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BLOCH SPACE AND THE NORM OF THE BERGMAN PROJECTION

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Abstract. In this note we calculate the norm of the Bergman projection from the space of essentially bounded functions to the Bloch space. We complement author's earlier results in [6] and their generalizations in [5].

1. Introduction

Bergman projection is no doubt a central object in the study of analytic function spaces. In the setting of the unit disk $\mathbf{D}$, it arises as the following integral operator

$$Pf(z) = \int_{\mathbf{D}} \frac{f(w)\,dA(w)}{(1-z\bar{w})^2}.$$

For $0 < p < \infty$ we denote by A^p the Bergman spaces, consisting of L^p-integrable analytic functions on $\mathbf{D}$ with respect to the normalized area measure $dA(z) = \pi^{-1}\,dx\,dy$ for $z = x + iy$. By H^∞ we mean the space of bounded analytic functions on the disk.

When $p = 2$, P is an orthogonal projection, and it is obvious that the norm of $P\colon L^2 \to A^2$ is one. However, this projection is also bounded $L^p \to A^p$ for $1 < p < \infty$, but unbounded for $p = 1$ and $p = \infty$.

There is a natural space, onto which P maps L^∞ boundedly. This is the Bloch space $\mathcal{B}$, which consists of analytic functions f such that

$$\|f\|_* = \sup_{z\in\mathbf{D}} (1-|z|^2)|f'(z)| < \infty$$

holds true. In [6] the author showed that

$$\|Pf\|_* \le (8/\pi)\|f\|_\infty,$$

where the constant $8/\pi$ is the best possible.

Note that the quantity $\|\cdot\|_*$ defined above is only a semi-norm. It clearly fails to distinguish constant functions, so it cannot be a norm. The most common way to overcome this inconvenience is to set

$$\|f\| = \|f\|_* + |f(0)|$$

to be the norm on $\mathcal{B}$.

In this note, we complement the result of [6]. We prove that

$$\|Pf\| \le (1 + 8/\pi)\|f\|_\infty,$$

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where the constant in the inequality is the best possible. The reader is referred to [8] for more information about analytic Bergman and Bloch spaces. It is fairly easy to see that the presented method works also for the more general setting in [5]; this will be commented in the last section.

We remark that calculating the exact norm of P on L^p spaces is a long-standing problem and only partial results are known, see [1] and [7]. Similar questions have been solved in the setting of Hardy and Segal–Bargmann spaces in the papers [3] and [2], respectively. Also, the paper [4] deals with a related question for the Beurling transform.

2. The result

Denote by $\|P\|$ the operator norm of $P\colon L^\infty \to \mathcal{B}$, that is, the optimal constant C in the inequality

$$\|Pf\| \leq C\|f\|_\infty.$$

We will show that $\|P\| = 1 + 8/\pi$. We will begin with simple lemma, which was actually proven in the recent paper by Kalaj and Markovic [5].

Lemma 2.1. *The norm of the projection P satisfies $\|P\| \leq 1 + 8/\pi$.*

Proof. Just note that $|Pf(0)| = |\int_{\mathbf{D}} f(z)\, dA(z)| \leq \|f\|_\infty$. □

In view of this lemma, it is enough to prove $\|P\| \geq 1 + 8/\pi$. The problem is then, whether the quantities $\|Pf\|_*$ and $|Pf(0)|$ can be maximized simultaneously. We will answer this problem in the affirmative.

The method in [6] uses functions

$$g_z(w) = \frac{w|1 - z\bar{w}|^3}{|w|(1 - \bar{z}w)^3},$$

which can be applied to maximize $\|Pf\|_*$. Here, these do not seem to work, because $|Pg_z(0)| < 1$ for all z. The next proposition is proven in [6], but we sketch the proof for the reader's convenience.

Proposition 2.2. *The optimal constant C for*

$$\|Pf\|_* \leq C\|f\|_\infty$$

is $8/\pi$. Moreover, it is obtained as the limit of

$$(1 - |z|^2)|(Pg_z)'(z)| = \int_{\mathbf{D}} \frac{2(1 - |z|^2)|w|\, dA(w)}{|1 - z\bar{w}|^3} = F(z)$$

as $|z| \to 1^-$.

Proof. Given $z \in \mathbf{D}$, the quantity $(1-|z|^2)|(Pg)'(z)|$ over $\|g\|_\infty \leq 1$ is maximized by g_z. This is transparent from the integral formula

$$|(Pg)'(z)| = \left| \int_{\mathbf{D}} \frac{2\bar{w}g(w)\, dA(w)}{(1 - z\bar{w})^3} \right|,$$

which is obtained by differentiating inside the integral.

Denote by $\varphi_z(w) = (z - w)/(1 - \bar{z}w)$ the usual Möbius transformation of the disk. By applying a change of variables $w \mapsto \varphi_z(w)$, one gets

$$\int_{\mathbf{D}} \frac{2(1 - |z|^2)|w|\, dA(w)}{|1 - z\bar{w}|^3} = \int_{\mathbf{D}} \frac{2|z - w|\, dA(w)}{|1 - z\bar{w}|^2}.$$

From this, one sees that $F(z)$ is subharmonic. Since it is easy to verify that $F(z)$ is also radial, the maximum of $F(z)$ is obtained as $|z| \to 1^-$. But

$$F(1) = \int_{\mathbf{D}} \frac{2dA(w)}{|1-w|} = 8/\pi,$$

and the proof is complete. □

The key to the proof of our result is noting that only the behaviour of g_z near the boundary matters. Define new test functions g_z^r with $\|g_z^r\|_\infty \le 1$ as follows:

$$g_z^r(w) = \begin{cases} g_z(w) & \text{if } |w| \ge r, \\ 1 & \text{if } |w| \le r^2. \end{cases}$$

Finally, define g_z^r on $\{r^2 < |w| < r\}$ so that g_z^r is continuous on $\overline{\mathbf{D}}$.

Lemma 2.3. *For every $r \in (0,1)$, we have*

$$(1-|z|^2)(Pg_z^r)'(z) \to 8/\pi$$

as $|z| \to 1^-$.

Proof. Because $|g_z(w) - g_z^r(w)| \le 2$ on $\mathbf{D}$ and $|g_z(w) - g_z^r(w)| = 0$ when $|w| > r$, we can estimate

$$(1-|z|^2)|(P(g_z - g_z^r))'(z)| \le \int_{B(0,r)} \frac{4(1-|z|^2)|w|dA(w)}{|1-z\bar{w}|^3}.$$

The right hand side goes to 0 as $|z| \to 1^-$. Therefore, one can conclude that

$$(1-|z|^2)(Pg_z^r)'(z) \to 8/\pi$$

as $|z| \to 1^-$. □

We have now collected sufficient amount of information to prove our main result without much trouble.

Theorem 2.4. *The operator norm of $P\colon L^\infty \to \mathcal{B}$ is $1 + 8/\pi$.*

Proof. For every $z \in \mathbf{D}$, we have

$$|Pg_z^r(0)| \ge r^4 - \int_{\mathbf{D}\setminus B(0,r^2)} dA(w) = 2r^4 - 1 \to 1$$

as $r \to 1$. Given $\epsilon > 0$, we may pick $r > 0$ such that $|Pg_z^r(0)| > 1 - \epsilon/2$ for every $z \in \mathbf{D}$. Fix such r. According to the previous lemma, one can pick $z \in \mathbf{D}$ such that

$$(1-|z|^2)(Pg_z^r)'(z) > 8/\pi - \epsilon/2.$$

But, one then ends up with a function g_z^r such that

$$\|P\| \ge \|Pg_z^r\| \ge |Pg_z^r(0)| + (1-|z|^2)|(Pg_z^r)'(z)| > 1 + 8/\pi - \epsilon.$$

Therefore $\|P\| \ge 1 + 8/\pi$, which together with Lemma 2.1 proves the claim. □

Denote by $\mathcal{B}_0$ the little Bloch space consisting of $f \in \mathcal{B}$ such that

$$\lim_{|z|\to 1^-} (1-|z|^2)|f'(z)| = 0.$$

We equip $\mathcal{B}_0$ also with the norm $\|\cdot\|$ used for the Bloch space. It is known that $P\colon C(\overline{\mathbf{D}}) \to \mathcal{B}_0$ is bounded and onto, where $C(\overline{\mathbf{D}})$ stands for the functions continuous on $\overline{\mathbf{D}}$. Note that $g_z^r \in C(\overline{\mathbf{D}})$, by definition. This shows that the operator norm of $P\colon C(\overline{\mathbf{D}}) \to \mathcal{B}_0$ is likewise $1 + 8/\pi$.

3. Several dimensions

In their recent paper [5], Kalaj and Markovic prove a generalization of [6]. This paper can currently be found in arXiv. Denote by $\mathbf{B}_n$ the unit ball of $\mathbf{C}^n$, and let $\alpha > -1$. On $\mathbf{B}_n$ we use the following measure

$$dV_\alpha = c_\alpha (1 - |z|^2)^\alpha dV(z).$$

Here dV is the normalized $2n$-dimensional Lebesgue measure on $\mathbf{B}_n$, and c_α is a constant such that dV_α is a probability measure.

The Bloch space $\mathcal{B}_n$ on $\mathbf{B}_n$ the set of those holomorphic functions f on such that

$$\|f\|_{*,n} = \sup_{z \in \mathbf{B}_n} (1 - |z|^2)|\nabla f(z)| < \infty,$$

where $\nabla = (\partial_{z_1}, \partial_{z_2}, \ldots, \partial_{z_n})$ is the usual complex gradient. The little Bloch space $\mathcal{B}_{0,n}$ is defined analogously to the case $n = 1$. The Bergman projection $P_{n,\alpha}$ is defined as

$$P_{n,\alpha} f(z) = \int_{\mathbf{B}_n} \frac{f(w)\, dV_\alpha(w)}{(1 - \langle z, w \rangle)^{1+n+\alpha}}.$$

Here

$$\langle z, w \rangle = \sum_{i=1}^n z_i \bar{w}_i$$

is the inner product of $\mathbf{C}^n$. Denote by Γ the Euler Gamma function. According to [5], the optimal constant $C(\alpha, n)$ in

$$\|P_{n,\alpha}\|_{*,n} \le C(\alpha, n) \|f\|_\infty$$

satisfies

$$C(\alpha, n) = \frac{\Gamma(2 + n + \alpha)}{\Gamma^2((2 + n + \alpha)/2)},$$

which equals $8/\pi$ if $\alpha = 0$ and $n = 1$, so it generalizes the main result of [6].

Of course $\mathcal{B}_n$ can be equipped with the norm

$$\|f\|_n = \|f\|_{*,n} + |f(0)|,$$

so we can discuss the operator norm of $P_{n,\alpha}$, which we denote by $\|P\|_{n,\alpha}$.

Like in [6], the authors of [5] use functions

$$g_z(w) = \frac{w_1}{|w_1|} \frac{|1 - \langle z, w \rangle|^{n+2+\alpha}}{(1 - \langle w, z \rangle)^{n+2+\alpha}}$$

to obtain $C(\alpha, n)$ as the limit of

$$(1 - |z|^2)|\nabla(P g_z)(z)|$$

when $z \to e_1 = (1, 0, \ldots, 0)$. By defining the functions g_z^r analogously to the previous section, one obtains

Corollary 3.1. *The operator norm of* $P_{n,\alpha}$ *from* $L^\infty(\mathbf{B}_n)$ *onto* $\mathcal{B}_n$ *equals* $C(\alpha, n) + 1$. *The same is true about* $P_{n,\alpha}\colon C(\overline{\mathbf{B}_n}) \to \mathcal{B}_{0,n}$.

The details of the proof are left as exercise for the reader. The proof is completely analogous to the case $n = 1$ and $\alpha = 0$.

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