_{2k+j}} \\ &= \frac{a_{2k+j}}{a_{2m+j}^{\frac{n_{2k+j}}{n_{2m+j}}(1-\delta)} \, a_{2(m+1)+j}^{\frac{n_{2k+j}}{n_{2(m+1)+j}}\delta}} \geq \frac{a_{2k+j}}{a_{2k+j}^{d^{2(m-k)}(1-\delta)} \, a_{2k+j}^{d^{2(m+1-k)}\delta}} \\ &\geq \frac{a_{2k+j}}{a_{2k+j}^{1-\delta} \, a_{2k+j}^{d\delta}} = a_{2k+j}^{\delta(1-d)} \geq \lambda^{\delta(1-d)} > 1 \end{aligned}$$

for $|z| \in I_{2m+j}$ when $1 \le k \le m$, and $m \in \mathbf{N}$; in particular, $|z|^{n_{2k+j}} > a_{2k+j}^{-1}$. Moreover, choose $t \in \mathbf{N}$ sufficiently large such that $1 - \lambda^{-1} \ge \tau^{-2t}$. Since $h_4(x) = 1 - (1-a)x^{-1} - a^{1/x} \ge 0$ for all $x \in [1, \infty)$, provided that $a \in (0,1)$, by applying (6) and (10), we obtain

$$\tag{18} \begin{aligned} |z| &\le a_{2(m+1)+j}^{-1/n_{2(m+1)+j}} \le \lambda^{-1/n_{2(m+1)+j}} \le 1 - \left(1 - \lambda^{-1}\right) n_{2(m+1)+j}^{-1} \\ &\le 1 - \tau^{-2t} n_{2(m+1)+j}^{-1} \le 1 - 1/n_{2(m+1+t)+j}, \quad |z| \in I_{2m+j}, \quad m \in \mathbf{N}. \end{aligned}$$

Therefore (9), (16) and (18) yield

$$\tag{19} \begin{aligned} |f_j(z)| &\ge C^* \prod_{k=1}^{m} a_{2k+j} \left| \frac{a_{2k+j}^{-1} + z^{n_{2k+j}}}{1 + a_{2k+j}^{-1} z^{n_{2k+j}}} \right| \\ &= C^* \frac{\omega\big(1 - 1/n_{2(m+1)+j}\big)}{\omega\big(1 - 1/n_{2+j}\big)} \prod_{k=1}^{m} \left| \frac{a_{2k+j}^{-1} + z^{n_{2k+j}}}{1 + a_{2k+j}^{-1} z^{n_{2k+j}}} \right| \\ &\ge C^* \frac{\omega\big(1 - 1/n_{2(m+1+t)+j}\big)}{\mu^t\, \omega\big(1 - 1/n_{2+j}\big)} \prod_{k=1}^{m} \frac{|z|^{n_{2k+j}} - a_{2k+j}^{-1}}{1 - a_{2k+j}^{-1} |z|^{n_{2k+j}}} \\ &\ge C^* \frac{\omega(|z|)}{\mu^t\, \omega\big(1 - 1/n_{2+j}\big)} \prod_{k=1}^{m} \frac{|z|^{n_{2k+j}} - a_{2k+j}^{-1}}{1 - a_{2k+j}^{-1} |z|^{n_{2k+j}}} \end{aligned}$$

for $|z| \in I_{2m+j}$ and $m \in \mathbf{N}$. For our purposes, it suffices to show that the product in the last line of (19) is uniformly bounded away from zero for $|z| \in E_j$. To simplify computations, we prove that the reciprocal of this product is uniformly bounded for such values of z. Now, since $\log x \le x - 1$ for $x \ge 1$, we have

$$\begin{aligned} \prod_{k=1}^{m} \frac{1 - a_{2k+j}^{-1} |z|^{n_{2k+j}}}{|z|^{n_{2k+j}} - a_{2k+j}^{-1}} &= \exp\left(\sum_{k=1}^{m} \log \frac{1 - a_{2k+j}^{-1} |z|^{n_{2k+j}}}{|z|^{n_{2k+j}} - a_{2k+j}^{-1}} \right) \\ &\le \exp\left((1+\mu) \sum_{k=1}^{m} \frac{1 - |z|^{n_{2k+j}}}{a_{2k+j} |z|^{n_{2k+j}} - 1} \right) \end{aligned}$$

for $|z| \in I_{2m+j}$ and $m \in \mathbf{N}$. By means of (10), (17), and the estimate $1 - e^{-x} \le x$ for $x \ge 0$, we conclude

$$\begin{aligned} \prod_{k=1}^{m} \frac{1 - a_{2k+j}^{-1} |z|^{n_{2k+j}}}{|z|^{n_{2k+j}} - a_{2k+j}^{-1}} &\le \exp\left(\frac{1+\mu}{\lambda^{\delta(1-d)} - 1} \sum_{k=1}^{m} (1 - |z|^{n_{2k+j}}) \right) \\ &\le \exp\left(\frac{1+\mu}{\lambda^{\delta(1-d)} - 1} \sum_{k=1}^{m} \left(1 - a_{2m+j}^{-n_{2k+j}/n_{2m+j}}\right) \right) \\ &\le \exp\left(\frac{1+\mu}{\lambda^{\delta(1-d)} - 1} \log \mu \sum_{k=1}^{m} \frac{n_{2k+j}}{n_{2m+j}} \right) \\ &\le \exp\left(\frac{1+\mu}{\lambda^{\delta(1-d)} - 1} \log \mu \sum_{k=0}^{\infty} \frac{1}{\tau^k} \right) \end{aligned}$$

for $|z| \in I_{2m+j}$ and $m \in \mathbf{N}$, which gives the desired uniform lower bound for the product in the last line of (19). Hence, by (14) and (19), we get

$$|f_j(z)| \asymp \omega(|z|), \quad |z| \in E_j = \bigcup_{m=1}^{\infty} I_{2m+j}. \tag{20}$$

2.4. The covering property of the sets where the products are maximal. It remains to prove that the sets E_0 and E_1 induce a covering of $[\min I_2, 1)$. Note that the closed intervals $\{I_{2m}\}_{m=1}^{\infty}$ are pairwise disjoint, which is also true for $\{I_{2m+1}\}_{m=1}^{\infty}$. Consequently, it is sufficient to show that

$$\min I_{2m+1} \le \max I_{2m}, \quad \min I_{2(m+1)} \le \max I_{2m+1}, \quad m \in \mathbf{N}. \tag{21}$$

We proceed to prove the first inequality in (21). By the definition of I_{2m+j}, the first inequality in (21) is equivalent to

$$a_{2m+1}^{-\frac{1-\delta}{n_{2m+1}}} a_{2m+3}^{-\frac{\delta}{n_{2m+3}}} \le a_{2m}^{-\frac{\delta}{n_{2m+1}}} a_{2m+2}^{-\frac{1}{n_{2m+2}}\left(1-\delta\frac{n_{2m}}{n_{2m+1}}\right)}, \quad m \in \mathbf{N}. \tag{22}$$

By taking the logarithm to the base a_{2m} on the both sides of (22), and then solving the resulting inequality with respect to δ, we conclude that the first inequality in (22) is valid if and only if $\delta \le T(m)$ for all $m \in \mathbf{N}$, where

$$T(m) = \frac{\log_{a_{2m}} a_{2m+2}^{-\frac{1}{n_{2m+2}}} - \log_{a_{2m}} a_{2m+1}^{-\frac{1}{n_{2m+1}}}}{\frac{1}{n_{2m+1}} - \log_{a_{2m}} a_{2m+1}^{-\frac{1}{n_{2m+1}}} + \log_{a_{2m}} a_{2m+3}^{-\frac{1}{n_{2m+3}}} + \log_{a_{2m}} a_{2m+2}^{-\frac{n_{2m}}{n_{2m+1}n_{2m+2}}}}.$$

Note that the denominator of $T(m)$ can be written in the form

$$\begin{aligned} &\log_{a_{2m}} a_{2m+3}^{-\frac{1}{n_{2m+3}}} - \log_{a_{2m}} a_{2m+1}^{-\frac{1}{n_{2m+1}}} \\ &\quad + \frac{n_{2m}}{n_{2m+1}}\left(\log_{a_{2m}} a_{2m+2}^{-\frac{1}{n_{2m+2}}} - \log_{a_{2m}} a_{2m}^{-\frac{1}{n_{2m}}}\right) > 0, \quad m \in \mathbf{N}, \end{aligned}$$

and hence $T(m)$ is strictly positive for all $m \in \mathbf{N}$. By means of (6) we get

$$T(m) \ge \frac{(d-1)\log_{a_{2m}} a_{2m+1}^{-\frac{1}{n_{2m+1}}}}{\frac{1}{n_{2m+1}} - \log_{a_{2m}} a_{2m+1}^{-\frac{1}{n_{2m+1}}}} = \frac{(1-d)\log_{a_{2m}} a_{2m+1}}{1+\log_{a_{2m}} a_{2m+1}}, \quad m \in \mathbf{N}.$$

This implies that, if

$$0 < \delta \le \frac{(1-d)\log_\mu \lambda}{1+\log_\lambda \mu}, \tag{23}$$

then the first inequality in (21) is satisfied for all $m \in \mathbf{N}$. The second inequality in (22) follows by a similar argument, and the choice (23) for δ is again adequate. We conclude that

$$|f_0(z)| + |f_1(z)| \asymp \omega(|z|) \tag{24}$$

for $|z| \ge \min I_2$. Finally, since $\left\{a_m^{-1/n_m}\right\}_{m=1}^{\infty}$ is strictly increasing, (24) holds also for $|z| \le \min I_2$, and hence f_0 and f_1 are analytic functions satisfying (2).

2.5. Asymptotic properties of the products. Product f_j for $j = 0, 1$ has exactly n_{2m+j} simple zeros on the each circle $\{z\colon |z| = s_{2m+j}\}$, where $s_{2m+j} = a_{2m+j}^{-1/n_{2m+j}}$ for $m \in \mathbf{N}$. Therefore, we obtain

$$\begin{aligned} n_{2m+j} \le n(s_{2m+j}, f_j, 0) &= \sum_{k=1}^{m} n_{2k+j} = n_{2m+j} \sum_{k=1}^{m} \frac{1}{\frac{n_{2m+j}}{n_{2k+j}}} \\ &\le n_{2m+j} \sum_{k=0}^{\infty} \frac{1}{\tau^k} \lesssim n_{2m+j}, \quad m \in \mathbf{N}, \end{aligned}$$

by (10). By applying the estimates $1 - x < \log x^{-1} < 2(1 - x)$, which are valid for $4^{-1} < x < 1$, it follows that $n(s_{2m+j}, f_j, 0) \asymp (1 - s_{2m+j})^{-1}$ for all $m \in \mathbf{N}$. Consequently,

$$n(r, f_j, 0) = \mathrm{O}\left((1 - r)^{-1}\right), \quad r \to 1^-.$$

Now we observe that $E_0 \cup E_1 = [\min I_2, 1)$, so it is not possible that $\underline{d}(E_0) = \underline{d}(E_1) = 0$, where

$$\underline{d}(F) = \liminf_{r \to 1^-} \frac{m\big(F \cap [r, 1)\big)}{1 - r}$$

is the *lower density* of the set $F \subset [0, 1)$, and where m denotes the Lebesgue measure. Consequently, for some $j = 0, 1$ we have $\underline{d}(E_j) > 0$, which together with the nature of the sets E_j, implies that $\underline{d}(E_j) > 0$ for both $j = 0, 1$. Consequently, (20) holds outside a set $E_j^\star = [0, 1) \setminus E_j$, which satisfies

$$\overline{d}(E_j^\star) = \limsup_{r \to 1^-} \frac{m\big(E_j^\star \cap [r, 1)\big)}{1 - r} < 1, \quad j = 0, 1.$$

So, (20), [5, Lemma 2] and Lemma 2 yield $M_p(r, f_j) \asymp \omega(r)$, as $r \to 1^-$, where the constants in the asymptotic relation are independent of $0 < p \le \infty$. This proves (3). On the other hand, for any $a \in \mathbf{C}$, Jensen's formula and (20) imply that $N(r, f_j, a) \asymp \log \omega(r)$ for $r \in [0, 1) \setminus E_0^\star$. The fact that the same estimate holds also without the exceptional set $E_0^\star$ follows again from [5, Lemma 2] and Lemma 2. Furthermore, $\log \omega(r) \asymp N(r, f_j, 0) \lesssim T(r, f_j) \le \log M_\infty(r, f_j) \asymp \log \omega(r)$, as $r \to 1^-$, again by Jensen's formula. This completes the proof of Theorem 1.

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