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MAPPINGS OF FINITE DISTORTION AND ASYMMETRY OF DOMAINS

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Abstract. We establish an anisotropic Fuglede inequality for images of balls under homeomorphisms with exponentially integrable distortion.

1. Introduction

In the last few years, the so-called *quantitative isoperimetric inequalities* have attracted a great interest (see for example [6, 3] and the references therein). In order to describe these results let us introduce, for any Borel set E in $\mathbf{R}^n$, with $0 < |E| < +\infty$, the *isoperimetric deficit of* E

$$\delta(E) = \frac{P(E)}{n\omega_n^{\frac{1}{n}}|E|^{\frac{n-1}{n}}} - 1 = \frac{P(E) - P(rB)}{P(rB)},$$

where P is the perimeter (surface measure of the boundary if E is smooth) and r is the radius of the ball having the same volume of E, i.e. $|E| = r^n\omega_n$. Notice that $\delta(E)$ is non-negative by the isoperimetric inequality, and equals 0 if and only if E is equivalent to a ball.

Fuglede [4] proved that if E is a convex set with volume ω_n, then

$$\min_{x\in\mathbf{R}^n} \operatorname{dist}_H(E, x + B) \le C(n)\delta(E)^{\alpha(n)}$$

where $\operatorname{dist}_H(E, x+B)$ denotes the Hausdorff distance between the sets E and $x+B$, and $\alpha(n)$ is a suitable exponent depending on the dimension n. When dealing with general non convex sets, one cannot expect the validity of inequalities like that proved by Fuglede, as can be seen by taking sets obtained by gluing thin long "tentacles" to the unit ball. Fuglede's result has been generalized to the class of John domains in [18], see also [5].

Fusco, Maggi and Pratelli [6] gave a sharp estimate for the Fraenkel asymmetry (see (1.1)) of a set E in terms of its isoperimetric deficit, proving a conjecture by Hall

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[9]. Very recently, an anisotropic version of the result in [6] was established by Figalli, Maggi and Pratelli in [3]. More precisely, given a convex set E , $0 < |E| < +\infty$, containing the origin, and an open set F, $0 < |F| < +\infty$, with smooth boundary ∂F oriented by its unit outer normal ν_F, the *anisotropic perimeter of F* is defined as

$$P_E(F) = \int_{\partial F} ||\nu_F(x)||_* \, d\mathcal{H}^{n-1}(x),$$

where

$$||\nu_F||_* = \sup\{y \cdot \nu_F \colon y \in E\}.$$

Note that, when E is the unit ball B, the perimeter $P_E(F)$ coincides with the usual notion of Euclidean perimeter. It is possible to extend the definition of anisotropic perimeter also to non-smooth sets, by using the notion of reduced boundary (see [1] or [3, Section 2.1]). Then, let us introduce the *isoperimetric deficit of F*, setting

$$\delta_E(F) = \frac{P_E(F)}{n|E|^{\frac{1}{n}}|F|^{\frac{1}{n'}}} - 1,$$

and the *relative asymmetry* of E and F as

$$A(E,F) = \inf_{b \in \mathbf{R}^n} \frac{|E \setminus (b + \kappa F)|}{|E|}, \quad \kappa = \Big(\frac{|E|}{|F|}\Big)^{1/n}, \tag{1.1}$$

where $\kappa F = \{\kappa y \colon y \in F\}$. Notice that $A(E,F) = A(F,E)$, $A(\lambda E, \mu F) = A(E,F)$ for every λ and $\mu > 0$ and that if E is the unit ball, the relative asymmetry coincides with the Fraenkel asymmetry of F.

In [3] it has been proved that, if F is a measurable set with finite perimeter and finite positive measure, then $A(F,E)$ can be controlled by the isoperimetric deficit. More precisely, $A(F,E) \le C\sqrt{\delta_E(F)}$ for a constant C depending only on n.

The aim of this paper is to prove an anisotropic version of the Fuglede type inequality in higher dimension, restricting ourselves to the class of domains which are images of the unit ball under global homeomorphisms with exponentially integrable distortion. We recall that $f \colon \Omega \subset \mathbf{R}^n \to \mathbf{R}^n$ is a *mapping of finite distortion*, if $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$, the Jacobian determinant $J_f \in L^1_{\mathrm{loc}}(\Omega)$, and if there is a measurable, almost everywhere finite function K_f such that

$$|Df(x)|^n \le K_f(x) J_f(x) \quad \text{for almost every } x \in \Omega.$$

If E and F are as above, we recall that the *Hausdorff distance* between ∂E and ∂F is

$$\mathrm{dist}_H(\partial E, \partial F) = \max\{\sup_{x \in \partial E} \inf_{y \in \partial F} |x-y|, \sup_{y \in \partial F} \inf_{x \in \partial E} |x-y|\}.$$

We define the *relative distance* $\mathrm{dist}_{\mathrm{rel}}(\partial E, \partial F)$,

$$\mathrm{dist}_{\mathrm{rel}}(\partial E, \partial F) = \inf_{b \in \mathbf{R}^n} \frac{\mathrm{dist}_H(\partial E, b + \kappa \partial F)}{|E|^{1/n}},$$

where κ is the constant in (1.1). Notice that the relative distance is also scaling and translation invariant.

In [17], it has been proved that the Hausdorff distance between the images of the unit ball under homeomorphisms with exponentially integrable distortion and the unit ball can be controlled by their Fraenkel asymmetry. Our main result states that we can control the relative distance between the images of the unit sphere under two homeomorphisms with exponentially integrable distortion with the relative asymmetry of the images of the unit ball. More precisely, we have

Theorem 1.1. *Let $f\colon B_2 \to fB_2$ and $g\colon B_2 \to gB_2$ be homeomorphisms of finite distortion satisfying*

$$\int_{B_2} \exp(\mu K_f(x)) + \exp(\mu K_g(x))\,\mathrm{d}x = \mathcal{K} < \infty$$

for some μ and $\mathcal{K} > 0$. Then

$$\operatorname{dist}_{\mathrm{rel}}(fS_1, gS_1)^{n+n^2/\mu} \le C(n, \mu, \mathcal{K})A(fB_1, gB_1). \tag{1.2}$$

As a consequence of Theorem 1.1, we obtain that the relative distance between the image of the unit sphere under a homeomorphism with exponentially integrable distortion and the boundary of a convex set can be estimated with the relative asymmetry between the convex set and the image of the unit ball. Namely, we have

Theorem 1.2. *Let $f\colon B_2 \to fB_2$ be a homeomorphism of finite distortion satisfying*

$$\int_{B_2} \exp(\mu K_f(x))\,\mathrm{d}x = \mathcal{K} < \infty$$

for some μ and $\mathcal{K} > 0$, and let E be a convex domain, $B_1 \subset E \subset B_\Lambda$. Then

$$\operatorname{dist}_{\mathrm{rel}}(fS_1, \partial E)^{n+n^2/\mu} \le C(n, \mu, \mathcal{K}, \Lambda)A(fB_1, E) \le C(n, \mu, \mathcal{K}, \Lambda)\sqrt{\delta_E(fB_1)}. \tag{1.3}$$

Notice that the second inequality in the previous theorem follows by the result in [3]. Let E be a convex set with finite positive Lebesgue measure and let $B_1 \subset E \subset B_\Lambda$. We notice that the estimate of Theorem 1.2 could blow up when Λ goes to infinity. In fact, we can take f to be the identity mapping and a sequence of convex sets E_j converging to a line segment. Then the right hand side of (1.3) tends to infinity while the left hand side remains bounded.

Theorems 1.1 and 1.2 do not hold in general if f is only defined on B_1. Indeed, for every $n \ge 2$ there exist $K \ge 1$ and a sequence of K-quasiconformal homeomorphisms $f_j\colon B_1 \to \Omega_j$ such that Ω_j is the union of B_1 and a truncated cone with vertex $2e_1$ and opening angle $1/j$. This can be seen using the fact that truncated cones with different opening angles smaller than $\pi/2$ are quasiconformally equivalent with distortion independent of the angles, see [8]. The exponent that appears in estimates (1.2) and (1.3) cannot be improved, except for the constant n^2, as shown in the example at Theorem 1.2 of [17].

We would like to mention that Theorem 1.2 gives an extension of the main Theorem 1.1 in [17]. Indeed, this is implied by Theorem 1.2, in case E is assumed to be a ball. The approach we use to prove Theorems 1.1 and 1.2 is quite similar to that used in [17] earlier. However, we are able to prove much more general results because of our new observation that the approach can be applied to give estimates for the distances of rather general domains (in particular without assuming that one of them is a ball).

Quantitative isoperimetric inequalities have been applied to prove new distortion estimates for quasiconformal maps with small distortion, see [15, 16]. More generally, we expect that Theorems 1.1 and 1.2 can be applied to give estimates for quasiconformal maps whose distortions are suitably controlled using a convex domain E.

2. Notations and preliminary results

We shall denote a ball in $\mathbf{R}^n$ with center x and radius r by $B_r(x)$, while when the ball is centered at the origin we shall omit the indication of the center, i.e. $B_r = B_r(0)$. The corresponding notations for spheres will be $S_r(x)$ and S_r.

We shall denote by $|Df|$ the operator norm of the differential matrix and by $(Df)^\sharp$ the adjugate of Df which is defined by the formula

$$Df \cdot (Df)^\sharp = \mathbf{I} \cdot J_f, \tag{2.1}$$

where, as usual, $J_f = \det Df$ and $\mathbf{I}$ is the identity matrix.

Recall that a homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ has finite outer distortion if its Jacobian J_f is strictly positive a.e. on the set where $|Df| \neq 0$. In case $J_f(x) \geq 0$ a.e., we define its outer distortion function as

$$K_f(x) = \begin{cases} \frac{|Df(x)|^n}{J_f(x)} & \text{for } J_f(x) > 0, \\ 1 & \text{otherwise.} \end{cases} \tag{2.2}$$

We note, for a homeomorphism with finite distortion, the following relation:

$$|(Df^{-1})^\sharp(y)| = K_f(f^{-1}(y))^{\frac{1}{n}} J_{f^{-1}}(y)^{\frac{n-1}{n}} \quad \text{a.e. } y \in f(\Omega) \tag{2.3}$$

(we refer to [11] for an exhaustive treatment of the mappings with finite distortion). We shall use following result concerning the modulus of continuity of a homeomorphism with exponentially integrable distortion.

Theorem 2.1. [14] *Let f be as in Theorem 1.1. If x and $y \in B_{5/4}$, then*

$$|f(x) - f(y)| \leq \frac{C(n,\mu,\mathcal{K})}{\log^{\mu/n} \frac{1}{|x-y|}} |fB_1|^{1/n}.$$

Moreover, we have the following distortion estimate.

Lemma 2.2. [17] *Let f be as in Theorem 1.1. If $B_t(x) \subset B_{5/4}$, then there exists a constant $C = C(n,\mu,\mathcal{K})$ such that*

$$\frac{\max_{y \in S_{3/2}} |f(x) - f(y)|}{\min_{y \in S_t(x)} |f(x) - f(y)|} \leq \exp\left(C(n,\mu,\mathcal{K}) t^{-\frac{1}{n-\frac{3}{2}}} \right).$$

We will use the following consequence of the previous results.

Lemma 2.3. *Let f be as above and assume $f(e_1) = 0$ and $|fB_1| = \omega_n$. Then there exists $m_0 > 0$, depending only on n, μ, and $\mathcal{K}$, such that for every $0 < t < m_0$,*

$$B_{s_t}(e_1) \subset f^{-1}B_t \subset B_{1/10}(e_1). \tag{2.4}$$

Here s_t satisfies

$$t = \frac{C(n,\mu,\mathcal{K})}{\log^{\mu/n} \frac{1}{s_t}},$$

where $C(n,\mu,\mathcal{K})$ is the constant in Theorem 2.1. Moreover,

$$|B_1 \cap f^{-1}B_t|^{(n-1)/n} \leq C(n)\mathcal{H}^{n-1}(B_1 \cap f^{-1}S_t). \tag{2.5}$$

Proof. The first inclusion in (2.4) follows directly from Theorem 2.1. Also, under the assumption $|fB_1| = \omega_n$, there exists $b \in S_{3/2}$ such that $|f(e_1)-f(b)| = |f(b)| \geq 1$. Lemma 2.2 with $x = e_1$ yields

$$\frac{1}{\min_{y\in S_{1/10}(e_1)} |f(y) - f(e_1)|} \leq \frac{\max_{b\in S_{3/2}} |f(b)|}{\min_{y\in S_{1/10}(e_1)} |f(y)|} \leq \varphi(n, \mu, \mathcal{K}).$$

The second inclusion follows once m_0 is chosen to be small enough depending only on n, μ, and $\mathcal{K}$.

To prove (2.5), we notice that the second inclusion in (2.4) guarantees that

$$|B_1 \cap f^{-1}B_t| \leq |B_1 \setminus f^{-1}B_t|.$$

Therefore, (2.5) follows by the following relative isoperimetric inequality in B_1 (see [1, (3.43)]):

$$\min\{|B_1 \cap f^{-1}B_t|^{(n-1)/n}, |B_1 \setminus f^{-1}B_t|^{(n-1)/n}\} \leq C(n)\mathcal{H}^{n-1}(B_1 \cap f^{-1}S_t). \qquad \square$$

3. Proof of Theorem 1.1

We assume that $A(fB_1, gB_1) > 0$. Since the distortions K_f and K_g are not affected by postcompositions with affine maps, we may assume that $|fB_1| = |gB_1| = \omega_n$ and

$$A(fB_1, gB_1) = \omega_n^{-1}|fB_1 \setminus gB_1| = \omega_n^{-1}|gB_1 \setminus fB_1|.$$

We denote $m = \operatorname{dist}_H(fS_1, gS_1)$. In order to prove the theorem, we need to show that

$$m^{n+n^2/\mu} \leq C(n, \mu, \mathcal{K})|fB_1 \setminus gB_1|.$$

Now either there exists a point $y_0 \in fS_1$ such that

$$\operatorname{dist}_H(y_0, gS_1) = m, \tag{3.1}$$

or a point $z_0 \in gS_1$ such that

$$\operatorname{dist}_H(z_0, fS_1) = m. \tag{3.2}$$

From now on we assume that (3.1) holds, otherwise we change the roles of f and g. By precomposing f with a rotation and postcomposing with a translation, if necessary, we may assume that $y_0 = 0$ and $f^{-1}(y_0) = e_1$.

Since $\operatorname{dist}_H(0, gS_1) = m$, we have that either

$$B_m \cap fB_1 \subset \mathbf{R}^n \setminus gB_1,$$

or

$$B_m \setminus fB_1 \subset gB_1,$$

depending on whether or not $0 \in \mathbf{R}^n \setminus gB_1$. In the first case we have

$$|fB_1 \setminus gB_1| \geq |B_m \cap fB_1 \setminus gB_1| = |B_m \cap fB_1|,$$

so in order to prove the theorem it suffices to show that

$$m^{n+n^2/\mu} \leq C(n, \mu, \mathcal{K})|B_m \cap fB_1|. \tag{3.3}$$

In the second case

$$|gB_1 \setminus fB_1| \geq |B_m \setminus fB_1|,$$

so in this case the theorem follows if

$$m^{n+n^2/\mu} \leq C(n, \mu, \mathcal{K})|B_m \setminus fB_1|. \tag{3.4}$$

The proofs of estimates (3.3) and (3.4) are very similar, and therefore we only give the proof of (3.3).

Proof of (3.3). We first assume that $m \le m_0$, where m_0 is as in Lemma 2.3. We choose $t_0 = m/2$ and an increasing sequence $(t_j)_{j=1}^k$ of radii inductively such that if

$$|f^{-1}(B_m) \cap B_1| > 2|f^{-1}(B_{t_{j-1}}) \cap B_1|,$$

then we choose $m/2 < t_j < m$ such that

$$|f^{-1}(B_{t_j}) \cap B_1| = 2|f^{-1}(B_{t_{j-1}}) \cap B_1|, \tag{3.5}$$

otherwise $j = k$ and we choose $t_k = m$. We claim that

$$k \le C(n) \log \frac{1}{s_{m/2}} \le C(n, \mu, \mathcal{K}) m^{-n/\mu}. \tag{3.6}$$

Here $s_{m/2}$ is as in Lemma 2.3. The second inequality in (3.6) follows from Lemma 2.3. In order to prove the first inequality we use the definition of (t_j) to write (note that $t_k = m$)

$$2^{k-1} \le \prod_{j=1}^{k} \frac{|f^{-1}(B_{t_j}) \cap B_1|}{|f^{-1}(B_{t_{j-1}}) \cap B_1|} = \frac{|f^{-1}(B_m) \cap B_1|}{|f^{-1}(B_{m/2}) \cap B_1|}. \tag{3.7}$$

Since

$$B_{s_{m/2}}(e_1) \subset f^{-1}(B_{m/2}) \quad \text{and} \quad f^{-1}(B_m) \subset B_{1/10}(e_1),$$

by Lemma 2.3, the right term in (3.7) can be estimated by $C(n)s_{m/2}^{-n}$. Taking logarithms gives the first inequality in (3.6).

By [2] (see also [7, 10, 13]), $f^{-1} \in W^{1,n}(fB_2, \mathbf{R}^n)$. Therefore,

$$\mathcal{H}^{n-1}(f^{-1}(S_t \cap fB_1)) \le \int_{S_t \cap fB_1} |(Df^{-1})^\sharp(y)| \, \mathrm{d}\mathcal{H}^{n-1}(y) \tag{3.8}$$

for almost every $m/2 < t < m$. Indeed, recall that the change of variables formula holds for $h\colon G \subset \mathbf{R}^k \to \mathbf{R}^n$, $k \le n$, whenever $h \in W^{1,p}$ for some $p > k$, see [12]. We conclude that (3.8) holds since the restriction of our f^{-1} to (the relative components of) $S_t \cap fB_1$ belongs to $W^{1,n}$ for almost every $m/2 < t < m$.

Let $V_t = f^{-1}(B_t) \cap B_1$ and $V_j = V_{t_j}$. We integrate both sides of (3.8) over the interval (t_{j-1}, t_j), $j = 1, \ldots, k$. By the relative isoperimetric inequality (2.5), and our choice (3.5) of the radii t_j, the left integral is estimated from below by

$$C(n)(t_j - t_{j-1})|V_j|^{(n-1)/n}.$$

Using the equality at (2.3), Hölder's inequality and the area formula, the right integral is estimated from above as follows:

$$\begin{aligned} \int_{fB_1 \cap B_{t_j} \setminus B_{t_{j-1}}} |(Df^{-1})^\sharp(y)| \, \mathrm{d}y &= \int_{fB_1 \cap B_{t_j} \setminus B_{t_{j-1}}} K_f(f^{-1}(y))^{1/n} J_{f^{-1}}(y)^{(n-1)/n} \, \mathrm{d}y \\ &\le |fB_1 \cap B_{t_j} \setminus B_{t_{j-1}}|^{1/n} \Big(\int_{V_j} K_f(x)^{1/(n-1)} \, \mathrm{d}x \Big)^{(n-1)/n}. \end{aligned}$$

We denote $|fB_1 \cap B_{t_j} \setminus B_{t_{j-1}}|$ by q_j. Combining the estimates and taking the measure of V_j to the right yields

$$(t_j - t_{j-1}) \le C q_j^{1/n} \Big(|V_j|^{-1} \int_{V_j} K_f(x)^{1/(n-1)} \, \mathrm{d}x \Big)^{(n-1)/n}, \tag{3.9}$$

where C depends only on n. Applying Jensen's inequality to the convex function $t \mapsto \exp(\mu t^{n-1})$, and using the integrability assumption on K_f, gives

$$
\begin{aligned}
(3.10)\qquad |V_j|^{-1}\int_{V_j} K_f(x)^{1/(n-1)}\,\mathrm{d}x &\le \mu^{-1}\Big(\log\big(|V_j|^{-1}\int_{V_j}\exp(\mu K_f(x))\,\mathrm{d}x\big)\Big)^{1/(n-1)} \\
&\le \mu^{-1}\Big(\log\Big(\mathcal{K}|V_j|^{-1}\Big)\Big)^{1/(n-1)}.
\end{aligned}
$$

Since $|V_j| \ge C(n)s_{m/2}^n$, where $s_{m/2}$ is as in Lemma 2.3, the second inequality in (3.6) yields

$$\Big(\log|V_j|^{-1}\Big)^{1/(n-1)} \le C(n,\mu,\mathcal{K})m^{-n/(\mu(n-1))}.$$

Combining with (3.9) and (3.10) gives

$$t_j - t_{j-1} \le Cq_j^{1/n}m^{-1/\mu}.$$

Finally, we add over j and use the Cauchy–Schwarz inequality:

$$(3.11)\qquad \frac{m}{2} = \sum_{j=1}^{k} t_j - t_{j-1} \le Cm^{-1/\mu}\sum_{j=1}^{k} q_j^{1/n} \le Cm^{-1/\mu}k^{(n-1)/n}\Big(\sum_{j=1}^{k} q_j\Big)^{1/n}.$$

Since

$$\sum_{j=1}^{k} q_j \le |B_m \cap fB_1|,$$

using estimates (3.11) and (3.6), we get

$$m \le C(n,\mu,\mathcal{K})|B_m \cap fB_1|^{1/n}m^{-n/\mu},$$

which implies (3.3).

We next assume that $m \ge m_0$. By applying the previous argument with $m = m_0$, we see that

$$(3.12)\qquad |B_m \cap fB_1| \ge |B_{m_0} \cap fB_1| \ge C_1(n,\mu,\mathcal{K}).$$

On the other hand, Lemma 2.2 shows that we always have

$$(3.13)\qquad m \le \operatorname{diam} fS_1 \le C_2(n,\mu,\mathcal{K})|fB_1| = C_3(n,\mu,\mathcal{K}).$$

Combining (3.12) and (3.13) gives (3.3). The proof is complete. □

4. Proof of Theorem 1.2

We use the following result of Gehring and Väisälä [8, Theorem 5.2] (they only consider the case $n = 3$, but the proof extends to all dimensions $n \ge 2$).

Theorem 4.1. *Let $B_1 \subset E \subset B_\Lambda$ be a convex domain. Then there exists a K-quasiconformal mapping $g\colon \mathbf{R}^n \to \mathbf{R}^n$, where K depends only on n and Λ, such that $gB_1 = E$.*

Proof of Theorem 1.2. We shall apply Theorems 1.1 and 4.1. By scaling invariance, we may assume that $|fB_1| = |E| = \omega_n$ and

$$A(fB_1, E) = \omega_n^{-1}|fB_1 \setminus E|.$$

By Lemmas 2.2 and 2.3, there exist $x \in \mathbf{R}^n$ and $t > 1$, depending only on n, μ, and $\mathcal{K}$, such that

$$B_{t^{-1}}(x) \subset fB_1 \subset B_t(x).$$

Let Λ be the smallest constant for which there exist $x \in \mathbf{R}^n$ and $R > 0$ such that

$$B_R(x) \subset E \subset B_{\Lambda R}(x).$$

Then, if Λ is large enough depending on t, the convexity of E implies that $A(fB_1, E) \geq 1/100$. We have $R < 1$ because $|E| = \omega_n$, and so

$$\operatorname{dist}_{\mathrm{rel}}(fS_1, \partial E) \leq \operatorname{diam} fS_1 + \operatorname{diam} \partial E \leq C(n, \mu, \mathcal{K}) + \Lambda.$$

Therefore,

$$\operatorname{dist}_{\mathrm{rel}}(fS_1, \partial E) \leq C(n, \mu, \mathcal{K}, \Lambda) A(fB_1, E)$$

when $\Lambda \geq \Lambda_0(n, \mu, \mathcal{K})$.

Now let $\Lambda \leq \Lambda_0(n, \mu, \mathcal{K})$. An application of Theorem 4.1 gives a $K(n, \mu, \mathcal{K})$-quasiconformal homeomorphism $g\colon B_2 \to gB_2$ such that $gB_1 = E$. We can now apply Theorem 1.1 to f and g, since

$$\int_{B_2} \exp(\mu K_g(x))\, \mathrm{d}x \leq C(n, \mu, \mathcal{K}).$$

This gives the first inequality at (1.3). The second inequality follows from [3, Theorem 1.1]. □

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