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TANGENTIAL PROPERTIES OF QUASIHYPERBOLIC GEODESICS IN BANACH SPACES

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Abstract. The main result states that in a large class of Banach spaces including all L_p spaces with $1 < p < \infty$, quasihyperbolic geodesics of domains are smooth.

1. Introduction

1.1. Quasihyperbolic metric. Throughout the paper, E will denote a real Banach space with $\dim E \ge 2$, and $G \subsetneq E$ will be a domain (open connected nonempty set). We recall that the *quasihyperbolic length* of a rectifiable arc $\gamma \subset G$ or a path γ in G is the number

$$l_k(\gamma) = \int_\gamma \frac{|dx|}{\delta(x)},$$

where $\delta(x) = \delta_G(x) = d(x, E \setminus G) = d(x, \partial G)$. For $a, b \in G$, the *quasihyperbolic distance* $k(a, b) = k_G(a, b)$ is defined by

$$k(a, b) = \inf l_k(\gamma)$$

over all rectifiable arcs γ joining a and b in G. An arc γ from a to b is a *quasihyperbolic geodesic* in G or briefly a *geodesic* if $l_k(\gamma) = k(a, b)$. A geodesic between given points $a, b \in G$ need not exist (see [Vä1, 3.5]), but it exists if either $\dim E < \infty$ [GO, Lemma 1] or if G is a convex domain in a reflexive space [Vä2, 2.1].

The quasihyperbolic metric has turned out to be a useful tool, for example, in the theory of quasiconformal and related maps.

In this paper we are interested in the smoothness properties of quasihyperbolic geodesics. In the main result 4.7 we show that in a large class of Banach spaces, including all L_p spaces with $1 < p < \infty$, quasihyperbolic geodesics in all domains are smooth, that is, they have a continuous tangent.

2. Preliminaries

In this section we present preparatory material needed in later sections.

2.1. Notation. The norm of a Banach space E will be written as $|\cdot|$, occasionally as $\|\cdot\|$. For $a \in E,\ r > 0$ we let $B(a, r)$ and $\bar{B}(a, r)$ denote the open and closed ball in E with center a and radius r, and we write $S(a, r) = \partial B(a, r) = \{x \in E \colon |x-a| = r\}$. The center a may be omitted if $a = 0$. In particular, $S(1)$ is the unit sphere. For $x, y \in E,\ x \ne y$, the line segment with endpoints x, y is $[x, y]$, the line through x, y is $\langle x, y\rangle$, and the ray from x through y is $[x, y\rangle$. For half open and open segments and for open rays we use the notations $[a, b)$, (a, b) and $(a, b\rangle$. For $x \in E,\ x \ne 0$ we write $\hat{x} = x/|x|$.

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The distance between nonempty sets $A, B \subset E$ is $d(A, B)$ and the boundary of a set $A \subset E$ is ∂A. For real numbers r, s we set $r \vee s = \max\{r, s\},\ r \wedge s = \min\{r, s\}$.

2.2. Curves and arcs. A set $C \subset E$ is a *curve* if it is homeomorphic to a real interval, which may be open, half open or closed. We assume that all curves are *oriented*, that is, equipped with one of the two natural orderings, written as $x \le y$. A curve is an *arc* if it is homeomorphic to a closed interval. An arc has two *endpoints*. If $a < b$ are the endpoints, then a is the *left* and b the *right* endpoint. We write $\gamma\colon a \curvearrowright b$ if γ is an arc with endpoints a and b and with the orientation $a < b$. We let $l(\gamma)$ denote the length of γ in the norm metric.

2.3. Deviations. In a Hilbert space, the angle $\operatorname{ang}(x, y)$ between nonzero vectors $x, y \in E$ is well-defined. We define some numbers, called *deviations*, which act as substitutes for the angle in normed spaces.

Let $x, y \in E,\ x \ne 0 \ne y$. We write

$$\begin{aligned} \operatorname{dev}(x, y) &= |\hat{x} - \hat{y}|, \\ \operatorname{dev}_1(x, y) &= d(\hat{x}, [0, y\rangle). \end{aligned}$$

If E is a Hilbert space and $\operatorname{ang}(x, y) = \alpha$, then

$$\operatorname{dev}(x, y) = 2\sin(\alpha/2), \quad \operatorname{dev}_1(x, y) = \begin{cases} \sin\alpha & \text{if } 0 \le \alpha \le \pi/2, \\ 1 & \text{if } \pi/2 \le \alpha \le \pi. \end{cases} \tag{2.4}$$

2.5. Lemma. *Let $x, y \in E$ be nonzero vectors. Then*

$$\operatorname{dev}_1(x, y) \le \operatorname{dev}(x, y) \le 2\operatorname{dev}_1(x, y).$$

Proof. The first inequality is trivial. For the second inequality, we may assume that $|x| = |y| = 1$. Set $s = \operatorname{dev}(x, y) = |x - y|$ and let $u \in [0, y\rangle$. We must show that $|x - u| \ge s/2$.

If $|u| \le 1 - s/2$, then $|x - u| \ge |x| - |u| \ge s/2$. If $||u| - 1| \le s/2$, then $|x - u| \ge |x - y| - |y - u| \ge s/2$. Finally, if $|u| \ge 1 + s/2$, then $|x - u| \ge |u| - |x| \ge s/2$. □

2.6. Remark. Clearly $\operatorname{dev}(x, y) = \operatorname{dev}(y, x)$, but dev_1 is not symmetric. However, Lemma 2.5 implies that $\operatorname{dev}_1(x, y) \le 2\ \operatorname{dev}_1(y, x)$.

2.7. Standard estimates. We recall some well-known estimates for the quasihyperbolic metric. Let $a, b \in G \subset E$. The j-metric of a domain G is defined by

$$j(a, b) = \log\left(1 + \frac{|a - b|}{\delta(a) \wedge \delta(b)}\right).$$

We have always

$$k(a, b) \ge j(a, b) \ge \log\left(1 + \frac{|a - b|}{\delta(a)}\right) \ge \log\frac{\delta(b)}{\delta(a)}; \tag{2.8}$$

see e.g. [GP, 2.1] or [Vä1, 3.7(1)].

Next, if $0 < t \le 1$ and $|a - b| \le t\delta(a)$, we have

$$k(a, b) \le l_k[a, b] \le \frac{|a - b|}{(1 - t)\delta(a)}. \tag{2.9}$$

Furthermore, if either $|a - b| \le \delta(a)/2$ or $k(a, b) \le 1$, then

$$k(a, b)/2 \le \frac{|a - b|}{\delta(a)} \le 2k(a, b); \tag{2.10}$$

see [Vä1, 3.9]. If $k(a,b) = \lambda \le 1/4$, then the first inequality of (2.10) can be improved to

$$\lambda(1-2\lambda) \le \frac{|a-b|}{\delta(a)}. \tag{2.11}$$

Indeed, the second inequality of (2.10) gives $|a-b| \le 2\lambda\delta(a)$, and (2.11) follows from (2.9).

2.12. Strictly, uniformly and strongly convex spaces. We recall that a space E is *strictly convex* if the unit sphere $S(1)$ does not contain a line segment. Equivalently, the equality $|x+y| = |x| + |y|$ implies that x and y have the same direction; see [FZ, 8.11].

A space E is *uniformly convex* if there is a continuous strictly increasing function $\psi\colon [0,2] \to [0,1]$ such that $\psi(0) = 0$ and such that

$$|x+y|/2 \le 1 - \psi(|x-y|)$$

for all $x, y \in S(1)$. We also say that E is ψ-uniformly convex. A uniformly convex space is always strictly convex. The converse holds if $\dim E < \infty$. A linear subspace of a ψ-uniformly convex space is also ψ-uniformly convex.

The function $\delta\colon [0,2] \to [0,1]$, defined by

$$\delta(t) = \inf\{1 - |x+y|/2 \colon x, y \in S(1),\ |x-y| = t\},$$

is the *convexity modulus* of a uniformly convex space E. It is easy to show that δ is a homeomorphism of $[0,2]$ onto $[0,1]$, and thus

$$\delta(t) = \max\{\psi(t) \colon E \text{ is } \psi\text{-uniformly convex}\}.$$

The inverse function ψ^{-1} is defined on $[0,\psi(2)]$ but we extend it to the whole $[0,1]$ by setting $\psi^{-1}(t) = 2$ for $\psi(2) < t \le 2$. Then $\psi(t) \le s$ implies $t \le \psi^{-1}(s)$ for all $0 \le t \le 2,\ 0 \le s \le 1$.

We say that E is *ψ-strongly convex* if E is ψ-uniformly convex and if the function $t \mapsto \psi^{-1}(t)/t$ is integrable on $(0,1]$.

2.13. Examples. 1. The convexity modulus of a Hilbert space is $\psi_H(t) = 1 - \sqrt{1-t^2/4} \le t^2/4$, and $\psi(t) \le \psi_H(t)$ for every ψ-uniformly convex space; see [BL, p. 409].

2. If $s \ge 2,\ m > 0$ and if $\psi(t) \ge mt^s$ for all t, then the function ψ is said to be of *power type* s, and a ψ-uniformly convex space is also said to be of power type s. Then

$$\psi^{-1}(t)/t \le m^{1/s} t^{1/s-1},$$

whence E is ψ-strongly convex. For $1 < p < \infty$, each L_p-space is of power type $p \vee 2$ and hence strongly convex; see [LZ, p. 63].

3. Suppose that $s > 1,\ m > 0$ and that E is ψ-uniformly convex with $\psi(t) = m\exp(-t^{-1/s})$. Then ψ is not of power type. However, E is strongly convex, since the function

$$\frac{\psi^{-1}(t)}{t} = \frac{1}{t(\log\frac{m}{t})^s}$$

is integrable on $(0,\psi(2)]$.

4. On the other hand, if $\psi(t) \le e^{-1/t}$, then

$$\frac{\psi^{-1}(t)}{t} \ge \frac{1}{t \log \frac{1}{t}}$$

is not integrable, whence a ψ-uniformly convex space is not ψ-strongly convex.

From the definition we readily obtain:

2.14. Lemma. *Suppose that E is ψ-uniformly convex and that $a \in E,\ u, v \in S(a, r)$. Then*

$$|a - (u + v)/2| \le r - r\psi(|u - v|/r). \qquad \square$$

2.15. Lemma. *Suppose that E is ψ-uniformly convex, that $L \subset E$ is a line and that $a, b \in L, u \in E \setminus L$. Set $r = |u - a|,\ s = |u - b|$. Then*

$$d(u, L) \le (r \vee s)\psi^{-1}(1 - |a - b|/(r + s)).$$

Proof. We may assume that $a = 0$ and that $\dim E = 2$. Let H and H' be the components of $E \setminus L$ with $u \in H$. We first show that there is a point $u' \in H'$ such that $|u' - a| = r,\ |u' - b| = s$. The points $x = -r\hat{b}$ and $y = r\hat{b}$ are the endpoints of the semicircle $\gamma' = S(r) \cap \bar{H}'$. It suffices to show that

$$|x - b| \ge s, \quad |y - b| \le s, \tag{2.16}$$

because this will imply that there is a point $u' \in \gamma' \cap S(b, s)$.

We have

$$|x - b| = |b| + r, \quad |y - b| = ||b| - r|.$$

From the triangle $0bu$ we obtain the inequalities

$$|r - s| \le |b| \le r + s.$$

Hence $|x - b| \ge |r - s| + r \ge s$. If $r \ge |b|$, then $|y - b| = r - |b| \le r - (r - s) = s$. If $r < |b|$, then $|y - b| = |b| - r \le s$. We have proved (2.16).

For $z = (u + u')/2$ we obtain by 2.14

$$|a - z| \le r - r\psi(|u - u'|/r), \quad |b - z| \le s - s\psi(|u - u'|/s),$$

whence

$$|a - b| \le |a - z| + |b - z| < (r + s)\left(1 - \psi\left(\frac{|u - u'|}{r \vee s}\right)\right).$$

Since $d(u, L) \le |u - u'|$, this implies the lemma. $\square$

Lemma 2.17. *Let $y \in G$ and $z \in \partial G$ with $|y - z| = \delta(y)$. Then for each $x \in (y, z)$, $[y, x]$ is a geodesic and*

$$k(x, y) = \log \frac{\delta(y)}{\delta(x)} = \log \frac{|y - z|}{|x - z|}.$$

If E is strictly convex, then $[y, x]$ is the only geodesic $y \curvearrowright x$.

Proof. For $u \in [y, z]$ we have $\delta(u) = |u - z|$, whence

$$k(x, y) \le l_k[x, y] = \int_{\delta(x)}^{\delta(y)} \frac{dt}{t} = \log \frac{\delta(y)}{\delta(x)}.$$

The converse inequality follows from (2.8).

Assume that E is strictly convex and that $\gamma\colon y \curvearrowright x$ is a geodesic. Let $u \in \gamma$. As $\delta(u) \le \delta(x) + |x-u|$ and $\delta(y) \le \delta(u) + |u-y|$, we get by (2.8)

$$
\begin{aligned}
k(x,u) &\ge \log\frac{\delta(x)+|x-u|}{\delta(x)} \ge \log\frac{\delta(u)}{\delta(x)},\\
k(u,y) &\ge \log\frac{\delta(u)+|u-y|}{\delta(u)} \ge \log\frac{\delta(y)}{\delta(u)}.
\end{aligned}
\tag{2.18}
$$

Since γ is a geodesic, we have

$$k(x,y) = k(x,u) + k(u,y) \ge \log\frac{\delta(y)}{\delta(x)} = k(x,y).$$

Hence each inequality in (2.18) holds as an equality, whence

$$\delta(u) = \delta(x) + |x-u|, \quad \delta(y) = \delta(u) + |u-y|.$$

As $\delta(y) - \delta(x) = |x-y|$, we obtain $|x-y| = |x-u| + |u-y|$, which implies that $u \in [y,x]$ by strict convexity. Hence $\gamma = [y,x]$. □

2.19. Remark on ball convexity. If E is a Hilbert space, then by the important ball convexity theorem of Martin ([Ma, 2.2] or [Vä3, 2.4]), the quasihyperbolic geodesics in a domain $G \subset E$ are *ball convex*, that is, a geodesic $\gamma\colon a \curvearrowright b$ lies in a ball $B \subset G$ whenever $a, b \in B$. The corresponding result is not true in arbitrary Banach spaces; see Example 5.8. However, it seems to be an open problem, whether quasihyperbolic geodesics are *ball starlike*, that is, a geodesic $\gamma\colon a \curvearrowright b$ lies in a ball $B = B(a, t\delta(a)$ whenever $t \le 1$ and $|b-a| < t\delta(a)$.

3. Approximative smoothness of geodesics

3.1. Smooth curves. Let $C \subset E$ be a curve. Recall that we consider C as an ordered set. Let $z \in C$ and assume that z is not the right endpoint of C. A unit vector $v \in E$ is the *right tangent vector* of C at z if

$$v = \lim_{\substack{x\to z\\ x\in C,\, x>z}} \frac{x-z}{|x-z|},$$

or equivalently, if $\operatorname{dev}(x-z, v) \to 0$ as $x \to z$ and $x > z$ on C. Similarly, if z is not the left endpoint and if the left limit of $(z-x)/|z-x|$ exists, it is the *left tangent vector* of γ at $z \in \gamma(a,b]$ (observe the sign). If both of these exist and are equal, their common value is the *tangent vector* of γ at x.

We say that a curve C is *smooth* if it has a continuous tangent. More precisely,

(1) the tangent vector $v(z)$ exists at each interior point $z \in C$,
(2) the right tangent vector $v(a)$ exists at the possible left endpoint a of C,
(3) the left tangent vector $v(b)$ exists at the possible right endpoint b of C,
(4) the function $v\colon C \to E$ is continuous.

Equivalently, a curve C is smooth if it has a smooth parametrization $f\colon J \to C$ with $f'(x) \ne 0$. The Hilbert case was proved in [Vä3, Appendix] and the Banach case is rather similar.

3.2. Smoothness of geodesics. Martin [Ma, 4.8] proved in 1985 that quasihyperbolic geodesics in a domain G in the Euclidean space $\mathbf{R}^n$ are smooth. Another proof, valid in all Hilbert spaces, was given in [Vä3, 2.8]. These proofs were based on Martin's idea of the ball convexity of quasihyperbolic geodesics, which is no longer valid in arbitrary Banach spaces. In fact, a geodesic in a half plane of $\mathbf{R}^2$ with the max norm need not have a tangent at all points; see 5.7. However, it is an open problem whether quasihyperbolic geodesics in uniformly convex Banach spaces are smooth.

In 3.4 we give a weaker result, which might be called *approximative smoothness.* It shows that if γ is a geodesic in a domain of a uniformly convex space and if $a \in \gamma$, then for small $t > 0$, the set $\gamma \cap B(a, t\delta(a))$ lies in a narrow neighborhood of a line, which might be considered as an approximate tangent of γ. However, the direction of this line is allowed to vary as $t \to 0$.

In Section 4 we show in 4.7 that in a strongly convex space, every quasihyperbolic geodesic in a domain is smooth.

The proofs are based on the following key result:

3.3. Theorem. *Suppose that E is ψ-uniformly convex and that $\gamma\colon a \curvearrowright b$ is a geodesic in $G \subset E$ with $k(a,b) = \lambda \le 1/4$. Then*

$$d(u, \langle a, b\rangle) < |a-b|\varepsilon(\lambda)$$

for all $u \in \gamma$, where $\varepsilon(\lambda) = 3\psi^{-1}(4\lambda) \to 0$ as $\lambda \to 0$.

Proof. The proof is based on the following rough idea: Since a small quasihyperbolic distance $k(a,b)$ is close to the scaled norm distance $|a-b|/\delta(a)$, also the quasihyperbolic geodesic $\gamma\colon a \curvearrowright b$ is close to the norm geodesic $[a,b]$.

For each $z \in \gamma$ we have $k(a,z) \le k(a,b) = \lambda$. By (2.10) this yields $|z-a| \le 2\delta(a)k(a,z) \le 2\lambda\delta(a)$. Hence $\delta(z) \le \delta(a) + |z-a| \le (1+2\lambda)\delta(a)$, which implies that

$$l(\gamma) \le \lambda(1+2\lambda)\delta(a).$$

As $|a-b| \le 2\lambda\delta(a)$, (2.11) gives

$$\lambda\delta(a) \le \frac{|a-b|}{1-2\lambda}.$$

Setting $r = |a-u|,\ s = |u-b|$ we obtain

$$r + s \le l(\gamma) \le \frac{(1+2\lambda)|a-b|}{1-2\lambda} \le 3|a-b|.$$

The theorem follows from this and from Lemma 2.15. □

3.4. Theorem. *Suppose that E is ψ-uniformly convex and that $a \in G \subset E$, $x, y \in \bar{B}(a, t\delta(a))$, $0 < t < 1/9$. Let $\gamma\colon x \curvearrowright y$ be a geodesic containing a. Then $d(a, \langle x, y\rangle) \le t\delta(a)\varepsilon(t)$, where $\varepsilon(t) = 6\psi^{-1}(9t) \to 0$ as $t \to 0$.*

Proof. By (2.9) we have

$$\lambda = k(x,y) \le \frac{|x-y|}{(1-t)\delta(a)} \le \frac{2t}{1-t}.$$

As $t \le 1/9$, we get $\lambda \le 9t/4 \le 1/4$. Hence 3.3 gives

$$d(a, \langle x, y\rangle) \le 3|x-y|\psi^{-1}(4\lambda) \le 6t\delta(a)\psi^{-1}(9t). \qquad \square$$

We shall apply Theorem 3.4 in 3.7 to prove that in a uniformly convex space E, a geodesic γ in $G \subset E$ has a tangent at an interior point $a \in \gamma$ whenever it has a one-sided tangent at a. For this we need some elementary geometry in normed spaces.

3.5. Lemma. *If $x, y \in S(1)$ and $|x+y| \ge 1$, then $d(0, [x,y]) \ge 1/4$.*

Proof. Set $z = (x+y)/2$ and let $u \in [x, y]$. We must show that $|u| \ge 1/4$. We may assume that $u \in [x, z]$. If $|x-u| \le 3/4$, then $|u| \ge |x|-|x-u| \ge 1/4$. If $|x-u| \ge 3/4$, then $|u-z| = |x-z|-|x-u| \le 1-3/4 = 1/4$, whence $|u| \ge |z|-|u-z| \ge 1/4$. □

3.6. Lemma. *Suppose that* $p, x, y \in S(1)$, $|x-p| \le \alpha$, $d(0, [x, y]) \le \beta$, $\alpha+\beta < 1/4$. *Then* $|y+p| \le 8(\alpha+\beta)$.

Proof. Case 1. $x = p$. As $\beta < 1/4$, we have $|x+y| < 1$ by 3.5. The function $f\colon \mathbf{R} \to \mathbf{R}$, defined by $f(t) = |y+tx|$, is convex, and $f(0) = 1 > f(1)$. Hence $f(t) \ge 1$ for $t \le 0$, which implies that $\operatorname{dev}_1(y, -x) = |y-x| \operatorname{dev}_1(y-x, -x)$. By 2.5 this yields

$$\begin{aligned} |y+x| &= \operatorname{dev}(y, -x) \le 2 \operatorname{dev}_1(y, -x) = 2|y-x| \operatorname{dev}_1(y-x, -x) \\ &\le 4|y-x| \operatorname{dev}_1(-x, y-x) = 4|y-x| d(0, [x, y]) \le 8\beta. \end{aligned}$$

Case 2. x is arbitrary. Let $z \in [x, y]$ and $z' \in [y, p\rangle$ be points such that

$$|z| = d(0, [x, y]), \quad |z-z'| = d(z, [y, p\rangle).$$

Then $|z| \le \beta$ and $|z-z'| \le d(x, [y, p\rangle) \le |x-p| \le \alpha$, whence

$$d(0, [p, y]) \le |z'| \le |z'-z| + |z| \le \alpha+\beta.$$

By Case 1 this gives $|y+p| \le 8(\alpha+\beta)$. □

3.7. Theorem. *Suppose that E is uniformly convex and that a is an interior point of a geodesic γ in a domain $G \subset E$. If γ has a left or a right tangent at a, it has a tangent at a.*

Proof. Assume, for example, that γ has a left tangent vector v at a. Let $0 < t < 1/9$ and suppose that γ meets $S(a, t\delta(a))$ on both sides of a. Write

$$s = t\delta(a), \quad p = a - sv, \quad q = a + sv.$$

Let $x, y \in \gamma \cap S(a, s)$ be points such that $x < a < y$ on γ. Then $|x-p| \le s\varepsilon(s)$ where $\varepsilon(s) \to 0$ as $s \to 0$. We must find an estimate $|y-q| \le s\varepsilon^*(s)$ with $\varepsilon^*(s) \to 0$.

As $|z-a| > s$ for $z \in \langle x, y\rangle \setminus [x, y]$, Theorem 3.4 gives $d(a, [x, y]) \le s\varepsilon_1(s)$ with $\varepsilon_1(s) \to 0$. Assume that s is so small that $\varepsilon_1(s) + \varepsilon(s) < 1/4$. Using an auxiliary homothety carrying $S(a, s)$ onto $S(1)$ and Lemma 3.6 we obtain

$$|y-q| \le 8s(\varepsilon_1(s) + \varepsilon(s)) = s\varepsilon^*(s).$$ □

4. Strongly convex spaces

Recall from 2.12 that a ψ-uniformly convex space is ψ-strongly convex if $\psi^{-1}(t)/t$ is integrable on the interval $0 < t \le \psi(2)$.

In this section we show in 4.7 that in a ψ-strongly convex space, every quasihyperbolic geodesic γ in a domain $G \subset E$ is smooth. Moreover, the tangent vector $v\colon \gamma \to S(1)$ is uniformly continuous in the quasihyperbolic metric with a continuity modulus depending only on ψ.

We first give some auxiliary elementary results on real functions and on the geometry of normed spaces.

4.1. Lemma. *Let $f\colon (0, \lambda] \to \mathbf{R}$ be a positive increasing function. Then the following conditions are equivalent:*

(1) $\int_0^\lambda \frac{f(t)}{t}\, dt < \infty$,

(2) *the series* $\sum_{j=0}^{\infty} f(2^{-j}\lambda)$ *is convergent.*

If these conditions hold, then

$$\frac{1}{2}\sum_{j=1}^{\infty} f(2^{-j}\lambda) \le \int_0^{\lambda} \frac{f(t)}{t}\,dt \le \sum_{j=0}^{\infty} f(2^{-j}\lambda). \tag{4.2}$$

Proof. For $t \in I_j = [2^{-j-1}\lambda, 2^{-j}\lambda]$ we have

$$f(2^{-j-1}\lambda) \le f(t) \le f(2^{-j}\lambda), \quad 2^j/\lambda \le 1/t \le 2^{j+1}/\lambda.$$

Multiplying, integrating over $t \in I_j$ and summing over $j \ge 0$ we obtain the lemma. □

4.3. Lemma. *Suppose that $L, L_0 \subset E$ are parallel lines with $0 \in L_0$ and $d(L, L_0) < 1$. Let $x_0 \in L_0 \cap S(1)$ and let $[b', b] = L \cap \bar{B}(1)$, where $b - b' = \lambda x_0$, $\lambda > 0$. Then $d(b, L_0) = d(b, [0, x_0\rangle)$.*

Proof. Let $y \in L$ be a point with $|y| = d(0, L)$. Then $y = (1-t)b + tb'$ for some $t \in [0, 1]$. Define an affine isometry $A\colon E \to E$ by $Ax = b - x$. Then $AL = L_0$, $A(0) = b$, $Ay = t\lambda x_0 \in [0, x_0\rangle$, whence $d(b, L_0) = d(0, L) = |y| = |Ay - b| \ge d(b, [0, x_0\rangle)$. □

4.4 Lemma. *Suppose that $L_0 \subset E$ is a line through the origin and that $x_0 \in L_0 \cap S(1)$. Let $0 < r < 1$ and let Z be the tube $\{x \in E\colon d(x, L_0) \le r\}$. Then $Z \setminus B(1)$ has two components V and V' where $x_0 \in V$, and for each $y \in V$ we have $d(y, L_0) = d(y, [0, x_0\rangle)$, and hence $\operatorname{dev}_1(y, x_0) = d(y, L_0)/|y|$.*

Proof. The tube Z is the union of all lines L parallel to L_0 with $d(L, L_0) \le r$. For such L, the set $L \setminus B(1)$ is the union of two rays R'_L, R_L with endpoints b'_L, b_L such that $b_L - b'_L$ has the same direction as x_0. Then V is the union of all rays R_L. By 4.3 we have $d(b_L, L_0) = d(b_L, [0, x_0\rangle)$, and the lemma follows. □

4.5. Notation. Suppose that E is ψ-uniformly convex. We set

$$\tau_\psi = \psi(1/12)/8, \quad f_\psi(t) = 12\psi^{-1}(8t) \text{ for } 0 \le t \le \tau_\psi.$$

Then $f_\psi(\tau_\psi) = 1$. Clearly $\tau_\psi \le 1/8$, but in fact, τ_ψ is much smaller. As $\psi(t) \le \psi_H(t) \le t^2/4$ by 2.12, we have $\tau_\psi \le 1/4608$.

The next result is a version of Theorem 3.3. Instead of the distance between a point and a line we now estimate the deviation between two vectors.

4.6. Theorem. *Suppose that E is ψ-uniformly convex and that $0 < t \le \tau_\psi$. Let $a \in G \subset E$, let $\gamma\colon a \curvearrowright x_0 \in S(a, t\delta(a))$ be a geodesic, and let x_1 be the last point of γ in $S(t\delta(a)/2)$. Then*

$$\operatorname{dev}(y - a, x_0 - a) \le f_\psi(t)$$

for all $y \in \gamma[x_1, x_0]$.

Proof. Setting $\lambda = k(a, x_0)$ we have $t/2 \le \lambda \le 2t$ by (2.10). Write $L_0 = \langle a, x_0\rangle$ and $Z = \{x \in E\colon d(x, L_0) \le t\delta(a)/4\}$. As $t \le \tau_\psi$, Theorem 3.3 gives

$$d(y, L_0) < 3t\delta(a)\psi^{-1}(8t) \le t\delta(a)/4.$$

From 4.4 it follows that $\gamma[x_1, x_0]$ lies in a component V of $Z \setminus B(a, t\delta(a)/2)$ with the property that $\operatorname{dev}_1(y - a, x_0 - a) = d(y, L_0)/|y - a|$ for all $y \in V$. For $y \in \gamma[x_1, x_0]$ we have $|y - a| \ge t\delta(a)/2$, whence $\operatorname{dev}_1(y - a, x_0 - a) \le 6\psi^{-1}(8t) = f_\psi(t)/2$. As $\operatorname{dev} \le 2\operatorname{dev}_1$ by 2.5, this yields the theorem. □

We next iterate Theorem 4.6 to obtain our main result:

4.7. Theorem. *Suppose that E is ψ-strongly convex and that $\gamma\colon a_0 \curvearrowright b_0$ is a quasihyperbolic geodesic in $G \subset E$. Then γ is smooth.*

Proof. We first show that γ has a tangent at every point (one-sided at the endpoints a_0, b_0). By Theorem 3.7 and by symmetry, it suffices to show that the right tangent vector $v(a)$ exists at an arbitrary point $a \in \gamma[a_0, b_0)$. Using the notation of 4.5 we fix a number $t \in (0, \tau_\psi]$ such that there is a point $x_0 \in \gamma(a, b_0] \cap S(a, t\delta(a))$.

For each positive integer j let x_j be the last point of γ in $S(a, 2^{-j}t\delta(a))$. Let $y \in \gamma[x_{j+1}, x_j]$. Applying Theorem 4.6 with the substitution $x_0, x_1, t \mapsto x_j, x_{j+1}, 2^{-j}t$ we obtain

$$\operatorname{dev}(y - a, x_j - a) \le f_\psi(2^{-j}t). \tag{4.8}$$

In particular, setting $w_j = x_j - a$ we have

$$|\hat{w}_{j+1} - \hat{w}_j| = \operatorname{dev}(w_{j+1}, w_j) \le f_\psi(2^{-j}t).$$

The function $f_\psi(t)/t$ is integrable on $(0, \tau_\psi]$ by strong convexity. As the series $\sum_j f_\psi(2^{-j}t)$ is convergent by 4.1, the sequence $(\hat{w}_j)$ is a Cauchy sequence in $S(1)$ and hence converges to a limit $v(a) \in S(1)$. Moreover, (4.8) implies that $\operatorname{dev}(y-a, v(a)) \to 0$ as $y \to a$ on $\gamma(a, b_0]$, whence $v(a)$ is the right tangent vector of γ at a.

The continuity of $v\colon \gamma \to S(1)$ follows from Theorem 4.12 below. □

By Example 2.13.2 we get:

Corollary 4.9. *If E is a uniformly convex space of power type (see 2.13.2), then every geodesic in a domain $G \subset E$ is smooth. In particular, this is true in the spaces $L_p(\mu)$ for some measure space (Ω, μ), $1 < p < \infty$.* □

4.10. Remark. Corollary 4.9 has been independently and by different methods obtained by [RT2].

4.11. Notation. Suppose that E is ψ-strongly convex. Let τ_ψ and f_ψ be as in 4.5. We define $h_\psi\colon [0, \tau_\psi] \to \mathbf{R}$ by

$$h_\psi(t) = f_\psi(t) + 2\int_0^t \frac{f_\psi(s)}{s}\, ds.$$

The function h_ψ is continuous, strictly increasing, and $h(0) = 0$.

4.12. Theorem. *Let E and γ be as in 4.7 and let $x, y \in \gamma$ be points with $k(x, y) \le \tau_\psi/2$. Then*

$$|v(x) - v(y)| \le 2h_\psi(2k(x, y)).$$

Proof. Assume, for example, that $x < y$ on γ. Set $\lambda = k(x, y)$ and $t = |x-y|/\delta(x)$. By (2.10) we have $t \le 2\lambda \le \tau_\psi$. In view of (4.2), the proof of Theorem 4.7 shows that

$$\operatorname{dev}(v(x), y - x) \le \sum_{j=0}^{\infty} f_\psi(2^{-j}t) \le h_\psi(t) \le h_\psi(2\lambda).$$

Changing the roles of x and y we obtain

$$\operatorname{dev}(v(y), y - x) = \operatorname{dev}(-v(y), x - y) \le h_\psi(2\lambda),$$

and the theorem follows. □

4.13. Uniform continuity. From Theorem 4.12 it easily follows that the tangent vector v is in fact uniformly continuous in the quasihyperbolic metric of γ. Moreover, the continuity modulus depends only on ψ. See e.g. [Vä1, 2.5].

4.14. Theorem. *Suppose that E is ψ-strongly convex. Then there is an increasing continuous function $\varphi_\psi\colon [0,\infty) \to \mathbf{R}$, depending only on ψ, with $\varphi_\psi(0) = 0$ and with the following property:*

Let γ be a quasihyperbolic geodesic in a domain $G \subset E$. Then the tangent vector v of γ satisfies the inequality

$$|v(x) - v(y)| \le \varphi_\psi(k(x,y))$$

for all $x, y \in \gamma$. □

In a Hilbert space the inequality of 4.14 holds as a Lipschitz condition:

4.15. Theorem. *Let E be a Hilbert space and let γ be a quasihyperbolic geodesic in a domain $G \subset E$. Then the tangent vector v of γ satisfies the condition*

$$|v(x) - v(y)| \le k(x,y)$$

for each pair $x, y \in \gamma$.

Proof. This was proved in [Vä4, 2.16] in the stronger form $\operatorname{ang}(v(x), v(y)) \le k(x,y)$ for $E = \mathbf{R}^n$, and the proof is valid in every Hilbert space. □

4.16. Correction. The proof of Theorem 2.16 of [Vä4] was based on Theorem 2.15, in which one should replace the condition $s < d/2$ by $s < d/3$. We give a corrected and more detailed proof for the first part of Theorem 2.15.

Set $r = d - s$, and let $z \in \mathbf{R}^n$ be a point with $|z - a| = |z - b| = r$. Let x_0 be the unique point in $S(z, r)$ for which $(a + b)/2 \in [z, x_0]$. Setting $d_0 = \delta(x_0)$ and $t = |x_0 - a| = |x_0 - b|$ we have $|d_0 - d| \le t < s < d/3$, whence $r < d_0$ and $d_0 \le d + s < d + d - 2s = 2r$. Hence we may apply the cap convexity theorem 2.13 of [Vä4] with the substitution $d \mapsto d_0$, which gives $\gamma \subset \bar{B}(z, r)$, and therefore $\gamma \subset \bar{Y}(a, b, r)$.

In the proof of [Vä4, 2.16] one should also replace the condition $s < d/2$ by $s < d/3$.

5. An example: The max norm in the plane

5.1. Introduction. In this section we consider in some detail the plane $\mathbf{R}^2$ with the max norm (or l_∞-norm)

$$\|x\| = |x_1| \vee |x_2|. \tag{5.2}$$

We consider the half plane

$$H = \{x \in \mathbf{R}^2 \colon x_2 > 0\}$$

and determine the quasihyperbolic geodesics in H. Some routine calculations are omitted.

This space has been recently considered by Rasila and Talponen [RT1, Sec. 5]. Moreover, the l_1-norm $|x_1| + |x_2|$ is in dimension 2 isometric to the max norm, and it has been considered in [TV] and in [Vä2, 5.1].

5.3. Notation. In this section, we use $\|x\|$ to denote the max norm (5.2). The plane $(\mathbf{R}^2, \|\cdot\|)$ will also be written as E or l^2_∞. For distinct points $a, b \in E$ we set

$$M(a,b) = \frac{|b_2 - a_2|}{|b_1 - a_1|}.$$

If $a_1 = b_1$, then $M(a,b) = \infty$. If $L \subset E$ is a line, we set $M_L = M(a,b)$ for arbitrary points $a, b \in L$.

The quasihyperbolic geodesics of $H \subset l^2_\infty$ are described in the following theorem and in Figure 1.

5.4. Theorem. *Let $a, b \in H$, $a \neq b$.*

(1) *If $M(a,b) \geq 1$, then*

$$k(a,b) = \left|\log \frac{b_2}{a_2}\right|. \tag{5.5}$$

An arc $\gamma\colon a \curvearrowright b$ is a geodesic iff

$$M(x,y) \geq 1 \quad \text{for all } x, y \in \gamma,\ x \neq y. \tag{5.6}$$

In particular, the segment $[a,b]$ is a geodesic.

(2) *If $M(a,b) = 1$, then (5.5) is true, and $[a,b]$ is the only geodesic $a \curvearrowright b$.*

(3) *If $M(a,b) < 1$, then the geodesic $\gamma\colon a \curvearrowright b$ is unique, and it is the broken line $[a,z] \cup [z,b]$ where z is the unique point for which $M(a,z) = M(b,z) = 1$ and $z_2 > a_2 \vee b_2$. Moreover,*

$$k(a,b) = \log \frac{z_2^2}{4a_2 b_2}.$$

Explicitly, if $a_1 < b_1$, then

$$2z_1 = a_1 + b_1 - a_2 + b_2, \quad 2z_2 = -a_1 + b_1 + a_2 + b_2,$$

$$k(a,b) = \log \frac{(-a_1 + b_1 + a_2 + b_2)^2}{4a_2 b_2}.$$

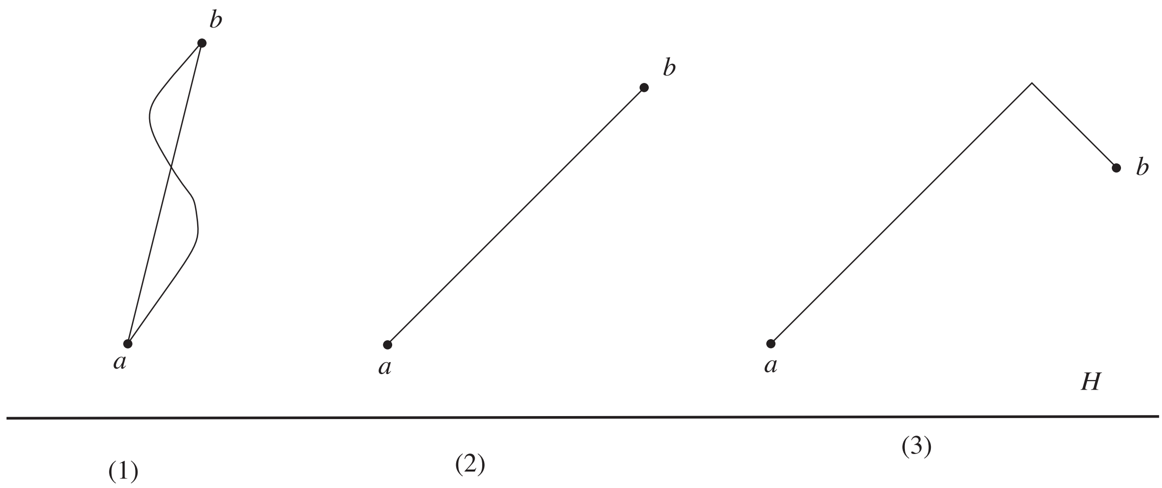

Figure 1. Geodesics in H.

Proof. We first prove some facts. From (2.8) we readily obtain:

Fact 1. Always $k(a,b) \geq |\log \frac{b_2}{a_2}|$.

Fact 2. $M(a,b) \geq 1$ iff $k(a,b) = |\log \frac{b_2}{a_2}|$.

Assume, for example, that $b_2 > a_2$ and that $k(a,b) = \log \frac{b_2}{a_2}$. Then (2.8) implies that $j(a,b) = \log \frac{b_2}{a_2}$, which is true iff $M(a,b) \geq 1$. Conversely, if $M(a,b) \geq 1$, then the line element ds on $[a,b]$ is dx_2, whence

$$k(a,b) \leq l_k[a,b] = \int_{a_2}^{b_2} \frac{dt}{t} = \log \frac{b_2}{a_2}.$$

Fact 3. If $M(a,b) \geq 1$, then an arc $\gamma\colon a \curvearrowright b$ is a geodesic iff γ satisfies the condition (5.6).

We may assume that $a_2 < b_2$. First, if (5.6) holds, then the second projection $p_2\colon \gamma \to \mathbf{R}$, $p_2 x = x_2$, is an isometry, Hence its inverse $g = (p_2|\gamma)^{-1}\colon [a_2, b_2] \to \gamma$ is a length parametrization of γ. By Fact 2 we get

$$l_k(\gamma) = \int_{a_2}^{b_2} \frac{dt}{t} = \log \frac{b_2}{a_2} = k(a,b).$$

Conversely, assume that $\gamma\colon a \curvearrowright b$ is a geodesic. If (5.6) is not true, there are $x, y \in \gamma$ such that $a \leq x < y \leq b$ and such that $M(x,y) < 1$. By Fact 2 this gives the contradiction

$$k(a,b) = k(a,x) + k(x,y) + k(y,b) > \log \frac{x_2}{a_2} + \log \frac{y_2}{x_2} + \log \frac{b_2}{y_2} = \log \frac{b_2}{a_2} = k(a,b),$$

and Fact 3 is proved.

Part (1) of the theorem follows now from Facts 2 and 3, and part (2) is a corollary of (1). To prove (3) we need the following

Fact 4. Let $L \subset E$ be a line with $M_L = 1$, and let $q\colon E \to L$ be the vertical projection defined by $(qx)_1 = x_1$. Let K be the half plane $\{x \in E\colon x_2 < (qx)_2\}$ with $\partial K = L$, and suppose that $g\colon [t_0, t_1] \to \bar{K} \cap H$ is a rectifiable path. Then $l_k(q \circ g) \leq l_k(g)$, with equality iff $\operatorname{im} g \subset L$.

The map q is 1-Lipschitz in the norm metric, and $\delta(qx) \geq \delta(x)$ for all $x \in \bar{K}$. Hence $l_k(q \circ g) \leq l_k(g)$. If $g(u) \notin L$ for some $u \in [t_0, t_1]$, then $\delta(qg(t)) > \delta(g(t))$ in a neighborhood of u, whence $l_k(q \circ g) < l_k(g)$, and Fact 4 follows.

To prove (3) set $\gamma_0 = [a,z] \cup [z,b]$ and suppose that $\gamma\colon a \curvearrowright b$ is a geodesic. We may assume that $a_1 < b_1$. Let y be the first point of γ with $y_1 = z_1$. If $y_2 = z_2$, then $\gamma = \gamma_0$ by (2). If $y_2 > z_2$, then $l_k(\gamma) > l_k(\gamma_0)$ by (1). If $y_2 < z_2$, then it follows from (2) that no subarc of $\gamma[a,y]$ lies above the line $\langle a, z\rangle$. By Fact 4 we have $l_k(\gamma[a,y]) > l_k[a,z]$. Similarly $l_k(\gamma[y',b]) > l_k[z,b]$ where y' is the last point of γ with $y'_1 = z_1$. Hence $l_k(\gamma) > l_k(\gamma_0)$, and (3) is proved. □

5.7. Remark. Case (1) of Theorem 5.4 shows that a quasihyperbolic geodesic need not have one-sided tangents at all points.

5.8. Example. Let $0 < r < 1/2$ and let $x = (0,2)$, $a = (-1+r, 3-r)$, $b = (1-r, 3-r)$. Then $a, b \in B(x,1) \subset H$ but a geodesic $\gamma\colon a \curvearrowright b$ contains the point $(0, 4-2r) \notin B(x,1)$. Hence the ball convexity theorem [Ma, 2.2] does not hold in l^2_∞.

5.9. Question. For $1 \leq p \leq \infty$ let l^2_p be the plane $\mathbf{R}^2$ with the l_p-norm. Let γ be a subarc of the semicircle $\{x \in l^2_p\colon |x| = 1,\ x_2 > 0\}$. Is γ a quasihyperbolic geodesic in the half plane $H \subset l^2_q$ where $1/p + 1/q = 1$?

The case $q = \infty$ follows from Theorem 5.4 and the case $q = 1$ from [Vä2, 5.1]. For $p = q = 2$ we have the classical Poincaré half plane.

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