QUASICONFORMAL EXTENSION OF
HARMONIC MAPPINGS IN THE PLANE

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Abstract. Let f be a sense-preserving harmonic mapping in the unit disk. We give a sufficient condition in terms of the pre-Schwarzian derivative of f to ensure that it can be extended to a quasiconformal map in the complex plane.

Introduction

A well-known criterion due to Becker [5] states that if a locally univalent analytic function ϕ in the unit disk $\mathbf{D}$ satisfies

$$(1) \qquad \sup_{z \in \mathbf{D}} \left| \frac{\phi''(z)}{\phi'(z)} \right| (1 - |z|^2) \le 1,$$

then ϕ is, indeed, univalent in $\mathbf{D}$. Becker and Pommerenke [6] proved later that the constant 1 is sharp.

The quotient $P\phi = \phi''/\phi'$ is called *the pre-Schwarzian derivative* of ϕ, a function that is well-defined in the unit disk for every locally univalent function ϕ in $\mathbf{D}$.

In that paper [5], the author also proves that if

$$(2) \qquad \sup_{z \in \mathbf{D}} |P\phi(z)| (1 - |z|^2) \le k < 1,$$

then not only ϕ is univalent but it has a continuous extension $\widetilde{\phi}$ to $\overline{\mathbf{D}}$ and $\widetilde{\phi}(\partial \mathbf{D})$ is a quasicircle. Indeed, using Löwner's chains, Becker shows that ϕ has a K-quasiconformal extension to the whole complex plane $\mathbf{C}$, where $K = (1+k)/(1-k)$. Moreover, Ahlfors [3] gives an explicit quasiconformal extension. Namely, the function

$$\Phi(z) = \begin{cases} \widetilde{\phi}(z), & |z| \le 1, \\ \phi\left(\frac{1}{\overline{z}}\right) + u\left(\frac{1}{\overline{z}}\right), & |z| > 1, \end{cases}$$

where, for $z \in \mathbf{D} \setminus \{0\}$, $u(z) = \phi'(z)(1-|z|^2)/\overline{z}$. This mapping Φ is K-quasiconformal (with K as above) in the complex plane whenever (2) holds and coincides with ϕ in $\mathbf{D}$.

Let now f be a locally univalent harmonic mapping defined in the unit disk and $f = h + \overline{g}$ its canonical representation, where h and g are analytic in $\mathbf{D}$ and $g(0) = 0$. The (second) complex dilatation of f is $\omega = g'/h'$. Since f is locally univalent, by

Lewy's Theorem (see [12]), the Jacobian $J_f = |h'|^2 - |g'|^2$ of f does not vanish in the unit disk. Hence, either $J_f > 0$ or $J_f < 0$ in $\mathbf{D}$. Any harmonic function f with $J_f > 0$ is called *sense-preserving*. If $J_f < 0$, f is said to be *sense-reversing*. Note that f is sense-preserving if and only if $\overline{f} = g + \overline{h}$ is sense-reversing.

Throughout this paper we will consider sense-preserving harmonic mappings. It is easy to check that, in this case, $f = h + \overline{g}$ with $J_f = |h'|^2(1 - |\omega|^2) > 0$. Equivalently, the analytic part h of f is locally univalent and the second complex dilatation ω is an analytic function in $\mathbf{D}$ with $\|\omega\|_\infty = \sup_{z \in \mathbf{D}} |\omega(z)| \leq 1$. Note that $\omega \equiv 0$ if and only if f is analytic.

In a recent preprint [9], we have introduced a definition of the *pre-Schwarzian derivative* P_f of locally univalent harmonic mappings $f = h + \overline{g}$ in the unit disk. If f is sense-preserving, then

$$P_f = \frac{h''}{h'} - \frac{\overline{\omega}\omega'}{(1 - |\omega|^2)}$$

in $\mathbf{D}$. Using this definition for the pre-Schwarzian derivative of f, we proved (see [9, Theorem 9]) that if $f = h + \overline{g}$ is a sense-preserving harmonic mapping in the unit disk with second complex dilatation ω and

$$(3) \qquad |P_f(z)|(1 - |z|^2) + \frac{|\omega'(z)|(1 - |z|^2)}{1 - |\omega(z)|^2} \leq 1$$

for all $|z| < 1$, then f is univalent in $\mathbf{D}$. The constant 1 is sharp.

The second term in (3) is the modulus of the *hyperbolic derivative* ω^* of ω defined by

$$\omega^*(z) = \frac{\omega'(z)(1 - |z|^2)}{1 - |\omega(z)|^2}, \quad z \in \mathbf{D}.$$

As we mentioned before, if ϕ is analytic, then it is a sense-preserving harmonic mapping with dilatation $\omega \equiv 0$. Hence, (3) generalizes Becker's criterion of univalence (1) to sense-preserving harmonic mappings.

The purpose of this paper is to show that analogous results to those due to Becker and Ahlfors that we have mentioned before also hold for the family of sense-preserving harmonic mappings in the unit disk. Concretely, we prove the following theorems.

Theorem 1. *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in the unit disk with second complex dilatation ω. Assume that*

$$(4) \qquad \left| P_f(z)(1 - |z|^2) \right| + |\omega^*(z)| \leq k < 1, \quad z \in \mathbf{D}.$$

Then, the harmonic mapping f has a continuous and injective extension $\widetilde{f}$ to $\overline{\mathbf{D}}$. Moreover, the function

$$(5) \qquad F(z) = \begin{cases} \widetilde{f}(z), & |z| \leq 1, \\ f\left(\frac{1}{\overline{z}}\right) + U\left(\frac{1}{\overline{z}}\right), & |z| > 1, \end{cases}$$

is a homeomorphic extension of f to the whole complex plane onto itself. The function U that appears in (5) is defined by

$$U(z) = \frac{h'(z)(1 - |z|^2)}{\overline{z}} + \frac{\overline{g'(z)}(1 - |z|^2)}{z}, \quad z \in \mathbf{D} \setminus \{0\}.$$

An extra condition on the second complex dilatation ω of the mapping f is needed in the following theorem. We will justify in Section 3 below why this condition cannot be omitted.

Theorem 2. *Let the sense-preserving harmonic mapping f satisfy* (4). *If, in addition, $\|\omega\|_\infty < 1$, then $\widetilde{f}(\partial \mathbf{D})$ is a quasicircle and f can be extended to a quasiconformal map in $\mathbf{C}$. Indeed, the function defined by* (5) *is an explicit K-quasiconformal extension of f whenever*

$$k < \frac{1 - \|\omega\|_\infty}{1 + \|\omega\|_\infty}.$$

The constant K equals

$$(6) \qquad K = \frac{(1 + k) + (1 - k)\|\omega\|_\infty}{(1 - k) - (1 + k)\|\omega\|_\infty}.$$

1. Background

1.1. Quasiconformal maps. A complex valued function f on a domain $\Omega \subset \overline{\mathbf{C}}$ is *absolutely continuous on lines* (ACL) *in* Ω if for every rectangle of the form $R = \{x + iy \colon a < x < b, \ c < y < d\}$ with $\overline{R} \subset \Omega \setminus \{\infty, f^{-1}(\infty)\}$, the map f is absolutely continuous on almost every horizontal and vertical line in R.

Let K be a real positive number with $K \geq 1$. A sense-preserving homeomorphism $f \colon \Omega \to \mathbf{C}$ is called *quasiconformal* (or *K-quasiconformal*) if f is ACL in Ω and $|f_{\overline{z}}| \leq k|f_z|$ almost everywhere in Ω, where $k = (K - 1)/(K + 1)$. The 1-quasiconformal mappings are the conformal mappings.

The *complex dilatation* of f is the ratio $\mu_f = f_{\overline{z}}/f_z$. If f is sense-preserving, then $0 \leq |\mu_f| < 1$. Hence, a sense-preserving ACL homeomorphism f is quasiconformal if and only if μ_f is bounded away from 1 in the given region. The *second complex dilatation* ω_f of f is defined by $\omega_f = \overline{f_{\overline{z}}}/f_z$. Since $|\omega_f| = |\mu_f|$, we conclude that f is quasiconformal if and only if $|\omega_f| \leq k < 1$.

Let now f be a harmonic mapping defined in a simply connected domain $\Omega \subset \mathbf{C}$. It is not difficult to check that f has the representation (unique up to an additive constant) $f = h + \overline{g}$ for two analytic functions h and g in Ω.

For harmonic mappings f in the unit disk, it is convenient to choose the additive constant in such a way that $g(0) = 0$. In this case, the representation $f = h + \overline{g}$ is unique and it is called the canonical representation of f. Such function f is quasiconformal if it is absolutely continuous on lines in $\mathbf{D}$ and $|\omega_f| = |g'/h'| \leq k < 1$.

We refer the reader to the books by Lehto and Virtanen [11], Ahlfors [1], and Astala, Iwaniec, and Martin [4], and to the book by Duren [7] for the general theory of quasiconformal mappings and harmonic mappings in the plane, respectively.

1.2. Quasicircles. A Jordan curve Γ is a *quasicircle* if there exists a constant M such that

$$\min\{\operatorname{diam} \Gamma_1, \operatorname{diam} \Gamma_2\} \leq M \, |w_1 - w_2|, \quad \text{for } w_1, w_2 \in \Gamma,$$

where Γ_1 and Γ_2 are the components of $\Gamma \setminus \{w_1, w_2\}$.

Ahlfors [2] proved that Γ is a quasicircle if and only if there exists a real positive constant C such that for all $w_1, w_2, w_3, w_4 \in \Gamma$,

$$|(w_1, w_2, w_3, w_4)| \leq C,$$

where (w_1, w_2, w_3, w_4) is the *cross ratio* defined by

$$(7) \qquad (w_1, w_2, w_3, w_4) = \frac{w_1 - w_3}{w_1 - w_4} \cdot \frac{w_2 - w_4}{w_2 - w_3}.$$

A *quasidisk* is a domain bounded by a quasicircle. In the case when f is a quasiconformal function in the unit disk, $f(\mathbf{D})$ is a quasidisk if and only if f has a quasiconformal extension to $\mathbf{C}$ (see [2] or [11, Theorem 8.3 on p. 98]).

1.3. Pre-Schwarzian derivatives of harmonic mappings. Let $f = h + \overline{g}$ be a locally univalent harmonic mapping in the unit disk with second complex dilatation $\omega = g'/h'$. The *pre-Schwarzian derivative* P_f of f is defined in $\mathbf{D}$ by

$$P_f = \frac{\partial}{\partial z} \log |J_f|,$$

where $J_f = |h'|^2 - |g'|^2$ is the Jacobian of the function f.

Notice that if f is analytic (this is, if $\omega \equiv 0$), then P_f coincides with the classical definition of pre-Schwarzian derivative of analytic mappings. It is also easy to check that $P_f = P_{\overline{f}}$. Hence, there is no loss of generality if we assume that $f = h + \overline{g}$ is sense-preserving. In this case, the pre-Schwarzian derivative of the function $f = h + \overline{g}$ equals

$$(8) \qquad P_f = \frac{\partial}{\partial z} \log \left(|h'|^2 \cdot (1 - |\omega|^2) \right) = \frac{h''}{h'} - \frac{\overline{\omega} \omega'}{(1 - |\omega|^2)}.$$

A straightforward computation shows that given any $a \in \mathbf{D}$ and any sense-preserving harmonic mapping f in the unit disk with canonical decomposition $f = h + \overline{g}$, the function $f_a = f + a\overline{f}$ is also a sense-preserving harmonic mapping. Let $f_a = h_a + \overline{g_a}$ be the canonical decomposition of f_a and ω_a its second complex dilatation. Then,

$$h_a = h + ag, \quad g_a = g + \overline{a}h, \quad \text{and} \quad \omega_a = \varphi_a \circ \omega,$$

where φ_a is the automorphism of the unit disk defined by

$$(9) \qquad \varphi_a(z) = \frac{\overline{a} + z}{1 + az}, \quad z \in \mathbf{D}.$$

The Jacobian J_{f_a} of f_a equals $(1 - |a|^2) J_f$. Hence, $P_f = P_{f_a}$.

As in the classical (analytic) case, the Schwarzian derivative S_f of f is

$$S_f = \frac{\partial}{\partial z} P_f - \frac{1}{2} (P_f)^2.$$

The reader can find the main properties and several results related to the pre-Schwarzian and Schwarzian derivatives of locally univalent harmonic mappings defined in this section in [9].

1.4. Hyperbolic derivative of self-maps of the unit disk. Let ω be a self-map of the unit disk, this is, an analytic function in $\mathbf{D}$ with $\omega(\mathbf{D}) \subset \mathbf{D}$. As we mentioned before, the *hyperbolic derivative* of such a function ω is

$$(10) \qquad \omega^*(z) = \frac{\omega'(z)(1 - |z|^2)}{1 - |\omega(z)|^2}, \quad z \in \mathbf{D}.$$

Using the Schwarz–Pick lemma, we see that $|\omega^*| \leq 1$ in $\mathbf{D}$ and that if there exists $z_0 \in \mathbf{D}$ with $|\omega^*(z_0)| = 1$, then ω maps the unit disk conformally onto itself and $|\omega^*| \equiv 1$ in $\mathbf{D}$.

Given two self-maps ω and φ of the unit disk, the *chain rule for the hyperbolic derivative* holds:

$$(\varphi \circ \omega)^* (z) = \varphi^*(\omega(z)) \cdot \omega^*(z).$$

In particular, if φ is a conformal self-map of $\mathbf{D}$ onto $\mathbf{D}$, then $|(\varphi \circ \omega)^*| \equiv |\omega^*|$ in the unit disk.

The exact value of the upper bound of the modulus of hyperbolic derivative of the so-called *lens-maps* ℓ_α will be of particular importance for our purposes. For $0 < \alpha < 1$, the mapping ℓ_α is the self-map of the unit disk defined by

$$\ell_\alpha(z) = \frac{\ell(z)^\alpha - 1}{\ell(z)^\alpha + 1}, \tag{11}$$

where $\ell(z) = (1+z)/(1-z)$. Note that $\|\ell_\alpha\|_\infty = 1$ for all $0 < \alpha < 1$. In [10], it is explicitly checked that

$$\sup_{z \in \mathbf{D}} |\ell_\alpha^*(z)| = \alpha. \tag{12}$$

2. Proof of Theorem 1

As we mentioned in Section 1.3, we have that for all $a \in \mathbf{D}$, $P_{f_a} = P_f$, where $f_a = f + a\overline{f}$. It is easy to check that $f_a = h_a + \overline{g_a}$ with $h_a = h + ag$ and $g_a = g + \overline{a}h$. The second complex dilatation ω_a of f_a equals $\omega_a = \varphi_a \circ \omega$ (φ_a being the automorphism of the unit disk defined by (9)). Hence, by the chain rule for the hyperbolic derivative, we get $|\omega_a^*| = |\omega^*|$ in the unit disk, where ω_a^* and ω^* are the hyperbolic derivatives of ω_a and ω, respectively, defined by (10).

2.1. Step 1. We first prove that for all $a \in \overline{\mathbf{D}}$, the functions $h_a = h + ag$ have a continuous and injective extension to the whole complex plane. To do so, we bear in mind the last paragraph and use (4) to obtain that for all $z \in \mathbf{D}$,

$$k \geq \left| P_{f_a}(z)(1 - |z|^2) \right| + |\omega_a^*(z)|$$
$$= \left| \frac{h''(z) + ag''(z)}{h'(z) + ag'(z)} - \frac{\overline{\omega_a(z)}\omega_a'(z)}{1 - |\omega_a(z)|^2} \right| (1 - |z|^2) + |\omega_a^*(z)|. \tag{13}$$

Making use of the triangle inequality, we get from (13):

$$\left| \frac{h_a''(z)}{h_a'(z)}(1 - |z|^2) \right| \leq \left| \frac{h''(z) + ag''(z)}{h'(z) + ag'(z)} - \frac{\overline{\omega_a(z)}\omega_a'(z)}{1 - |\omega_a(z)|^2} \right| (1 - |z|^2)$$
$$+ \left| \frac{\overline{\omega_a(z)}\omega_a'(z)}{1 - |\omega_a(z)|^2} \right| (1 - |z|^2)$$
$$\leq \left| P_{f_a}(z)(1 - |z|^2) \right| + |\omega_a^*(z)| \leq k,$$

which implies, by the classical result due to Becker for analytic functions, that for each $a \in \mathbf{D}$, the function h_a is univalent and can be extended to a continuous and injective mapping $\widetilde{h}_a$ in the closed unit disk. Moreover (see [3]), the function

$$H_a(z) = \begin{cases} \widetilde{h}_a(z), & |z| \leq 1, \\ h_a\left(\frac{1}{\overline{z}}\right) + u_a\left(\frac{1}{\overline{z}}\right), & |z| > 1, \end{cases} \tag{14}$$

(where, for $z \in \mathbf{D} \backslash \{0\}$, $u_a(z) = h'_a(z)(1-|z|^2)/\overline{z}$) is K-quasiconformal in the complex plane with $K = (1+k)/(1-k)$. Hence, in particular, H_a is continuous and injective in $\mathbf{C}$.

Take a complex number $\lambda = e^{i\theta} \in \partial\mathbf{D}$ and define

$$H_\lambda(z) = \lim_{r \to 1^-} H_{re^{i\theta}}(z), \quad z \in \mathbf{C}.$$

According to [11, Theorem 5.3], the functions H_λ are either constant functions, functions from $\mathbf{C}$ onto two points, or K-quasiconformal functions in the complex plane. But using Hurwitz's theorem, we see that for each $\lambda \in \partial\mathbf{D}$, the function $h + \lambda g$ is either a constant or univalent in $\mathbf{D}$. Such a function $h + \lambda g$ cannot be constant for any $|\lambda| = 1$ because it would imply that $|h'| \equiv |g'|$ in the unit disk and this is a contradiction with the condition $|g'(0)| < |h'(0)|$ that holds for any sense-preserving harmonic mapping. Therefore, we conclude that $h + \lambda g$ are univalent for all $\lambda \in \partial\mathbf{D}$. Since H_λ coincides with $h + \lambda g$ in $\mathbf{D}$, we obtain that H_λ is a K-quasiconformal function in the complex plane for each $|\lambda| = 1$. Thus H_λ is continuous and one-to-one in $\mathbf{C}$ for all $|\lambda| = 1$ (hence for all $|\lambda| \leq 1$).

2.2. Step 2. The next step is to construct an explicit candidate for a continuous and injective extension of f to the whole complex plane.

We define, for each $z \in \mathbf{C}$, the function

$$G(z) = H_1(z) - H(z)$$

(where $H = H_0$ and H_1 denote the corresponding extension of $h_0 = h$ and $h_1 = h+g$, respectively, as in (14)), to obtain a continuous extension G of g to $\mathbf{C}$ given by

$$G(z) = \begin{cases} g(z), & |z| < 1, \\ \widetilde{g}(z) = \widetilde{h}_1(z) - \widetilde{h}(z), & |z| = 1, \\ g\left(\frac{1}{\overline{z}}\right) + v\left(\frac{1}{\overline{z}}\right), & |z| > 1, \end{cases}$$

where

$$v(z) = \frac{g'(z)(1 - |z|^2)}{\overline{z}}, \quad z \in \mathbf{D} \backslash \{0\}.$$

Note that for each $z \in \mathbf{C}$ and all $a \in \overline{\mathbf{D}}$,

$$H_a(z) = H(z) + aG(z).$$

Hence, since H_a are univalent in the complex plane for all $a \in \overline{\mathbf{D}}$, we see that $H + aG$ is injective in $\mathbf{C}$ for all such a.

We now construct the function defined by (5):

$$F(z) = H(z) + \overline{G(z)} = \begin{cases} f(z), & |z| < 1, \\ \widetilde{h}(z) + \overline{\widetilde{g}(z)}, & |z| = 1, \\ f\left(\frac{1}{\overline{z}}\right) + U\left(\frac{1}{\overline{z}}\right), & |z| > 1, \end{cases}$$

where U is defined in $\mathbf{D} \backslash \{0\}$ by

$$U(z) = \frac{h'(z)(1 - |z|^2)}{\overline{z}} + \frac{\overline{g'(z)}(1 - |z|^2)}{z}.$$

2.3. Step 3. We are going to show that the function F as in (5) is injective in the complex plane. We use the arguments from [8].

Assume, in order to get a contradiction, that $F(z_1) = F(z_2)$ for two different points z_1 and z_2 in $\mathbf{C}$. Then, we have

(15) $$H(z_1) - H(z_2) = \overline{G(z_2) - G(z_1)}.$$

By hypothesis, H is univalent in the complex plane. Hence, $H(z_1) \neq H(z_2)$. Denote by $\theta = \arg\{H(z_1) - H(z_2)\} \in [0, 2\pi)$. Thus, we obtain from (15) that

$$e^{-i\theta}(H(z_1) - H(z_2)) = e^{-i\theta}\left(\overline{G(z_1) - G(z_2)}\right)$$

is a positive real number. Therefore, after taking conjugates on the right hand side of the last equality, we get $H(z_1) - H(z_2) = e^{2i\theta}(G(z_1) - G(z_2))$, which contradicts the fact that the function $H - e^{2i\theta}G$ is univalent in $\mathbf{C}$.

2.4. Step 4. Finally, we will check that the function F defined by (5) is a homeomorphism of $\mathbf{C}$ onto itself.

That the function F is continuous in the complex plane follows from the fact that both H and G are continuous in $\mathbf{C}$. Moreover, a straightforward computation (that mainly uses the triangle inequality and that f is sense-preserving) gives $\lim_{|z|\to\infty} |F(z)| = \infty$. Hence, we can extend continuously (and in a one-to-one manner) the function F to the Riemann sphere $\overline{\mathbf{C}}$ in such a way that $F(\infty) = \infty$. Therefore (see for instance [13, Theorem 5.6 on p. 167]), F is a homeomorphism of the Riemann sphere onto $F(\overline{\mathbf{C}})$, so that $F(\overline{\mathbf{C}}) = \overline{\mathbf{C}}$. Since $F(\infty) = \infty$, we conclude that F is a homeomorphism of the complex plane onto itself.

3. Proof of Theorem 2

We would first like to remark that the hypotheses $\|\omega\|_\infty < 1$ in this theorem cannot be removed. To do so, consider the sense-preserving harmonic mapping $f = z + \overline{g}$, where g' equals the lens-map ℓ_α defined by (11). Note that the second complex dilatation ω of f equals ℓ_α. Then, on the one side, using (12) we obtain

$$\sup_{z \in \mathbf{D}}\left(|P_f(z)|(1 - |z|^2) + |\omega^*(z)|\right) = \sup_{z \in \mathbf{D}}\left(|\overline{\ell_\alpha}\ell_\alpha^*(z)| + |\ell_\alpha^*(z)|\right) \leq 2\alpha.$$

Therefore, by choosing any $\alpha < 1/2$, we have that (4) holds (with $k = 2\alpha < 1$). On the other side, the function f is not quasiconformal since $\|\ell_\alpha\|_\infty = 1$.

Going back to the proof, we first notice that it is enough to prove Theorem 2 for sense-preserving harmonic mappings f that have a canonical decomposition of the form $f = h + \overline{g}$, where both f and g are analytic in $\overline{\mathbf{D}}$. This is because given any positive real number $0 < r < 1$, we can define the sense-preserving harmonic function $f_r(z) = f(rz) = h(rz) + \overline{g(rz)}$. Each of these functions f_r has second complex dilatation $\omega_r = \omega(rz)$, so that $\|\omega_r\|_\infty \leq \|\omega\|_\infty < 1$.

Also, it is easy to check that if f satisfies (4), then

$$\sup_{z \in \mathbf{D}}\left\{|P_{f_r}(z)|\left(1 - |z|^2\right) + |\omega_r^*(z)|\right\} \leq r \sup_{z \in \mathbf{D}}|P_f(z)|\left(1 - |z|^2\right) + r \sup_{z \in \mathbf{D}}|\omega^*(z)| \leq rk < k,$$

so that f_r satisfies (4) as well.

Now, if we check that f_r can be extended to a K-quasiconformal function in the complex plane for some constant $K \geq 1$ that does not depend on r, then we can use [11, Theorem 5.3] and argue as in Subsection 2.1 to conclude that f can be also extended to a K-quasiconformal function in $\mathbf{C}$. Hence, from now on, we can assume

that the functions h and g in the canonical decomposition of f are both analytic functions in $\overline{\mathbf{D}}$.

3.1. Some useful lemmas. We will use the following three lemmas.

Lemma 1. *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in $\mathbf{D}$ with complex dilatation $\omega \not\equiv 0$. Assume that $\|\omega\|_\infty < 1$ and that f satisfies (4). Then, the analytic functions $h_a = h + ag$ are univalent in $\mathbf{D}$ for all $0 \le |a| < \delta$, where*

$$1 < \delta = \frac{1 + k\|\omega\|_\infty}{k + \|\omega\|_\infty} \le \frac{1}{\|\omega\|_\infty}.$$

Moreover, h_a has a continuous and injective extension $\widetilde{h}_a$ to $\overline{\mathbf{D}}$.

Proof. Notice that the fact that h_a is univalent in the unit disk for all $|a| \le 1$ has been already proved in Subsection 2.1. Indeed, we showed in that subsection that h_a has a continuous and injective extension to $\overline{\mathbf{D}}$ of the form $\widetilde{h}_a = \widetilde{h} + a\widetilde{g}$ for suitable functions $\widetilde{h}$ and $\widetilde{g}$ that coincides with h and g in the unit disk, respectively. Hence, we can extend these functions h_a for any value of a in the complex plane via the formula $\widetilde{h}_a = \widetilde{h} + a\widetilde{g}$. Note that if we can prove that the function h_a (for any $a \in \mathbf{C}$) can be extended continuously to the closed unit disk, then the extension will coincide with $\widetilde{h}_a$.

We are going to show that these functions h_a keep the univalence property in $\mathbf{D}$ for $1 < |a| < \delta$ and that the corresponding extensions $\widetilde{h}_a$ are injective in $\overline{\mathbf{D}}$.

Since $|a| < \delta \le 1/\|\omega\|_\infty$ and $h_a' = h'(1 + a w)$, we see that h_a is locally univalent in the unit disk. Therefore, we can compute the pre-Schwarzian derivative of h_a to obtain

$$(16) \qquad \frac{h_a''}{h_a'} = \frac{h''}{h'} + \frac{a\omega'}{1 + a\omega}.$$

Keeping in mind Formula (8) for the pre-Schwarzian derivative of f and (16), we get

$$\frac{h_a''}{h_a'} = P_f + \omega'\left(\frac{\overline{\omega}}{1 - |\omega|^2} + \frac{a}{1 + a\omega}\right)$$

$$= P_f + \frac{\omega'}{1 - |\omega|^2}\left(\overline{\omega} + a\frac{1 - |\omega|^2}{1 + a\omega}\right) = P_f + \frac{\omega'}{1 - |\omega|^2} \cdot \frac{\overline{\omega} + a}{1 + a\omega},$$

which implies that for all $z \in \mathbf{D}$,

$$(17) \qquad \left|\frac{h_a''(z)}{h_a'(z)}\right| (1 - |z|^2) \le |P_f(z)| (1 - |z|^2) + |\omega^*(z)| \cdot \left|\frac{\omega(z) + \overline{a}}{1 + a\omega(z)}\right|.$$

Now, note that since f is supposed to satisfy (4), we get that its second complex dilatation ω satisfies $|\omega^*| \le k$ in the unit disk. Moreover, we also have for all $|z| < 1$,

$$|P_f(z)| (1 - |z|^2) \le k - |\omega^*(z)|.$$

Then, we obtain from (17):

$$(18) \qquad \left|\frac{h_a''(z)}{h_a'(z)}\right| (1 - |z|^2) \le k + |\omega^*(z)|\left(\left|\frac{\omega(z) + \overline{a}}{1 + a\omega(z)}\right| - 1\right) \le k\left|\frac{\omega(z) + \overline{a}}{1 + a\omega(z)}\right|.$$

In view of the inequality (18), the proof of the lemma will follow from the classical Becker's criterion for analytic functions once we show that

$$(19) \qquad \sup_{z \in \mathbf{D}} \left| \frac{\omega(z) + \overline{a}}{1 + a\omega(z)} \right| < \frac{1}{k}$$

for all $1 < |a| < \delta$. To check that (19) holds, we first note that for $\lambda = a/|a|$,

$$\left| \frac{\omega(z) + \overline{a}}{1 + a\omega(z)} \right| = \left| \frac{\lambda\omega(z) + \lambda\overline{a}}{1 + \overline{\lambda}a\lambda\omega(z)} \right| = \left| \frac{\lambda\omega(z) + |a|}{1 + |a|\lambda\omega(z)} \right|.$$

Therefore,

$$\sup_{z \in \mathbf{D}} \left| \frac{\lambda\omega(z) + \lambda\overline{a}}{1 + \overline{\lambda}a\lambda\omega(z)} \right| = \sup_{z \in \mathbf{D}} \left| \frac{\lambda\omega(z) + |a|}{1 + |a|\lambda\omega(z)} \right| \leq \sup_{|z| \leq \|\omega\|_\infty} \left| \frac{z + |a|}{1 + |a|z} \right|.$$

Now, using again that $|a| < \delta \leq 1/\|\omega\|_\infty$, we have that the linear fractional transformation

$$T(z) = \frac{z + |a|}{1 + |a|z}$$

maps the disk $D(0, \|\omega\|_\infty)$ of center at $z = 0$ and radius $\|\omega\|_\infty$ onto a disk D, say. Moreover, the Taylor coefficients of T are real numbers, hence D is symmetric with respect to the real axes. A straightforward calculation shows that the Möbius transformation T maps the interval $(-\|\omega\|_\infty, \|\omega\|_\infty)$ onto the interval

$$\left(\frac{|a| + \|\omega\|_\infty}{1 + |a|\|\omega\|_\infty} , \frac{|a| - \|\omega\|_\infty}{1 - |a|\|\omega\|_\infty} \right),$$

so that

$$\sup_{|z| \leq \|\omega\|_\infty} \left| \frac{z + |a|}{1 + |a|z} \right| = \frac{|a| - \|\omega\|_\infty}{1 - |a|\|\omega\|_\infty} < \frac{1}{k}$$

since

$$|a| < \delta = \frac{1 + k\|\omega\|_\infty}{k + \|\omega\|_\infty}.$$

Thus (19) holds, which completes the proof. $\qquad\qquad \square$

Lemma 2. *Let $f = h + \overline{g}$ satisfy the hypothesis of Lemma 1. Assume in addition that both h and g are analytic functions in $\overline{\mathbf{D}}$. Then,*

$$\sup_{\alpha, \beta \in \overline{\mathbf{D}}, \, \alpha \neq \beta} \left| \frac{g(\alpha) - g(\beta)}{h(\alpha) - h(\beta)} \right| \leq \frac{1}{\delta} = \frac{k + \|\omega\|_\infty}{1 + k\|\omega\|_\infty} < 1.$$

Proof. Fix an arbitrary point $\beta \in \overline{\mathbf{D}}$ and define the function

$$\psi_\beta(\alpha) = \begin{cases} \frac{g(\alpha) - g(\beta)}{h(\alpha) - h(\beta)}, & \alpha \neq \beta, \\ \omega(\alpha), & \alpha = \beta, \end{cases} \qquad \alpha \in \overline{\mathbf{D}},$$

where ω is the second complex dilatation of f.

The function ψ_β is continuous in $|\alpha| \leq 1$, so that there exists $\alpha_0 \in \overline{\mathbf{D}}$ such that $S = \sup_{\alpha \in \overline{\mathbf{D}}} |\psi_\beta(\alpha)| = |\psi_\beta(\alpha_0)|$.

If $\alpha_0 = \beta$, then $S \leq \|\omega\|_\infty$ and the result follows because $\delta \leq 1/\|\omega\|_\infty$.

Assume that $\alpha_0 \neq \beta$. In order to get a contradiction, suppose that $S > 1/\delta$. Then, there exist a positive real number $\varepsilon < \delta$, $|\lambda| = 1$, and $\alpha_1 \in \overline{\mathbf{D}}$ with

$$\frac{g(\alpha_1) - g(\beta)}{h(\alpha_1) - h(\beta)} = \frac{\lambda}{\delta - \varepsilon},$$

which implies that the function $h - \overline{\lambda}(\delta - \varepsilon)\, g$ is not univalent in $\overline{\mathbf{D}}$. This is in contradiction with the result obtained in Lemma 1. $\qquad\square$

The third lemma is related to the analytic part h of f.

Lemma 3. *Let $f = h + \overline{g}$ ba a sense-preserving harmonic mapping in the unit disk for which (4) holds. For any positive real number $r < 1$, define the function h_r by $h_r(z) = h(rz)$, $z \in \mathbf{D}$. Then, there exists a constant $M > 0$ such that for all $r \in (0, 1]$ and all four different points w_i, $i = 1, 2, 3, 4$, in $\Gamma_r = h_r(\partial\mathbf{D})$, the modulus of the cross ratio (w_1, w_2, w_3, w_4) defined by (7) is bounded by M.*

Proof. Using the arguments in the proof of Theorem 1, since we are assuming that f satisfies (4), we have that the analytic part h of f can be extended continuously to an injective function $\widetilde{h}$ in $\overline{\mathbf{D}}$ with the property that $\widetilde{h}(\partial\mathbf{D})$ is a quasicircle.

Hence, given any four different points w_i, $i = 1, 2, 3, 4$, in $\Gamma = \widetilde{h}(\partial\mathbf{D})$, the modulus of the cross ratio $|(w_1, w_2, w_3, w_4)|$ is bounded by a constant M. This implies that for any four different points $|\zeta_i| = 1$, $i = 1, 2, 3, 4$, we have

$$(20) \qquad \left| \frac{\widetilde{h}(\zeta_1) - \widetilde{h}(\zeta_3)}{\widetilde{h}(\zeta_1) - \widetilde{h}(\zeta_4)} \cdot \frac{\widetilde{h}(\zeta_2) - \widetilde{h}(\zeta_4)}{\widetilde{h}(\zeta_2) - \widetilde{h}(\zeta_3)} \right| \leq M.$$

Now, if we fix four different points α_i, $i = 1, 2, 3, 4$, on the boundary of the unit disk and define the function

$$\hbar(\lambda) = \frac{\widetilde{h}(\lambda\alpha_1) - \widetilde{h}(\lambda\alpha_3)}{\widetilde{h}(\lambda\alpha_1) - \widetilde{h}(\lambda\alpha_4)} \cdot \frac{\widetilde{h}(\lambda\alpha_2) - \widetilde{h}(\lambda\alpha_4)}{\widetilde{h}(\lambda\alpha_2) - \widetilde{h}(\lambda\alpha_3)}, \quad \lambda \in \overline{\mathbf{D}},$$

we conclude (using (20) and the Maximum Modulus Principle) that for all $0 < r < 1$, the inequality

$$(21) \qquad \left| \frac{h(r\alpha_1) - h(r\alpha_3)}{h(r\alpha_1) - h(r\alpha_4)} \cdot \frac{h(r\alpha_2) - h(r\alpha_4)}{h(r\alpha_2) - h(r\alpha_3)} \right| \leq M$$

holds. Since α_i, $i = 1, 2, 3, 4$ were arbitrary and since $w \in h_r(\partial\mathbf{D})$ if and only if $w = h(r\alpha)$ for some $\alpha \in \partial\mathbf{D}$, the conclusion of the lemma follows from (21). $\qquad\square$

We now have all the tools to prove Theorem 2.

The proof of Theorem 2. Note that if $\omega \equiv 0$ in the unit disk, Theorem 2 holds by the classical theorems due to Becker and Ahlfors that we have mentioned in the introduction. Therefore, we will assume that $\omega \not\equiv 0$ in $\mathbf{D}$.

We first show that if

$$(22) \qquad k < \frac{1 - \|\omega\|_\infty}{1 + \|\omega\|_\infty},$$

then the function F defined by (5) is K-quasiconformal in the whole complex plane, where K is the constant defined by (6). To do so, it suffices to prove (since h satisfies (2)) that for all $|z| \neq 1$, the dilatation $\mu_F = F_{\overline{z}}/F_z$ of F is bounded by $(K-1)/(K+1)$. (Notice that this is enough for our purposes since $|z| = 1$ is a removable set in $\mathbf{C}$.)

If $|z| < 1$, then $|\mu_F(z)| = |\omega(z)| \leq \|\omega\|_\infty$.

Let now w have modulus bigger than 1. Hence, $w = 1/\overline{z}$ for some $|z| < 1$ and a straightforward computation shows that

$$
\begin{aligned}
(23) \quad |\mu_F(w)| &= \left| \frac{h'(z) + U_z(z)}{g'(z) + U_{\overline{z}}(z)} \right| = \left| \frac{z^2 h''(z)(1 - |z|^2) - \overline{z g'(z)}}{z^2 g''(z)(1 - |z|^2) - \overline{z h'(z)}} \right| \\
&= \left| \frac{z^2 \dfrac{h''(z)}{h'(z)}(1 - |z|^2) - \overline{z\omega(z)}\dfrac{\overline{h'(z)}}{h'(z)}}{z^2 \dfrac{g''(z)}{h'(z)}(1 - |z|^2) - \overline{z}\dfrac{\overline{h'(z)}}{h'(z)}} \right| \leq \frac{\left| \dfrac{h''(z)}{h'(z)} \right|(1 - |z|^2) + \|\omega(z)\|_\infty}{1 - \left| \dfrac{g''(z)}{h'(z)} \right|(1 - |z|^2)}.
\end{aligned}
$$

On the one hand, using that f satisfies (4), we get

$$
\left| \frac{h''(z)}{h'(z)} - \frac{\overline{\omega(z)}\,\omega'(z)}{1 - |\omega(z)|^2} \right|(1 - |z|^2) \leq k - |\omega^*(z)|,
$$

which implies (by the triangle inequality)

$$
(24) \quad \left| \frac{h''(z)}{h'(z)}(1 - |z|^2) \right| \leq k - |\omega^*(z)| + |\omega(z)\,\omega^*(z)| \leq k - (1 - \|\omega\|_\infty)|\omega^*(z)|.
$$

On the other hand, we have $g'' = (\omega\, h')' = \omega' h' + \omega h''$ in the unit disk. Hence, for all $z \in \mathbf{D}$ we obtain

$$
\begin{aligned}
\frac{g''(z)}{h'(z)}(1 - |z|^2) &= \left(\omega(z) \frac{h''(z)}{h'(z)} + \omega'(z) \right)(1 - |z|^2) \\
&= \omega(z) \frac{h''(z)}{h'(z)}(1 - |z|^2) + \frac{\omega'(z)(1 - |z|^2)}{1 - |\omega(z)|^2}(1 - |\omega(z)|^2) \\
&= \omega(z) P_f(z)(1 - |z|^2) + \frac{\omega'(z)(1 - |z|^2)}{1 - |\omega(z)|^2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(25) \quad \left| \frac{g''(z)}{h'(z)} \right|(1 - |z|^2) &\leq |\omega(z)|\,|P_f(z)|(1 - |z|^2) + |\omega^*(z)| \\
&\leq \|\omega\|_\infty (k - |\omega^*(z)|) + |\omega^*(z)| \\
&= k\,\|\omega\|_\infty + (1 - \|\omega\|_\infty)|\omega^*(z)|.
\end{aligned}
$$

Using (24) and (25) in (23), we deduce

$$
\begin{aligned}
(26) \quad |\mu_F(w)| &\leq \frac{k - (1 - \|\omega\|_\infty)|\omega^*(z)| + \|\omega\|_\infty}{1 - (k\|\omega\|_\infty + (1 - \|\omega\|_\infty)|\omega^*(z)|)} \\
&= \frac{k + \|\omega\|_\infty - (1 - \|\omega\|_\infty)\,|\omega^*(z)|}{1 - k\|\omega\|_\infty - (1 - \|\omega\|_\infty)|\omega^*(z)|}.
\end{aligned}
$$

Define the function $\rho(x) \colon [0, k] \to \mathbf{R}$ by

$$
\rho(x) = \frac{k + \|\omega\|_\infty - (1 - \|\omega\|_\infty)\,x}{1 - k\|\omega\|_\infty - (1 - \|\omega\|_\infty)x}.
$$

Bearing in mind that (22) holds, we see that $\rho'(x) < 0$ for all $x \in [0, k]$. Thus,

$$\rho(x) \leq \rho(0) = \frac{k + \|\omega\|_\infty}{1 - k\|\omega\|_\infty} = \frac{K - 1}{K + 1}.$$

We then conclude by (26) that for all $|w| > 1$ the inequality $|\mu_F(w)| \leq (K-1)/(K+1)$ holds. A straightforward computation shows that $\|\omega\|_\infty \leq (K - 1)/(K + 1)$ as well. Therefore, we obtain, in fact, $|\mu_F(z)| \leq (K - 1)/(K + 1)$ for all $|z| \neq 1$. This proves that the function F defined by (5) is K-quasiconformal whenever (22) holds.

To finish the proof of Theorem 2, we are to show that even if (22) does not hold (so that we cannot ensure that the function defined by (5) is quasiconformal), the harmonic mapping f can be extended to some quasiconformal map in the complex plane.

By considering the dilations f_r defined by $f_r(z) = f(rz)$ and using [11, Theorem 5.3] one more time, we have that the result will follow once we prove that it holds under the additional assumption that f has a canonical decomposition of the form $h + \overline{g}$, where h and g are analytic functions in $\overline{\mathbf{D}}$, provided that we can show that each of the functions f_r has a K-quasiconformal extension F_r to $\mathbf{C}$, where K is a constant that does not depend on r (and is not necessarily equal to that constant K defined by (6)). To do so, we are going to check that the cross ratio of any four different points in $\Gamma_r = f_r(\partial\mathbf{D})$ is uniformly bounded by a universal positive constant C for all $r \in (0, 1)$. This will be enough for our purposes (see [2]) since, in this case, we conclude that Γ_r is a quasicircle so that f_r admits a K-quasiconformal reflection, where the constant K only depends on C (hence, K is independent of r, as needed).

Note that, at this point, we know that for each $\lambda \in \overline{\mathbf{D}}$, the analytic function $\varphi_{r,\lambda} = h_r + \lambda g_r$ has a K-quasiconformal extension $\Phi_{r,\lambda}$ to $\mathbf{C}$. Hence, the Jordan curve $\Phi_{r,\lambda}(\partial\mathbf{D})$ is a quasicircle for each $|\lambda| \leq 1$. As a consequence, we see that given any four different arbitrary points $w_i \in \Phi_{r,\lambda}(\partial\mathbf{D})$, $i = 1, 2, 3, 4$, the cross ratio (w_1, w_2, w_3, w_4) defined by (7) is bounded by a constant $K_{r,\lambda}$ (that can depend on r and on λ).

Take four different arbitrary points w_i, $i = 1, 2, 3, 4$ in $\Gamma_r = f_r(\partial\mathbf{D})$. The fact that $w_i \in \Gamma_r$ implies that there exists $|\zeta_i| = 1$ such that $w_i = f_r(\zeta_i) = h_r(\zeta_i) + \overline{g_r(\zeta_i)} = h_i + \overline{g_i}$, $i = 1, 2, 3, 4$. Thus, we can write

$$
\begin{aligned}
|(w_1, w_2, w_3, w_4)| &= \left| \frac{h_1 - h_3 + \overline{g_1 - g_3}}{h_1 - h_4 + \overline{g_1 - g_4}} \cdot \frac{h_2 - h_4 + \overline{g_2 - g_4}}{h_2 - h_3 + \overline{g_2 - g_3}} \right| \\
\text{(27)} \qquad &= \left| \frac{h_1 - h_3 + \lambda_{13}(g_1 - g_3)}{h_1 - h_4 + \lambda_{14}(g_1 - g_4)} \right| \cdot \left| \frac{h_2 - h_4 + \lambda_{24}(g_2 - g_4)}{h_2 - h_3 + \lambda_{23}(g_2 - g_3)} \right| \\
&= \left| \frac{h_{13}}{h_{14}} \cdot \frac{h_{24}}{h_{23}} \right| \cdot \left| \frac{1 + \lambda_{13}\Lambda_{13}}{1 + \lambda_{14}\Lambda_{14}} \right| \cdot \left| \frac{1 + \lambda_{24}\Lambda_{24}}{1 + \lambda_{23}\Lambda_{23}} \right|,
\end{aligned}
$$

where $\lambda_{ij}(g_i - g_j) = \overline{g_i - g_j}$ (so that $|\lambda_{ij}| = 1$), $h_{ij} = h_i - h_j$, and $\Lambda_{ij} = (g_i - g_j)/(h_i - h_j)$, with $i, j = 1, 2, 3, 4$.

Using Lemma 3, we see that there exists a constant M that does not depend on r such that for any four different points $|\zeta_i| = 1$, $i = 1, 2, 3, 4$,

$$\text{(28)} \qquad |(h_r(\zeta_1), h_r(\zeta_2), h_r(\zeta_3), h_r(\zeta_4))| = \left| \frac{h_{13}}{h_{14}} \cdot \frac{h_{24}}{h_{23}} \right| \leq M.$$

Hence, we deduce that the first factor in (27) is uniformly bounded by that constant M.

On the other hand, define

$$\delta_r = \frac{1 + k\|\omega_r\|_\infty}{k + \|\omega_r\|_\infty}$$

(where ω_r is the dilatation of f_r) and use Lemma 2 to obtain

(29)
$$\left|\frac{1 + \lambda_{13}\Lambda_{13}}{1 + \lambda_{14}\Lambda_{14}}\right| \cdot \left|\frac{1 + \lambda_{24}\Lambda_{24}}{1 + \lambda_{23}\Lambda_{23}}\right| \leq \left(\frac{1 + 1/\delta_r}{1 - 1/\delta_r}\right)^2$$

$$= \left(\frac{1 + k}{1 - k}\right)^2 \cdot \left(\frac{1 + \|\omega_r\|_\infty}{1 - \|\omega_r\|_\infty}\right)^2$$

$$\leq \left(\frac{1 + k}{1 - k}\right)^2 \cdot \left(\frac{1 + \|\omega\|_\infty}{1 - \|\omega\|_\infty}\right)^2.$$

Therefore, as a direct consequence of (28) and (29), we get from (27)

$$|(w_1, w_2, w_3, w_4)| \leq M \cdot \left(\frac{1 + k}{1 - k}\right)^2 \cdot \left(\frac{1 + \|\omega\|_\infty}{1 - \|\omega\|_\infty}\right)^2 = C.$$

This proves that f_r (hence f) admits a K-quasiconformal reflection, so that f has a quasiconformal extension to $\mathbf{C}$. This ends the proof of Theorem 2.

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