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BOUNDEDNESS OF PRETANGENT SPACES TO GENERAL METRIC SPACES

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Abstract. Let (X, d, p) be a metric space with a metric d and a marked point p. We define the set of w-strongly porous at 0 subsets of $[0, \infty)$ and prove that the distance set $\{d(x, p) \colon x \in X\}$ is w-strongly porous at 0 if and only if every pretangent space to X at p is bounded.

1. Introduction

Recent achievements in the metric space theory are closely related to some generalizations of the differentiation. A possible but not the only one initial point to develop the theory of a differentiation in metric spaces is the fact that every separable metric space admits an isometric embedding into the dual space of a separable Banach space. It provides a linear structure, and so a differentiation. This approach leads to a rather complete theory of rectifiable sets and currents on metric spaces [4, 5]. The concept of the upper gradient [13, 14, 16], Cheeger's notion of differentiability for Rademacher's theorem in certain metric measure spaces [7], the metric derivative in the studies of metric space valued functions of bounded variation [3, 6] and the Lipschitz type approach in [12] are the important examples of such generalizations. The generalizations of the differentiability mentioned above give usually nontrivial results only for the assumption that metric spaces have "sufficiently many" rectifiable curves.

A new intrinsic notion of differentiability for the mapping between the general metric spaces was produced in [10] (see also [11]). A basic technical tool in [10] is a pretangent and tangent spaces to an arbitrary metric space X at a point p. The development of this theory requires the understanding of interrelations between the infinitesimal properties of initial metric space and geometry of pretangent spaces to this initial. The main purpose of the present paper is to search the conditions under which all pretangent spaces to X at a point $p \in X$ are bounded.

For convenience we recall some terminology and results related to pretangent spaces to general metric spaces.

Let (X, d, p) be a pointed metric space with a metric d and a marked point p. Fix a sequence $\tilde{r}$ of positive real numbers r_n tending to zero. In what follows $\tilde{r}$ will be called a *normalizing sequence*. Let us denote by $\tilde{X}$ the set of all sequences of points from X and by $\mathbf{N}$ the set of positive integer (= natural) numbers.

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Definition 1.1. Two sequences $\tilde{x} = \{x_n\}_{n\in\mathbf{N}}$ and $\tilde{y} = \{y_n\}_{n\in\mathbf{N}}$, $\tilde{x}, \tilde{y} \in \tilde{X}$, are mutually stable with respect to $\tilde{r} = \{r_n\}_{n\in\mathbf{N}}$ if there is a finite limit

$$\lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} := \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y}). \tag{1.1}$$

We shall say that a family $\tilde{F} \subseteq \tilde{X}$ is *self-stable* (w.r.t. $\tilde{r}$) if every two $\tilde{x}, \tilde{y} \in \tilde{F}$ are mutually stable. A family $\tilde{F} \subseteq \tilde{X}$ is *maximal self-stable* if $\tilde{F}$ is self-stable and for an arbitrary $\tilde{z} \in \tilde{X}$ either $\tilde{z} \in \tilde{F}$ or there is $\tilde{x} \in \tilde{F}$ such that $\tilde{x}$ and $\tilde{z}$ are not mutually stable.

The standart application of Zorn's lemma leads to the following

Proposition 1.2. *Let (X, d, p) be a pointed metric space. Then for every normalizing sequence $\tilde{r} = \{r_n\}_{n\in\mathbf{N}}$ there exists a maximal self-stable family $\tilde{X}_{p,\tilde{r}}$ such that $\tilde{p} := \{p, p, \ldots\} \in \tilde{X}_{p,\tilde{r}}$.*

Note that the condition $\tilde{p} \in \tilde{X}_{p,\tilde{r}}$ implies the equality

$$\lim_{n\to\infty} d(x_n, p) = 0$$

for every $\tilde{x} = \{x_n\}_{n\in\mathbf{N}} \in \tilde{X}_{p,\tilde{r}}$.

Consider a function $\tilde{d}\colon \tilde{X}_{p,\tilde{r}} \times \tilde{X}_{p,\tilde{r}} \to \mathbf{R}$ where $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y})$ is defined by (1.1). Obviously, $\tilde{d}$ is symmetric and nonnegative. Moreover, the triangle inequality for d implies

$$\tilde{d}(\tilde{x}, \tilde{y}) \le \tilde{d}(\tilde{x}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{y})$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}_{p,\tilde{r}}$. Hence $(\tilde{X}_{p,\tilde{r}}, \tilde{d})$ is a pseudometric space.

Definition 1.3. The pretangent space to the space X (at the point p w.r.t. $\tilde{r}$) is the metric identification of the pseudometric space $(\tilde{X}_{p,\tilde{r}}, \tilde{d})$.

Since the notion of pretangent space is important for the paper, we remind this metric identification construction.

Define the relation $\sim$ on $\tilde{X}_{p,\tilde{r}}$ by $\tilde{x} \sim \tilde{y}$ if and only if $\tilde{d}(\tilde{x}, \tilde{y}) = 0$. Then $\sim$ is an equivalence relation. Let us denote by $\Omega^X_{p,\tilde{r}}$ the set of equivalence classes in $\tilde{X}_{p,\tilde{r}}$ under the equivalence relation $\sim$. It follows from general properties of pseudometric spaces (see, for example, [15]), that if ρ is defined on $\Omega^X_{p,\tilde{r}}$ by

$$\rho(\alpha, \beta) := \tilde{d}(\tilde{x}, \tilde{y})$$

for $\tilde{x} \in \alpha$ and $\tilde{y} \in \beta$, then ρ is a well-defined metric on $\Omega^X_{p,\tilde{r}}$. The metric identification of $(\tilde{X}_{p,\tilde{r}}, \tilde{d})$ is the metric space $(\Omega^X_{p,\tilde{r}}, \rho)$.

It should be observed that $\Omega^X_{p,\tilde{r}} \neq \varnothing$ because the constant sequence $\tilde{p}$ belongs to $\tilde{X}_{p,\tilde{r}}$. Thus every pretangent space $\Omega^X_{p,\tilde{r}}$ is a pointed metric space with natural distinguished point $\pi(\tilde{p})$, (see diagram (1.3) below).

Let $\{n_k\}_{k\in\mathbf{N}}$ be an infinite strictly increasing sequence of natural numbers. Let us denote by $\tilde{r}'$ the subsequence $\{r_{n_k}\}_{k\in\mathbf{N}}$ of the normalizing sequence $\tilde{r} = \{r_n\}_{n\in\mathbf{N}}$ and let $\tilde{x}' := \{x_{n_k}\}_{k\in\mathbf{N}}$ for every $\tilde{x} = \{x_n\}_{n\in\mathbf{N}} \in \tilde{X}$. It is clear that if $\tilde{x}$ and $\tilde{y}$ are mutually stable w.r.t. $\tilde{r}$, then $\tilde{x}'$ and $\tilde{y}'$ are mutually stable w.r.t. $\tilde{r}'$ and

$$\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = \tilde{d}_{\tilde{r}'}(\tilde{x}', \tilde{y}'). \tag{1.2}$$

If $\tilde{X}_{p,\tilde{r}}$ is a maximal self-stable (w.r.t. $\tilde{r}$) family, then, by Zorn's Lemma, there exists a maximal self-stable (w.r.t. $\tilde{r}'$) family $\tilde{X}_{p,\tilde{r}'}$ such that

$$\{\tilde{x}' \colon \tilde{x} \in \tilde{X}_{p,\tilde{r}}\} \subseteq \tilde{X}_{p,\tilde{r}'}.$$

Denote by $in_{\tilde{r}'}$ the map from $\tilde{X}_{p,\tilde{r}}$ to $\tilde{X}_{p,\tilde{r}'}$ with $in_{\tilde{r}'}(\tilde{x}) = \tilde{x}'$ for all $\tilde{x} \in \tilde{X}_{p,\tilde{r}}$. It follows from (1.2) that after metric identifications $in_{\tilde{r}'}$ passes to an isometric embedding $em' \colon \Omega^X_{p,\tilde{r}} \to \Omega^X_{p,\tilde{r}'}$ under which the diagram

$$\begin{tikzcd} \tilde{X}_{p,\tilde{r}} \arrow[r, "in_{\tilde{r}'}"] \arrow[d, "\pi"'] & \tilde{X}_{p,\tilde{r}'} \arrow[d, "\pi'"] \\ \Omega^X_{p,\tilde{r}} \arrow[r, "em'"] & \Omega^X_{p,\tilde{r}'} \end{tikzcd} \tag{1.3}$$

is commutative. Here π and π' are the natural projections, $\pi(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{p,\tilde{r}} \colon \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) = 0\}$ and $\pi'(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{p,\tilde{r}'} \colon \tilde{d}_{\tilde{r}'}(\tilde{x}, \tilde{y}) = 0\}$.

Let X and Y be metric spaces. Recall that a map $f \colon X \to Y$ is called an *isometry* if f is distance-preserving and onto.

Definition 1.4. A pretangent $\Omega^X_{p,\tilde{r}}$ is tangent if $em' \colon \Omega^X_{p,\tilde{r}} \to \Omega^X_{p,\tilde{r}'}$ is an isometry for every $\Omega^X_{p,\tilde{r}'}$.

The following lemma is a direct corollary of Lemma 5 from [1].

Lemma 1.5. *Let (X, d, p) be a pointed metric space, $\mathfrak{B}$ a countable subfamily of $\tilde{X}$ and let $\tilde{r} = \{r_n\}_{n\in\mathbf{N}}$ be a normalizing sequence. Suppose that $\tilde{b}$ and $\tilde{p}$ are mutually stable for every $\tilde{b} = \{b_n\}_{n\in\mathbf{N}} \in \mathfrak{B}$. Then there is an infinite subsequence $\tilde{r}' = \{r_{n_k}\}_{k\in\mathbf{N}}$ of $\tilde{r}$ such that the family*

$$\mathfrak{B}' := \{\tilde{b}' = \{b_{n_k}\}_{k\in\mathbf{N}} \colon \tilde{b} \in \mathfrak{B}\}$$

is self-stable w.r.t. $\tilde{r}'$.

2. Boundedness of pretangent spaces and local strong right porosity

Let us recall the definition of the right porosity. This definition and an useful collection of facts related to the notion of porosity can be found in [17]. Let E be a subset of $\mathbf{R}^+ = [0, \infty)$.

Definition 2.1. The local right porosity of E at 0 is the quantity

$$p^+(E, 0) := \limsup_{h\to 0^+} \frac{\lambda(E, 0, h)}{h}$$

where $\lambda(E, 0, h)$ is the length of the largest open subinterval of $(0, h)$ that contains no point of E. The set E is strongly porous on the right at 0 if $p^+(E, 0) = 1$.

It was proved in [1] that a bounded tangent space to X at p exists if and only if the distance set

$$S_p(X) := \{d(x, p) \colon x \in X\}$$

is strongly porous on the right at 0.

- It is therefore reasonable to ask for which pointed metric spaces (X, d, p) all pretangent spaces $\Omega^X_{p,\tilde{r}}$ are bounded?

- Is there a modification of the local strong porosity describing the boundedness of all pretangent spaces $\Omega^X_{p,\tilde{r}}$?

Our first goal is to introduce a desired modification of porosity.

Let $\tilde{\tau} = \{\tau_n\}_{n\in\mathbf{N}}$ be a sequence of real numbers. We shall say that $\tilde{\tau}$ is *almost decreasing* if the inequality $\tau_{n+1} \leq \tau_n$ holds for sufficiently large n. Write $\tilde{E}^d_0$ for the set of almost decreasing sequences $\tilde{\tau}$ with $\lim\limits_{n\to\infty} \tau_n = 0$ and having $\tau_n \in E \setminus \{0\}$ for $n \in \mathbf{N}$.

Define $\tilde{I}_E$ to be the set of sequences $\{(a_n, b_n)\}_{n\in\mathbf{N}}$ of open intervals $(a_n, b_n) \subseteq \mathbf{R}^+$ meeting the following conditions:

- each (a_n, b_n) is a connected component of the set $\operatorname{Ext} E = Int(\mathbf{R}^+ \setminus E)$, i.e., $(a_n, b_n) \cap E = \varnothing$ but for every $(a, b) \supseteq (a_n, b_n)$ we have
$$((a, b) \neq (a_n, b_n)) \Rightarrow ((a, b) \cap E \neq \varnothing).$$
- $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} \frac{b_n - a_n}{b_n} = 1$.

Define also the weak equivalence $\asymp$ on the set of sequences of strictly positive numbers as follows. Let $\tilde{a} = \{a_n\}_{n\in\mathbf{N}}$ and $\tilde{\gamma} = \{\gamma_n\}_{n\in\mathbf{N}}$. Then $\tilde{a} \asymp \tilde{\gamma}$ if there are constants $c_1, c_2 > 0$ such that
$$c_1 a_n < \gamma_n < c_2 a_n, \quad n \in \mathbf{N}.$$

Definition 2.2. Let 0 be an accumulation point of a set $E \subseteq \mathbf{R}^+$ and let $\tilde{\tau} \in \tilde{E}^d_0$. The set E is $\tilde{\tau}$-strongly porous at 0 if there is a sequence $\{(a_n, b_n)\}_{n\in\mathbf{N}} \in \tilde{I}_E$ such that
$$\tilde{\tau} \asymp \tilde{a},$$
where $\tilde{a} = \{a_n\}_{n\in\mathbf{N}}$.

Let E be a subset of $\mathbf{R}^+$ and let $0 \in E$.

Definition 2.3. The set E is w-strongly porous at 0 if for every sequence $\tilde{\tau} = \{\tau_n\}_{n\in\mathbf{N}} \in \tilde{E}^d_0$ there is a subsequence $\tilde{\tau}' = \{\tau_{n_k}\}_{k\in\mathbf{N}} \in \tilde{E}^d_0$ for which the set E is $\tilde{\tau}'$-strongly porous at 0.

Remark 2.4. It is clear that $E \subseteq \mathbf{R}^+$ is w-strongly porous at 0 if 0 is an isolated point of E and, on the other hand, if E is w-strongly porous at 0 then E is strongly porous at 0.

The following theorem gives a boundedness criterion for pretangent spaces.

Theorem 2.5. *Let (X, d, p) be a pointed metric space. All pretangent spaces to X at p are bounded if and only if the set $S_p(X)$ is w-strongly porous at 0.*

The proof of Theorem 2.5 is based on several auxiliary results. In the following proposition we consider the distance set $S_p(X)$ as a pointed metric space with the standard metric induced from $\mathbf{R}$ and the marked point 0.

Proposition 2.6. *Let (X, d, p) be a pointed metric space with the distance set $S_p(X)$. The following statements are equivalent.*

(i) *All pretangent spaces to X at p are bounded.*

(ii) *All pretangent spaces to $S_p(X)$ at 0 are bounded.*

Proof. (ii) $\Rightarrow$ (i) Write $\Omega^{S_p(X)}_{0,\tilde{r}}$ and $\tilde{S}_{0,\tilde{r}}(X)$ for pretangent spaces to $S_p(X)$ at 0 and, respectively, for the corresponding maximal self-stable families. Suppose that the inequality

$$\operatorname{diam} \Omega_{0,\tilde{r}}^{S_p(X)} < \infty$$

holds for every $\Omega_{0,\tilde{r}}^{S_p(X)}$. Let $\Omega_{p,\tilde{r}}^{X}$ be an arbitrary pretangent space to X with the corresponding maximal self-stable family $\tilde{X}_{p,\tilde{r}}$. Let $\tilde{x}, \tilde{y} \in \tilde{X}_{p,\tilde{r}}$. The membership relations $\tilde{x}, \tilde{y} \in \tilde{X}_{p,\tilde{r}}$ imply the existence of the finite limits

$$\lim_{n\to\infty} \frac{d(x_n, p)}{r_n} \qquad \text{and} \qquad \lim_{n\to\infty} \frac{d(y_n, p)}{r_n}.$$

As it was shown in [2, Proposition 2.2] the statement "*If $\tilde{a}$ and $\tilde{0}$ are mutually stable and $\tilde{b}$ and $\tilde{0}$ are mutually stable, then $\tilde{a}$ and $\tilde{b}$ are mutually stable*" holds for every normalizing sequence $\tilde{r}$ and every subspace E of the metric space $\mathbf{R}^+$ with $0 \in E$ and $\tilde{a}, \tilde{b} \in \tilde{E}$.

Consequently, we obtain that $\{d(x_n, p)\}_{n\in\mathbf{N}}$, $\{d(y_n, p)\}_{n\in\mathbf{N}} \in \tilde{S}_{0,\tilde{r}}(X)$. Using the triangle inequality, we obtain

$$\begin{aligned}\tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) &= \lim_{n\to\infty} \frac{d(x_n, y_n)}{r_n} \le \lim_{n\to\infty} \frac{d(x_n, p)}{r_n} + \lim_{n\to\infty} \frac{d(y_n, p)}{r_n} \\ &\le 2 \sup_{\tilde{z}\in\tilde{S}_{0,\tilde{r}}(X)} \tilde{d}_{\tilde{r}}(\tilde{0}, \tilde{z}) \le 2 \operatorname{diam} \Omega_{0,\tilde{r}}^{S_p(X)}.\end{aligned}$$

Hence

$$\operatorname{diam} \Omega_{p,\tilde{r}}^{X} = \sup_{\tilde{x},\tilde{y}\in\tilde{X}_{p,\tilde{r}}} \tilde{d}_{\tilde{r}}(\tilde{x}, \tilde{y}) \le 2 \operatorname{diam} \Omega_{0,\tilde{r}}^{S_p(X)}.$$

The boundedness of $\Omega_{p,\tilde{r}}^{X}$ follows.

(i) $\Rightarrow$ (ii) Suppose that all $\Omega_{p,\tilde{r}}^{X}$ are bounded but there is an unbounded $\Omega_{0,\tilde{r}}^{S_p(X)}$. Let $\tilde{S}_{0,\tilde{r}}(X)$ be the maximal self-stable family corresponding to $\Omega_{0,\tilde{r}}^{S_p(X)}$. Since $\Omega_{0,\tilde{r}}^{S_p(X)}$ is unbounded we can find a countable family of the sequences $\{d(p, b_n^j)\}_{n\in\mathbf{N}} \in \tilde{S}_{0,\tilde{r}}(X)$, $j \in \mathbf{N}$, such that

$$\infty > \lim_{n\to\infty} \frac{d(p, b_n^j)}{r_n} \ge j \tag{2.1}$$

for every $j \in \mathbf{N}$. By Lemma 1.5 there is a subsequence $\tilde{r}' = \{r_{n_k}\}_{k\in\mathbf{N}}$ such that the family of sequences $\{b_{n_k}^j\}_{k\in\mathbf{N}}$, $j \in \mathbf{N}$, is self-stable w.r.t. $\tilde{r}'$. Applying the Zorn Lemma we find a maximal self-stable family $\tilde{X}_{p,\tilde{r}'}$ such that $\{b_{n_k}^j\}_{k\in\mathbf{N}} \in \tilde{X}_{p,\tilde{r}'}$ for every j. Inequalities (2.1) imply that the pretangent space corresponding to $\tilde{X}_{p,\tilde{r}'}$ is unbounded, contrary to the supposition. $\square$

The next lemma was proved in [9, Corollary 2.4].

Lemma 2.7. *Let $E \subseteq \mathbf{R}^+$ and let $\tilde{\tau} = \{\tau_n\}_{n\in\mathbf{N}} \in \tilde{E}_0^d$. The set E is $\tilde{\tau}$-strongly porous if and only if there exists a sequence $\{(a_n, b_n)\}_{n\in\mathbf{N}} \in \tilde{I}_E$ such that $\limsup\limits_{n\to\infty} \frac{a_n}{\tau_n} < \infty$ and $\tau_n \le a_n$ for sufficiently large n.*

Proposition 2.8. *Let $E \subseteq \mathbf{R}^+$ and let $\tilde{\tau} = \{\tau_n\}_{n\in\mathbf{N}} \in \tilde{E}_0^d$. The following statements are equivalent.*

(i) *E is $\tilde{\tau}$-strongly porous at 0.*

(ii) *There is a constant* $k \in (1, \infty)$ *such that for every* $K \in (k, \infty)$ *there exists* $N_1(K) \in \mathbf{N}$ *such that*

$$(k\tau_n, K\tau_n) \cap E = \varnothing \tag{2.2}$$

for $n \geq N_1(K)$.

Proof. (i) $\Rightarrow$ (ii) Suppose that E is $\tilde{\tau}$-strongly porous at 0. By Lemma 2.7, there is a sequence

$$\{(a_n, b_n)\}_{n \in \mathbf{N}} \in \tilde{I}_E \tag{2.3}$$

such that $\limsup\limits_{n\to\infty} \frac{a_n}{\tau_n} < \infty$ and $\tau_n \leq a_n$ for sufficiently large n. Write

$$k = 1 + \limsup_{n\to\infty} \frac{a_n}{\tau_n},$$

then $k \geq 2$ and there is $N_0 \in \mathbf{N}$ such that

$$\tau_n \leq a_n < k\tau_n \tag{2.4}$$

for $n \geq N_0$. Let $K \in (k, \infty)$. Membership (2.3) implies the equality $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = \infty$. The last equality and (2.4) show that there is $N_1 \geq N_0$ such that

$$a_n < k\tau_n < K\tau_n \leq b_n$$

if $n \geq N_1$. Hence the inclusion

$$(k\tau_n, K\tau_n) \subseteq (a_n, b_n) \tag{2.5}$$

holds for $n \geq N_1$. Since

$$E \cap (a_n, b_n) = \varnothing, \tag{2.6}$$

(2.5) and (2.6) imply (2.2). Thus (ii) follows from (i).

(ii) $\Rightarrow$ (i) Assume that statement (ii) holds. Then for $K = 2k$ there is $N_0 \in \mathbf{N}$ such that

$$(k\tau_n, 2k\tau_n) \cap E = \varnothing$$

for $n \geq N_0$. Consequently, for every $n \geq N_0$, we can find a connected component (a_n, b_n) of $\operatorname{Ext} E$ meeting the inclusion

$$(k\tau_n, 2k\tau_n) \subseteq (a_n, b_n).$$

Define $(a_n, b_n) := (a_{N_0}, b_{N_0})$ for $n < N_0$. Since, for $n \geq N_0$, we have

$$\tau_n \in E,\ \tau_n < k\tau_n \text{ and } (a_n, k\tau_n) \cap E = \varnothing,$$

the double inequality $\tau_n \leq a_n < k\tau_n$ holds. Hence $\{\tau_n\}_{n\in\mathbf{N}} \asymp \{a_n\}_{n\in\mathbf{N}}$, i.e., to prove (i) it is sufficient to show that

$$\{(a_n, b_n)\}_{n \in \mathbf{N}} \in \tilde{I}_E. \tag{2.7}$$

All (a_n, b_n) are connected components of $\operatorname{Ext} E$, so that (2.7) holds if and only if

$$\lim_{n\to\infty} \frac{b_n}{a_n} = \infty. \tag{2.8}$$

Let K be an arbitrary point of (k, ∞). Applying (2.2) we can find $N_1(K) \in \mathbf{N}$ such that

$$(k\tau_n, K\tau_n) \subseteq (a_n, b_n)$$

for $n \geq N_1(K)$. Consequently, for such n, we have

$$\frac{b_n}{a_n} \geq \frac{K\tau_n}{k\tau_n} = \frac{K}{k}.$$

Letting $K \to \infty$ we see that (2.8) follows. □

Proof of Theorem 2.5. The theorem is trivial if p is an isolated point of X. Suppose that p is an accumulation point of X. Taking into account Proposition 2.6, we can also assume that $X \subseteq \mathbf{R}^+$ and $p = 0$.

Let X be w-strongly porous at 0 and let $\Omega^X_{0,\tilde{r}}$ be an arbitrary pretangent space to X at 0 with the corresponding maximal self-stable family $\tilde{X}_{0,\tilde{r}}$. To prove that $\Omega^X_{0,\tilde{r}}$ is bounded it suffices to show that

$$\sup_{\substack{\beta \in \Omega^X_{0,\tilde{r}} \\ \beta \neq \alpha}} \rho(\alpha, \beta) < \infty, \tag{2.9}$$

where $\alpha = \pi(\tilde{0})$ (see (1.3)). This inequality is vacuously true if $\Omega^X_{0,\tilde{r}}$ is one-point. In the case of $\operatorname{card}\Omega^X_{0,\tilde{r}} \geq 2$ we can find $c \in (0, \infty)$ and $\tilde{\tau} = \{\tau_n\}_{n\in\mathbf{N}} \in \tilde{X}_{0,\tilde{r}}$ such that

$$\rho(\pi(\tilde{\tau}), \alpha) = \lim_{n\to\infty} \frac{\tau_n}{r_n} = c. \tag{2.10}$$

Let $\mathbf{N}^i$ be the set of all infinite subsets of $\mathbf{N}$. Since X is w-strongly porous at 0, there is $A \in \mathbf{N}^i$ such that $\tilde{\tau}' = \{\tau_n\}_{n\in A}$ is almost decreasing and X is $\tilde{\tau}'$-strongly porous at 0. Note that the equivalence $\tilde{x} \asymp \tilde{\tau}$ holds for $\tilde{x} \in \tilde{X}_{0,\tilde{r}}$ if and only if $\pi(\tilde{x}) \neq \alpha$. Using (2.10) we can write (2.9) in the equivalent form

$$\sup_{\substack{\tilde{x}\in\tilde{X}_{0,\tilde{r}} \\ \tilde{x}\asymp\tilde{\tau}}} \inf_{\substack{B\subseteq A \\ B\in\mathbf{N}^i}} \limsup_{\substack{n\to\infty \\ n\in B}} \frac{x_n}{\tau_n} < \infty. \tag{2.11}$$

Let $\tilde{x} = \{x_n\}_{n\in\mathbf{N}}$ be an arbitrary element of $\tilde{X}_{0,\tilde{r}}$ for which $\pi(\tilde{x}) \neq \alpha$. Then $\tilde{x} \asymp \tilde{\tau}$ holds. Moreover, it is easy to find $B \subseteq A$, $B \in \mathbf{N}^i$, such that $\{x_n\}_{n\in B}$ is almost decreasing. Since X is $\tilde{\tau}'$-strongly porous at 0 and $\tilde{x} \asymp \tilde{\tau}$, the set X is also $\{x_n\}_{n\in B}$-strongly porous at 0. Let $\{(a_n, b_n)\}_{n\in B} \in \tilde{I}_X$ be a sequence such that $\{\tau_n\}_{n\in B} \asymp \{a_n\}_{n\in B} \asymp \{x_n\}_{n\in B}$. Lemma 2.7 implies that

$$x_n \leq a_n$$

for sufficiently large $n \in B$ and that

$$\limsup_{\substack{n\to\infty \\ n\in A}} \frac{a_n}{\tau_n} < \infty.$$

Thus

$$\limsup_{\substack{n\to\infty \\ n\in B}} \frac{x_n}{\tau_n} \leq \limsup_{\substack{n\to\infty \\ n\in B}} \frac{a_n}{\tau_n} \leq \limsup_{\substack{n\to\infty \\ n\in A}} \frac{a_n}{\tau_n} < \infty.$$

Inequality 2.11 follows.

Suppose now that all pretangent spaces to X at 0 are bounded but the set X is not w-strongly porous at 0. By Definition 2.3, there exists a decreasing sequence $\tilde{\tau} = \{\tau_n\}_{n\in\mathbf{N}}$ such that $\tau_n \in X \setminus \{0\}$ for every $n \in \mathbf{N}$, $\lim\limits_{n\to\infty} \tau_n = 0$ and X is not $\tilde{\tau}'$-strongly porous at 0 for every subsequence $\tilde{\tau}'$ of the sequence $\tilde{\tau}$. Since X is not $\tilde{\tau}$-strongly porous, by Proposition 2.8 for every $k_1 > 1$ there is $K_1 \in (k_1, \infty)$ such that $(k_1\tau_n, K_1\tau_n) \cap E \neq \varnothing$ for all n belonging to an infinite set $A_1 \subseteq \mathbf{N}$. Using this fact we can find a convergent subsequence

$$\frac{\tilde{x}^{(1)}}{\tilde{\tau}^{(1)}} := \left\{\frac{x_i^{(1)}}{\tau_i}\right\}_{i\in A_1} \quad \text{such that } k_1 < \frac{x_i^{(1)}}{\tau_i} < K_1, \ i \in A_1,$$

$\tilde{\tau}^{(1)} := \{\tau_i\}_{i\in A_1}$, $\tilde{x}^{(1)} := \{x_i^{(1)}\}_{i\in A_1}, x_i^{(1)} \in X \setminus \{0\}$. Let $k_2 > K_1 \vee 2$. Since X is not $\tilde{\tau}'$-strongly porous, there are $K_2 > k_2$ and an infinite $A_2 \subseteq A_1$ such that $(k_2\tau_n, K_2\tau_n) \cap E \neq \varnothing$ for $n \in \mathbf{N}$, so that we can construct a convergent subsequence

$$\frac{\tilde{x}^{(2)}}{\tilde{\tau}^{(2)}} := \left\{\frac{x_i^{(2)}}{\tau_i}\right\}_{i\in A_2} \quad \text{such that } k_2 < \frac{x_i^{(2)}}{\tau_i} < K_2, \ x_i^{(2)} \in X \setminus \{0\}, \ i \in A_2,$$

where $\tilde{\tau}^{(2)} = \{\tau_i^{(2)}\}_{i\in A_2}$ is a subsequence of $\tilde{\tau}^{(1)}$ and $\tilde{x}^{(2)} := \{x_i^{(2)}\}_{i\in A_2}$. Repeating this procedure we see that, for every $j \in \mathbf{N}$, there are some sequences $\tilde{x}^{(j+1)} = \{x_i^{(j+1)}\}_{i\in A_{j+1}}, x_i^{j+1} \in X \setminus \{0\}$ and $\tilde{\tau}^{(j+1)} = \{\tau_i\}_{i\in A_{j+1}}$, A_{j+1} is infinite subset of $A_j \subseteq \mathbf{N}$, such that

$$k_j \vee k_{j+1} < \frac{x_i^{(j+1)}}{\tau_i} < K_{j+1},$$

for $i \in A_{j+1}$ and

$$\frac{\tilde{x}^{(j+1)}}{\tilde{\tau}^{(i+1)}} := \left\{\frac{x_i^{(j+1)}}{\tau_i}\right\}_{i\in A_{j+1}}$$

is convergent. To complete the proof, it suffices to make use of Cantor's diagonal argument.

Let $B := \{n_1, \ldots, n_j, \ldots\}$ be an infinite subset of $\mathbf{N}$ such that $n_j \in A_j$ for every $j \in \mathbf{N}$. Let us define the subsequences $\tilde{y}^{(j)} = \{y_k^j\}_{k\in B}$ by the rule

$$y_k^{(j)} := \begin{cases} 0 & \text{if } \ k \in A_j \setminus B, \\ x_k^{(j)} & \text{if } \ k \in A \cap A_j. \end{cases}$$

Then the sequence $\left\{\frac{y_k^{(j)}}{\tau_k}\right\}_{k\in B}$ is convergent and

$$j \leq \lim_{\substack{k\to\infty \\ k\in B}} \frac{y_k^{(j)}}{\tau_k} \tag{2.12}$$

for every $j \in \mathbf{N}$. Since all $\{y_k^{(j)}\}_{k\in B}$ are mutually stable w.r.t. the normalizing sequence $\tilde{\tau}' = \{\tau_k\}_{k\in B}$, there is a maximal self-stable $\tilde{X}_{0,\tilde{r}'}$ such that $\{y_k^{(j)}\}_{k\in B} \in \tilde{X}_{0,\tilde{r}'}$ for $j \in \mathbf{N}$. Inequality (2.12) shows that the corresponding pretangent space $\Omega^X_{0,\tilde{r}'}$ is unbounded, contrary to the supposition. $\square$

Corollary 2.9. *Let (X, d, p) be a pointed metric space. If all pretangent spaces $\Omega^X_{p,\tilde{r}}$ are bounded, then at least one from these pretangent spaces is tangent.*

Proof. Suppose that all $\Omega^X_{p,\tilde{r}}$ are bounded, then, by Theorem 2.5, the set $S_p(X)$ is w-strongly porous at 0. Consequently $S_p(X)$ is strongly porous on the right at 0 (see Remark 2.4). As was noted above, $S_p(X)$ is strongly porous on the right at 0 if and only if there is a bounded tangent space $\Omega^X_{p,\tilde{r}}$. $\square$

We shall say that a set $E \subseteq \mathbf{R}^+$ is *completely strongly porous* at 0 if E is $\tilde{\tau}$-strongly porous at 0 for every $\tilde{\tau} \in \tilde{E}_0^d$. Some properties of completely strongly porous sets $E \subseteq \mathbf{R}^+$ are described in [9].

The following example shows that there exist w-strongly porous at 0 subsets of $\mathbf{R}^+$ which are not completely strongly porous at 0.

Example 2.10. Let $\tau_1 = 1$ and $\tau_{n+1} = 2^{-n^2}\tau_n$ for every $n \in \mathbf{N}$. Let $\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_k, \ldots$ be an infinite partition of $\mathbf{N}$,

$$\bigcup_{k=1}^{\infty} \mathbf{N}_k = \mathbf{N}, \quad \mathbf{N}_i \cap \mathbf{N}_j = \varnothing \quad \text{for} \quad i \neq j,$$

such that

$$\nu(1) < \nu(2) < \ldots < \nu(k) < \ldots$$

where

$$\nu(k) := \min_{n \in \mathbf{N}_k} n.$$

For every $n \in \mathbf{N}$ define τ_n^* as $2^{-\nu(m(n))}\tau_n$ where $m(n)$ is the index for which $n \in \mathbf{N}_{m(n)}$. Write $E_1 := \{\tau_n \colon n \in \mathbf{N}\}$, $E_1^* := \{\tau_n^* \colon n \in \mathbf{N}\}$ and

$$E := E_1 \cup E_1^* \cup \{0\},$$

here E_1 and E_1^* are the ranges of the sequences $\{\tau_n\}_{n \in \mathbf{N}}$ and $\{\tau_n^*\}_{n \in \mathbf{N}}$ respectively. Using Lemma 2.7 we can show that E is not $\tilde{\tau}$-strongly porous with $\tilde{\tau} = \{\tau_n\}_{n \in \mathbf{N}}$ define as above, so that E is not completely strongly porous.

Let us show that E is w-strongly porous at 0. Note that, for every $\tilde{y} = \{y_n\}_{n \in \mathbf{N}} \in \tilde{E}_0^d$, there are three possibilities:

(i) $y_n \in E_1$ holds for an infinite number of subscripts n;

(ii) there is $k \in \mathbf{N}$ such that

$$\operatorname{card}(\{y_n \colon n \in \mathbf{N}\} \cap \{\tau_n^* \colon n \in \mathbf{N}_k\}) = \infty;$$

(iii) there is an infinite strictly increasing sequence $\{k_i\}_{i \in \mathbf{N}}$ such that

$$\{y_n \colon n \in \mathbf{N}\} \cap \{\tau_n^* \colon n \in \mathbf{N}_{k_i}\} \neq \varnothing.$$

It follows directly from the definitions that

$$n \geq \nu(k) \geq k \tag{2.13}$$

for every $k \in \mathbf{N}$ and every $n \in \mathbf{N}_k$. This double inequality implies that $n \geq \nu(m(n))$. Using the last inequality and definitions of τ_n and τ_n^* we obtain

$$\tau_{n+1} = 2^{-n^2}\tau_n \leq 2^{-n}\tau_n \leq 2^{-\nu(m(n))}\tau_n = \tau_n^* < \tau_n. \tag{2.14}$$

In particular, (2.14) implies that $\tau_n^* = \tau_n$ is possible only for $n = 1$ and that

$$\tau_{n+1}^* < \tau_{n+1} < \tau_n^* < \tau_n$$

holds for every $n \geq 2$. Moreover we obtain from (2.14) that

$$\lim_{n \to \infty} \frac{\tau_{n+1}}{\tau_n^*} \leq \lim_{n \to \infty} \frac{2^{-n^2}}{2^{-n}} = 0. \tag{2.15}$$

Consequently $\{(\tau_{n+1}, \tau_n^*)\}_{n \in \mathbf{N}} \in \tilde{I}_E$. Using the last membership we can show that for $\tilde{y}$ satisfying (i), there is $\tilde{y}'$ such that E is $\tilde{y}'$-strongly porous at 0. If $\tilde{y}$ meets condition (ii), then using (2.15) and the equality $\tau_n^* = 2^{-\nu(k)}\tau_n$, $n \in \mathbf{N}_k$, we can also find the desired $\tilde{y}'$. Finally, if (iii) holds, then $\tilde{y}'$ can be constructed with the use of the relation

$$\lim_{k\to\infty} 2^{-\nu(k)} = 0,$$

wich follows from the second inequality in (2.13). We leave the details of the constructions of $\tilde{y}'$ to the reader.

Remark 2.11. Considering E from Example 2.10 as a pointed metric space with a marked point 0 we can show that the inequality $\operatorname{card}(\Omega^E_{0,\tilde{r}}) \leq 3$ holds for every $\Omega^E_{0,\tilde{r}}$. On the other hand, if (X, d, p) is a pointed metric space such that $\operatorname{card}(\Omega^E_{p,\tilde{r}}) \leq 2$ holds for every $\Omega^E_{p,\tilde{r}}$, then the distance set $\{d(x, p) \colon x \in X\}$ is completely strongly porous at 0.

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