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# $Q_K$ AND MORREY TYPE SPACES

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**Abstract.** In this paper, we obtain a characterization of spaces $Q_K$ in terms of fractional order derivatives of functions. We give a description of Morrey-type spaces similar to the well-known characterization of $BMOA$. A relationship between $Q_K$ spaces and Morrey type spaces in terms of the fractional order derivatives is established.

## 1. Introduction

There are two principal results obtained in this article. The first result is a characterization of the space $Q_K$ in terms of some fractional order derivatives of an analytic function in the unit disc $\mathbf{D}$. In [12] we characterized the $Q_K$ spaces in terms of higher order derivatives. The main difficulty here is to replace higher order derivatives by fractional order derivatives. The second result is a connection between the spaces $Q_K$ and Morrey type spaces $H^2_K$ introduced in Section 3. We will show that if $f$ is a member of $Q_K$, then some fractional order derivatives of $f$ belongs to $H^2_K$. Conversely, if $f$ is in the Morrey type spaces, then some fractional order derivatives of $f$ belongs to $Q_K$ space.

Before proceeding, it may be useful to recall a few fundamental definitions and establish some notation.

Let $K\colon [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function. The $Q_K$ space consists of analytic functions $f$ in $\mathbf{D}$ satisfying

$$(1.1) \qquad \|f\|_K = \left(\sup_{a\in\mathbf{D}} \int_{\mathbf{D}} |f'(z)|^2 K(g(z,a))\, dA(z)\right)^{1/2} < \infty,$$

where $g(z,a)$ is the Green function in $\mathbf{D}$ with singularity at $a \in \mathbf{D}$, and $dA(z)$ is the Euclidean area element on $\mathbf{D}$ so that $A(\mathbf{D}) = 1$.

It is clear that $Q_K$ is Möbius-invariant, i.e., $\|f \circ \varphi_a\|_K = \|f\|_K$ holds for all $a \in \mathbf{D}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$; see [3] and [4] for the theory of $Q_K$ spaces. In the case $K(t) = t^p, 0 < p < \infty$, the space $Q_K$ gives $Q_p$ space; see [1], [13] and [14]. Especially, $Q_K$ coincides with BMOA if $K(t) = t$. We know from [3] that $Q_K$ spaces are contained in the Bloch space $\mathcal{B}$, which consists of analytic functions $f$ such that

$$\|f\|_{\mathcal{B}} = \sup\{(1-|z|^2)|f'(z)|\colon z \in \mathbf{D}\} < \infty.$$

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In this paper we assume that

$$\int_0^{\frac{1}{e}} K\left(\log\frac{1}{r}\right) dr < \infty. \tag{1.2}$$

Otherwise, the space $Q_K$ contains constant functions only. By Theorem 2.1 in [3] we may assume that $K$ is defined on [0,1] and extend its domain to $[0,\infty)$ by setting $K(t) = K(1)$ for $t > 1$.

Further we need two conditions on $K$ as follows:

$$\int_0^1 \varphi_K(s)\frac{ds}{s} < \infty \tag{1.3}$$

and

$$\int_1^\infty \varphi_K(s)\frac{ds}{s^{1+p}} < \infty, \quad 0 < p < 2, \tag{1.4}$$

where

$$\varphi_K(s) = \sup_{0<t\le 1} K(st)/K(t), \quad 0 < s < \infty.$$

It is obvious that $K(t) = t^q, 0 \le q \le 1$, satisfies (1.3) and (1.4) for all $0 < p < 2$. For a subarc $I \subset \partial\mathbf{D}$, let $\theta$ be the midpoint of $I$ and denote

$$S(I) = \left\{ z \in \mathbf{D}\colon 1 - |I| < |z| < 1,\ |\theta - \arg z| < \frac{|I|}{2} \right\}$$

for $|I| \le 1$ and $S(I) = \mathbf{D}$ for $|I| > 1$, where $|I|$ denotes the length of $I$. For $0 < p < \infty$, we say that a positive measure $d\mu$ is a $p$-Carleson measure on $\mathbf{D}$ provided

$$\|\mu\|_p = \sup_{I\subset\partial\mathbf{D}} \frac{\mu(S(I))}{|I|^p} < \infty.$$

A positive measure $d\mu$ is said to be a $K$-Carleson measure on $\mathbf{D}$ if

$$\|\mu\|_K = \sup_{I\subset\partial\mathbf{D}} \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) d\mu(z) < \infty. \tag{1.5}$$

Clearly, if $K(t) = t^p$ , then $\mu$ is a $K$-Carleson measure on $\mathbf{D}$ if and only if $(1 - |z|^2)^p\, d\mu(z)$ is a $p$-Carleson measure on $\mathbf{D}$.

In addition, we may assume that $K(t) \approx K(2t)$. This means that $K(t) \lesssim K(2t) \lesssim K(t)$. Note, we say $K_1 \lesssim K_2$ (for two functions $K_1$ and $K_2$) if there exists a constant $C > 0$ (independent of $K_1$ and $K_2$) such that $K_1 \le CK_2$.

In the present work we need two basic characterizations of $Q_K$ spaces and we shall list them here for reference. First we mention the higher order derivative characterization of $Q_K$ spaces given by the first author and Zhu in [12].

**Theorem A.** *Suppose*

$$\int_0^1 \frac{\varphi_K(s)}{s}\, ds < \infty$$

*or*

$$\int_1^\infty \frac{\varphi_K(s)}{s^p}\, ds < \infty$$

*for some* $0 < p < 2$*. Then for any positive integer* $n$*, an analytic function* $f$ *in* $\mathbf{D}$ *belongs to* $Q_K$ *if and only if*

$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}|f^{(n)}(z)|^2(1-|z|^2)^{2n-2}K(1-|\varphi_a(z)|^2)\,dA(z)<\infty.$$

The second result we mention here is a characterization of $K$-Carleson measure given by the first author and Essen and Xiao in [4].

**Theorem B.** *Let* $K$ *satisfy (1.3). A positive measure* $d\mu$ *on* $\mathbf{D}$ *is a* $K$*-Carleson measure if and only if*

$$\sup_{a\in\mathbf{D}}\int_{\mathbf{D}}K(1-|\varphi_a(z)|^2)\,d\mu(z)<\infty.$$

By Theorems A and B, we have

**Theorem C.** *Let* $K$ *satisfy (1.3). An analytic function* $f$ *in* $\mathbf{D}$ *belongs to* $Q_K$ *if and only if* $|f^{(n)}(z)|^2(1-|z|^2)^{2n-2}\,dA(z)$ *is a* $K$*-Carleson measure.*

The following lemma will be used in the sections 2 and 3, and its proof will be given in Section 3.

**Lemma D.** *If* $K$ *satisfies the condition (1.4), then there exists a weight* $K_1$*, comparable with* $K$*, such that* $K_1(t)/t^p$ *is non-increasing. Moreover, for some enough small* $c>0$*,* $K_1(t)/t^{p-c}$ *is also non-increasing.*

## 2. Fractional order derivative and $Q_K$ spaces

For fixed $b>1$, define the $\alpha$-order derivative as follows:

$$f^{(\alpha)}(z)=\frac{\Gamma(b+\alpha)}{\Gamma(b)}\int_{\mathbf{D}}\frac{(1-|w|^2)^{b-1}}{(1-\overline{w}z)^{b+\alpha}}\overline{w}^{[\alpha-1]}f'(w)\,dA(w),\quad b+\alpha>0,$$

where $\Gamma$ is the Gamma function and $[\alpha]$ denotes the smallest integer which is larger than or equal to $\alpha$. Since

$$(z^n)^{(\alpha)}=\begin{cases}\frac{\Gamma(b+n+\alpha-1-[\alpha-1])\Gamma(n+1)}{\Gamma(b+n)\Gamma(n-[\alpha-1])}z^{n-1-[\alpha-1]}, & n\geq[\alpha-1]+1,\\ 0, & n<[\alpha-1]+1,\end{cases}$$

we know that if $\alpha=n, n=1,2,3\cdots$, then $f^{(\alpha)}$ is just the derivative of order $n$ of $f$.

The following is our first main result in this paper:

**Theorem 2.1.** *Let* $K$ *satisfy the conditions (1.3) and (1.4). If* $\alpha>1/2$*, then* $f\in Q_K$ *if and only if* $|f^{(\alpha)}(z)|^2(1-|z|^2)^{2(\alpha-1)}\,dA(z)$ *is a* $K$*-Carleson measure.*

Firstly, we give some results which will be used in our proof.

**Lemma 2.1.** *Let* $K$ *satisfy the conditions (1.3) and (1.4). Let* $b+\alpha\geq 1+p$*,* $b\geq p$ *and* $\alpha>0$*. There exists a* $\beta\in(0,1)$ *and a constant* $C$ *(independent of* $|I|$*, the length of arc* $I$ *on* $\partial\mathbf{D}$*) such that*

$$\int_{\mathbf{D}}\frac{K\left(\frac{1-|z|}{|I|}\right)(1-|w|^2)^{b-1}}{(1-|z|)^{1-\alpha+\beta}|1-\overline{w}z|^{b+\alpha}}\,dA(z)\leq C\frac{K\left(\frac{1-|w|}{|I|}\right)}{(1-|w|)^{\beta}}\tag{2.1}$$

*for all* $w\in\mathbf{D}$.

*Proof.* By Lemma D, there exists a small enough $c > 0$ such that $t^{c-p}K(t)$ is decreasing. Since $b+\alpha \geq 1+p$, $b \geq p$, $\alpha > 0$, we are able to choose $\beta \in (0, \min\{\alpha, 1\})$ such that $b - p + \beta + c > 1$. If $1 - |w| \geq |I|$, Lemma 4.2.2 in [15] gives

$$\int_{\mathbf{D}} \frac{K\left(\frac{1-|z|}{|I|}\right)(1-|w|^2)^{b-1}}{(1-|z|)^{1-\alpha+\beta}|1-\overline{w}z|^{b+\alpha}}\,dA(z) \lesssim \int_{\mathbf{D}} \frac{(1-|w|^2)^{b-1}}{(1-|z|)^{1-\alpha+\beta}|1-\overline{w}z|^{b+\alpha}}\,dA(z)$$
$$\lesssim \frac{1}{(1-|w|^2)^{\beta}} \lesssim \frac{K\left(\frac{1-|w|}{|I|}\right)}{(1-|w|^2)^{\beta}}.$$

It is easy to see that (2.1) holds when $1 - |w| < |I|$ and $|w| \leq 1/2$. Now we assume $1 - |w| < |I|$ and $|w| > 1/2$. Without loss of generality we may assume that $I$ is centered at $e^{i0} = 1$ and $\mathrm{Im}(w) = 0$. Let $\gamma = 1 - w$. We divide the unit disk $\mathbf{D}$ into $S_1 \bigcup S_2 \bigcup S_3$, where

$$S_1 = \{z\colon 0 < 1-|z| \leq \gamma,\ |\arg z| \leq \gamma/2\},$$
$$S_2 = \{z\colon \gamma < 1-|z| \leq 1,\ |\arg z| \leq \gamma/2\}$$

and

$$S_3 = \{z\colon 0 < 1-|z| \leq 1,\ |\arg z| > \gamma/2\}.$$

Then

$$\int_{S_1} \frac{K\left(\frac{1-|z|}{|I|}\right)(1-|w|^2)^{b-1}}{(1-|z|)^{1-\alpha+\beta}|1-\overline{w}z|^{b+\alpha}}\,dA(z) \lesssim \gamma^b \int_0^{\gamma} \frac{K(t/|I|)\,dt}{(\gamma+t(1-\gamma))^{b+\alpha}t^{1-\alpha+\beta}}$$
$$\leq \frac{1}{\gamma^{\alpha}} \int_0^{\gamma} \frac{K(t/|I|)}{t^{1-\alpha+\beta}}\,dt$$

and

$$\int_{S_2} \frac{K\left(\frac{1-|z|}{|I|}\right)(1-|w|)^{b-1}}{(1-|z|)^{1-\alpha+\beta}|1-\overline{w}z|^{b+a}}\,dA(z) \leq \gamma^b \int_{\gamma}^{1} \frac{K(t/|I|)\,dt}{(\gamma+t(1-\gamma))^{b+\alpha}t^{1-\alpha+\beta}}$$
$$\leq \frac{\gamma^{b-1}}{(1-\gamma)^{b+\alpha-1}} \int_{\gamma}^{1} \frac{K(t/|I|)}{t^{b+\beta}}\,dt.$$

On the other hand,

$$\int_{S_3} \frac{K\left(\frac{1-|z|}{|I|}\right)(1-|w|)^{b-1}}{(1-|z|)^{1-\alpha+\beta}|1-\overline{w}z|^{b+\alpha}}\,dA(z)$$
$$\leq 2\gamma^{b-1} \int_0^1 \frac{K(t/|I|)}{t^{1-\alpha+\beta}} \left(\int_{\gamma}^{\pi} \frac{d\theta}{[(\gamma+t(1-\gamma))^2+\sin^2(\theta/2)]^{\frac{b+\alpha}{2}}}\right) dt$$
$$\lesssim \gamma^{b-1} \int_0^1 \frac{K(t/|I|)\,dt}{(\gamma+t(1-\gamma))^{b+\alpha-1}t^{1-\alpha+\beta}}$$
$$\leq \frac{1}{\gamma^{\alpha}} \int_0^{\gamma} \frac{K(t/|I|)}{t^{1-\alpha+\beta}}\,dt + \frac{\gamma^{b-1}}{(1-\gamma)^{b+\alpha-1}} \int_{\gamma}^{1} \frac{K(t/|I|))\,dt}{t^{b+\beta}}.$$

Since $\beta < \alpha$, Lemma 2.1 in [4] gives

$$\frac{1}{\gamma^{\alpha}} \int_0^{\gamma} \frac{K(t/|I|)}{t^{1-\alpha+\beta}}\,dt = \frac{|I|^{\alpha-\beta}}{\gamma^{\alpha}} \int_0^{\gamma/|I|} \frac{K(s)}{s^{1-\alpha+\beta}}\,ds \lesssim \frac{1}{\gamma^{\beta}} \int_0^{\gamma/|I|} \frac{K(s)}{s}\,ds \approx \frac{K(\gamma/|I|)}{\gamma^{\beta}}.$$

Note that $b-p+\beta+c>1$. By Lemma D we have

$$\begin{aligned}\frac{\gamma^{b-1}}{(1-\gamma)^{b+\alpha-1}}\int_\gamma^1\frac{K(t/|I|)}{t^{b+\beta}}\,dt &\le \frac{\gamma^{b-1}}{(1-\gamma)^{b+\alpha-1}}K\left(\frac{\gamma}{|I|}\right)\left(\frac{\gamma}{|I|}\right)^{c-p}\int_\gamma^1\frac{dt}{t^{b+\beta}(t/|I|)^{c-p}}\\ &\le \frac{\gamma^{b-p-1+c}}{(1-\gamma)^{b+\alpha-1}}K\left(\frac{\gamma}{|I|}\right)\int_\gamma^1\frac{dt}{t^{b-p+\beta+c}}\lesssim\frac{K(\gamma/|I|)}{\gamma^\beta}.\end{aligned}$$

The above estimates give

$$\begin{aligned}&\int_{\mathbf{D}}\frac{K\left(\frac{1-|z|}{|I|}\right)(1-|w|)^{b-1}}{(1-|z|)^{1-\alpha+\beta}|1-\overline{w}z|^{b+\alpha}}\,dA(z)\lesssim\sum_{j=1}^{3}\int_{S_i}\frac{K\left(\frac{1-|z|}{|I|}\right)(1-|w|)^{b-1}}{(1-|z|)^{1-\alpha+\beta}|1-\overline{w}z|^{b+\alpha}}\,dA(z)\\ &\le\frac{1}{\gamma^\alpha}\int_0^\gamma\frac{K(t/|I|)}{t^{1-\alpha+\beta}}\,dt+\frac{\gamma^{b-1}}{(1-\gamma)^{b+\alpha-1}}\int_\gamma^1\frac{K(t/|I|)}{t^{b+\beta}}\,dt\lesssim\frac{K(\gamma/|I|)}{\gamma^\beta}.\end{aligned}$$

Hence (2.1) holds. The proof is complete. □

**Lemma 2.2.** *Let $K$ satisfy the conditions (1.3) and (1.4). Let $\psi$ be measurable on $\mathbf{D}$. If $d\mu(z)=|\psi(z)|^2dA(z)$ is a $K$-Carleson measure, then $|\psi(z)|(1-|z|^2)^{(p-1)/2}dA(z)$ is a $(p+1)/2$-Carleson measure.*

*Proof.* By Lemma D, we can choose a small $c$ such that $t^{-p+c}K(t)$ is decreasing. By Cauchy–Schwarz inequality we have

$$\begin{aligned}&\int_{S(I)}|\psi(z)|(1-|z|^2)^{(p-1)/2}\,dA(z)\\ &\le\left(\int_{S(I)}|\psi(z)|^2K\left(\frac{1-|z|}{|I|}\right)dA(z)\right)^{1/2}\left(\int_{S(I)}\frac{(1-|z|^2)^{p-1}}{K((1-|z|)/|I|)}\,dA(z)\right)^{1/2}\\ &\lesssim\|\mu\|_K^{1/2}\left(\int_{S(I)}|I|^{p-c}(1-|z|^2)^{c-1}\,dA(z)\right)^{1/2}\lesssim\|\mu\|_K^{1/2}|I|^{(p+1)/2}.\end{aligned}$$

The above estimates give the desired result. □

**Lemma 2.3.** *Let $K$ satisfy the conditions (1.3) and (1.4). Let $b+\alpha\ge 1+p, b\ge\max\{p,(1+p)/2\}$ and $\alpha>1/2$. Let $\psi$ be measurable on $\mathbf{D}$ and define an operator on $L^2(\mathbf{D})$ as:*

$$T\psi(z)=\int_{\mathbf{D}}\frac{(1-|w|^2)^{b-1}}{|1-\overline{w}z|^{b+\alpha}}|\psi(w)|\,dA(w).$$

*If $d\mu(z)=|\psi(z)|^2dA(z)$ is a $K$-Carleson measure, then $|T\psi(z)|^2(1-|z|^2)^{2(\alpha-1)}dA(z)$ is a $K$-Carleson measure.*

*Proof.* For the Carleson box $S(I)$, we have

$$\begin{aligned}&\int_{S(I)}|T\psi(z)|^2(1-|z|^2)^{2(\alpha-1)}K\left(\frac{1-|z|}{|I|}\right)dA(z)\\ &\le\int_{S(I)}K\left(\frac{1-|z|}{|I|}\right)(1-|z|^2)^{2(\alpha-1)}\\ &\quad\cdot\left\{\left(\int_{S(2I)}+\int_{\mathbf{D}\setminus S(2I)}\right)|\psi(w)|\frac{(1-|w|^2)^{b-1}}{|1-\overline{w}z|^{b+\alpha}}\,dA(w)\right\}^2dA(z)\lesssim E_1+E_2,\end{aligned}$$

where

$$E_1 = \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)(1-|z|^2)^{2(\alpha-1)}\left(\int_{S(2I)}|\psi(w)|\frac{(1-|w|^2)^{b-1}}{|1-\overline{w}z|^{b+\alpha}}\,dA(w)\right)^2 dA(z)$$

and

$$E_2 = \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)(1-|z|^2)^{2(\alpha-1)}\left(\int_{\mathbf{D}\setminus S(2I)}|\psi(w)|\frac{(1-|w|^2)^{b-1}}{|1-\overline{w}z|^{b+\alpha}}\,dA(w)\right)^2 dA(z).$$

To estimate $E_1$, consider

$$B(z,w) = \sqrt{\frac{K\left((1-|z|)/|I|\right)}{K\left((1-|w|)/|I|\right)}}\frac{(1-|w|^2)^{b-1}(1-|z|^2)^{\alpha-1}}{|1-\overline{w}z|^{b+\alpha}}$$

and the integral operator on $L^2(\mathbf{D})$

$$T_B\psi(z) = \int_{\mathbf{D}} B(z,w)|\psi(w)|\,dA(w).$$

Choose $\beta$ as in Lemma 2.1 such that $\beta < b$ and $\alpha+\beta > 1$. In fact, if $p \geq 1$, we choose $\beta \in (1/2, \min\{\alpha, 1\})$; if $0 < p < 1$, we choose $\beta \in ((1+p-c)/2, (1+p)/2)$, where $c$ is given as in Lemma 2.1.

Define

$$h(z) = \frac{\left(K\left(\frac{1-|z|}{|I|}\right)\right)^{1/2}}{(1-|z|^2)^{\beta}}.$$

By Lemma 4.2.2 in [15] and Lemma 2.1, we have

$$\int_{\mathbf{D}} B(z,w)h(w)\,dA(w) \lesssim h(z)$$

and

$$\int_{\mathbf{D}} B(z,w)h(z)\,dA(z) \lesssim h(w).$$

By Schur's Theorem (cf. [15]) we have

$$\int_{\mathbf{D}} |T_Bg(w)|^2\,dA(w) \lesssim \int_{\mathbf{D}} |g(w)|^2\,dA(w)$$

for all $g \in L^2(\mathbf{D})$. Thus the operator $T_B$ is bounded on $L^2(\mathbf{D})$ by Corollary 3.2.3 in [15]. Consider the function

$$g(w) = |\psi(w)|\left(K\left(\frac{1-|w|}{|I|}\right)\right)^{1/2}\chi_{S(2I)}(w),$$

where $\chi_{S(2I)}(w) = 1$ for $w \in S(2I)$ and 0 for $w \notin S(2I)$. We have

$$E_1 \lesssim \int_{\mathbf{D}} |T_Bg(w)|^2\,dA(w) \lesssim \int_{\mathbf{D}} |g(w)|^2\,dA(w) \leq \|\mu\|_K. \tag{2.2}$$

Next we estimate $E_2$. Since $|\psi(w)|^2\,dA(w)$ is a $K$-Carleson measure, by Lemma 2.2, $dv(w) = |\psi(w)|(1-|w|^2)^{(p-1)/2}\,dA(w)$ is $(p+1)/2$-Carleson measure. This deduces

$$\begin{aligned}
E_2 &= \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)(1-|z|^2)^{2(\alpha-1)} \\
&\quad \cdot \left(\sum_{n=1}^{\infty}\int_{S(2^{n+1}I)\setminus S(2^nI)} \frac{|\psi(w)|(1-|w|^2)^{b-1}}{|1-\overline{w}z|^{b+\alpha}}\,dA(w)\right)^2 dA(z) \\
&= \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)(1-|z|^2)^{2(\alpha-1)} \\
&\quad \cdot \left(\sum_{n=1}^{\infty}\int_{S(2^{n+1}I)\setminus S(2^nI)} \frac{dv(w)}{|1-\overline{w}z|^{\alpha+(p+1)/2}}\right)^2 dA(z) \\
&\lesssim \|v\|^2_{(p+1)/2}\int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)(1-|z|^2)^{2(\alpha-1)}\left(\sum_{n=1}^{\infty}\frac{(2^{n+1}|I|)^{(p+1)/2}}{(2^n|I|)^{\alpha+(p+1)/2}}\right)^2 dA(z) \\
&\lesssim \|v\|^2_{(p+1)/2}\frac{1}{|I|^{2\alpha}}\int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)(1-|z|^2)^{2(\alpha-1)}\,dA(z) \\
&\lesssim \|v\|^2_{(p+1)/2}\frac{1}{|I|^{2\alpha}}\int_{S(I)}(1-|z|^2)^{2(\alpha-1)}\,dA(z) \lesssim \|v\|^2_{(p+1)/2}.
\end{aligned}$$

Here we use the following estimate:

$$\frac{1}{|1-\overline{w}z|} \lesssim \frac{1}{2^n|I|}, \quad w \in S(2^{n+1}I)\setminus S(2^nI). \tag{2.3}$$

Combining our estimates for $E_1$ and $E_2$, we have

$$\int_{S(I)} |T\psi(z)|^2(1-|z|^2)^{2(\alpha-1)}K\left(\frac{1-|z|}{|I|}\right)dA(z) \lesssim \|\mu\|_K + \|v\|^2_{(p+1)/2}$$

for any $I \subset \partial\mathbf{D}$. By Theorem 3.1 in [4] we obtain that $|T\psi(z)|^2(1-|z|^2)^{2(\alpha-1)}\,dA(z)$ is a $K$-Carleson measure. The proof is complete. □

*Proof of Theorem 2.1.* Now we apply Theorem A to prove Theorem 2.1. Suppose $f \in Q_K$, then $|f'(z)|^2\,dA(z)$ is a $K$-Carleson measure. For $\alpha > 1/2$, the $\alpha$-order derivative of $f$ at $z \in \mathbf{D}$ is

$$f^{(\alpha)}(z) = \frac{\Gamma(b+\alpha)}{\Gamma(b)}\int_{\mathbf{D}} \frac{(1-|w|^2)^{b-1}}{(1-\overline{w}z)^{b+\alpha}}\overline{w}^{[\alpha-1]}f'(w)\,dA(w),$$

where $b > 1$ and $b+\alpha \geq 1+p,\ b \geq \max\{p,(1+p)/2\}$. By Lemma 2.3 we obtain that $|f^{(\alpha)}(z)|^2(1-|z|^2)^{2(\alpha-1)}\,dA(z)$ is a $K$-Carleson measure.

Conversely, assume that $|f^{(\alpha)}(z)|^2(1-|z|^2)^{2(\alpha-1)}\,dA(z)$ is a $K$-Carleson measure. We consider the Taylor series of $f$: $f(z) = \sum_{j=0}^{\infty} a_j z^j$. Note that

$$\frac{1}{(1-\overline{w}z)^{\lambda}} = \sum_{j=0}^{\infty}\frac{\Gamma(j+\lambda)}{j!\Gamma(\lambda)}\overline{w}^j z^j, \quad \lambda > 0.$$

Hence

$$
\begin{aligned}
f^{(\alpha)}(z) &= \frac{\Gamma(b+\alpha)}{\Gamma(b)} \int_{\mathbf{D}} \frac{\overline{w}^{[\alpha-1]}(1-|w|^2)^{b-1}}{(1-\overline{w}z)^{b+\alpha}} f'(w)\, dA(w) \\
&= \frac{\Gamma(b+\alpha)}{\Gamma(b)} \int_{\mathbf{D}} \frac{\overline{w}^{[\alpha-1]}(1-|w|^2)^{b-1}}{(1-\overline{w}z)^{b+\alpha}} \sum_{j=1}^{\infty} a_j (z^j)'\, dA(w) \\
&= \frac{\Gamma(b+\alpha)}{\Gamma(b)} \sum_{j=1}^{\infty} a_j \int_{\mathbf{D}} \frac{\overline{w}^{[\alpha-1]}(1-|w|^2)^{b-1}}{(1-\overline{w}z)^{b+\alpha}} (z^j)'\, dA(w) = \sum_{j=0}^{\infty} a_{j,\alpha} z^j,
\end{aligned}
$$

where

$$
\text{(2.5)} \qquad a_{j,\alpha} = a_{j+m+1} \left( \frac{\Gamma(b+j+\alpha)\Gamma(j+m+2)}{\Gamma(b+j+m+1)\Gamma(j+1)} \right), \quad j = 0, 1, \cdots, m = [\alpha - 1].
$$

Since $\alpha > 1/2$, $m \geq 0$, a simple computation gives the following equality

$$
\begin{aligned}
&\frac{\Gamma(b+m+1)}{\Gamma(b+\alpha-1)} \int_{\mathbf{D}} \frac{(1-|w|^2)^{b-1}}{(1-\overline{w}z)^{b+m+1}} f^{(\alpha)}(w)(1-|w|^2)^{\alpha-1}\, dA(w) \\
&= \frac{\Gamma(b+m+1)}{\Gamma(b+\alpha-1)} \int_{\mathbf{D}} \frac{(1-|w|^2)^{b+\alpha-2}}{(1-\overline{w}z)^{b+m+1}} \left( \sum_{j=0}^{\infty} a_{j,\alpha} w^j \right) dA(w) \\
&= \frac{\Gamma(b+m+1)}{\Gamma(b+\alpha-1)} \sum_{j=0}^{\infty} a_{j,\alpha} \int_{\mathbf{D}} (1-|w|^2)^{b+\alpha-2} \left( \sum_{k=0}^{\infty} \frac{\Gamma(k+b+m+1)}{k!\Gamma(b+m+1)} \overline{w}^k z^k \right) w^j\, dA(w) \\
&= \sum_{j=0}^{\infty} \frac{\Gamma(j+m+2)}{\Gamma(j+1)} a_{j+m+1} z^j = f^{(m+1)}(z).
\end{aligned}
$$

Since $|f^{(\alpha)}(w)|^2(1-|w|^2)^{2(\alpha-1)}dA(w)$ is a $K$-Carleson measure, Lemma 2.3 implies that $|f^{(m+1)}(z)|^2(1-|z|^2)^{2m}dA(z)$ is a $K$-Carleson measure. Hence $f \in Q_K$ by Theorem A. □

## 3. Morrey type spaces and $Q_K$ spaces

Denote $H^2_K$ the Morrey type space of all analytic functions $f \in H^2$ on $\mathbf{D}$ such that

$$
\text{(3.1)} \qquad \|f\|_{H^2_K} = \left( \sup_{I \subset \partial \mathbf{D}} \frac{1}{K(|I|)} \int_I |f(\zeta) - f_I|^2\, |d\zeta| \right)^{1/2} < \infty,
$$

where

$$
f_I = \frac{1}{|I|} \int_I f(\zeta)\, d\zeta
$$

and the Hardy space $H^2$ consists of analytic functions $f$ in $\mathbf{D}$ satisfying

$$
\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta < \infty.
$$

See [8] and [16] about the Morrey space.

The Poisson extension of a function $f \in L^1(\partial \mathbf{D})$ from $\partial \mathbf{D}$ to $\mathbf{D}$ is denoted by $\hat{f}$ and defined as follows:

$$
\hat{f}(z) = \frac{1}{2\pi} \int_{\partial \mathbf{D}} f(\zeta) \frac{1-|z|^2}{|\zeta - z|^2}\, |d\zeta|, \quad z \in \mathbf{D}.
$$

If $f \in H_K^2$, then $\hat{f}(z) = f(z)$ for $z \in \mathbf{D}$. Wu and Xie in [10] characterized functions in the Morrey space in terms of $p$-Carleson measures. Furthermore, they reveal a simple relation between $Q_p$ space and Morrey space. In this section, we will give a series of characterizations of the Morrey type space $H_K^2$ and build a relationship between the spaces $H_K^2$ and $Q_K$.

**Theorem 3.1.** *Let $K$ satisfy the conditions (1.3) and (1.4). Then the following are equivalent.*

(1) $f \in H_K^2$.

(2) $\sup_{I \subset \partial \mathbf{D}} \dfrac{1}{K(|I|)} \displaystyle\int_{S(I)} |f'(z)|^2 (1-|z|^2)\, dA(z) < \infty$.

(3) $\sup_{a \in \mathbf{D}} \dfrac{1-|a|^2}{K(1-|a|)} \displaystyle\int_{\mathbf{D}} |f'(z)|^2 (1-|\varphi_a(z)|^2) dA(z) < \infty$.

(4) $\sup_{a \in \mathbf{D}} \dfrac{1-|a|^2}{K(1-|a|)} \displaystyle\int_{\mathbf{D}} |f'(z)|^2 g(z,a)\, dA(z) < \infty$.

(5) $\sup_{a \in \mathbf{D}} \dfrac{1-|a|^2}{K(1-|a|)} |\widehat{f - f(a)}|^2(a) < \infty$.

(6) $\sup_{a \in \mathbf{D}} \dfrac{1-|a|^2}{K(1-|a|)} \left( \widehat{|f|^2}(a) - |f(a)|^2 \right) < \infty$.

To prove Theorem 3.1, we need the following Lemmas. The first lemma, Lemma D, was proved in [11] but here we state it again.

**Lemma 3.1.** *If $K$ satisfies the condition (1.4), then there exists a weight $K_1$, comparable with $K$, such that $K_1(t)/t^p$ is non-increasing. Moreover, for some enough small $c > 0$, $K_1(t)/t^{p-c}$ is also non-increasing.*

*Proof.* If $K$ satisfies the condition (1.4), we will claim that

$$\liminf_{t \to 0} K(t)/t^p > 0. \tag{3.2}$$

If $s > 1$, then

$$K(1)/K(1/s) \le \varphi_K(s)$$

and by (1.4)

$$\int_1^\infty K(1/s)^{-1} \frac{ds}{s^{1+p}} = \int_0^1 K(s)^{-1} \frac{ds}{s^{1-p}} < \infty.$$

So, we have

$$t^p/K(t) \lesssim K(t)^{-1} \int_0^t \frac{ds}{s^{1-p}} \le \int_0^1 K(s)^{-1} \frac{ds}{s^{1-p}} < \infty.$$

Then we obtain the claim.

We define

$$K_1(t) = t^p \int_t^\infty \frac{K(s)}{s^{1+p}}\, ds, \quad 0 < t < \infty.$$

It is easy to see that $K_1(t)/t^p$ is non-increasing. Since $K$ is nondecreasing, it follows that $K_1(t) \ge K(t), 0 < t < \infty$. We note that for $t \in (0,1)$,

$$\int_t^1 \frac{K(s)}{s^{1+p}}\, ds \le K(t) \int_t^1 \frac{\varphi_K(s/t)}{s^{1+p}}\, ds \le \frac{K(t)}{t^p} \int_1^\infty \frac{\varphi_K(u)}{u^{1+p}}\, du,$$

and by (3.2),

$$\int_1^\infty \frac{K(s)}{s^{1+p}}\,ds = K(1) \lesssim \frac{K(t)}{t^p}.$$

Hence, we obtain that

$$K_1(t) \le K(t)\left(\int_1^\infty \frac{\varphi_K(s)}{s^{1+p}}\,ds + 1\right), \quad 0<t<1. \tag{3.3}$$

For $t \in [1,\infty)$, we have

$$K_1(t) = t^p \int_1^\infty \frac{\varphi_K(s)}{s^{1+p}}\,ds = K(1) = K(t) \tag{3.4}$$

By (3.3) and (3.4) we get that $K_1 \approx K$.

Note that if $c$ is sufficiently small, then we have

$$(t^{c-p}K_1(t))' = t^{c-1-p}(cK_1(t) - K(t)) < 0, \quad 0<t<\infty. \qquad \square$$

**Lemma 3.2.** *Let $K$ satisfy the condition (1.4). Then*

$$K(rt) \le t^p K(r), \quad 0 \le r \le 1, 1 \le t < \infty.$$

*Proof.* By Lemma 3.1 we know that $t^{-p}K(t)$ is non-increasing. Thus

$$K(rt) = t^p \frac{(rt)^{-p}K(rt)K(r)}{r^{-p}K(r)} \le t^p K(r),$$

and we get the desired result. $\square$

*Proof of Theorem 3.1.* We first show (1) $\Leftrightarrow$ (5). If (1) holds, without loss of generality, we assume that $|a| > 3/4$. Let $I_a$ be the subarc of $\partial\mathbf{D}$ with the mid-pointer $a/|a|$ and length $1-|a|$. Moreover, let $J_n = 2^n I_a$ for $n = 0,1,\ldots,N-1$, where $N$ is the smallest positive integer such that $2^N|I_a| \ge 1$. Let $J_N$ be the unit circle. Then we have the following estimate:

$$\frac{1-|a|^2}{|1-\overline{a}\zeta|^2} \approx \frac{1}{|I_a|}, \quad \zeta \in I_a \tag{3.5}$$

and

$$\frac{1-|a|^2}{|1-\overline{a}\zeta|^2} \approx \frac{1}{2^{2n}|I_a|}, \quad \zeta \in J_{n+1}\setminus J_n,\ n = 0,1,2,\cdots,N-1. \tag{3.6}$$

For a fixed point $a \in \mathbf{D}$ with $|a| > 3/4$, we obtain the following estimate.

$$\begin{aligned}
|\widehat{f - f(a)}|^2(a) &= \frac{1}{2\pi}\int_{\partial\mathbf{D}} |f(\zeta)-f(a)|^2 \frac{1-|a|^2}{|\zeta-a|^2}\,|d\zeta| \\
&= \frac{1}{2\pi}\int_{\partial\mathbf{D}} |(f(\zeta)-f_{I_a}) - (f(a)-f_{I_a})|^2 \frac{1-|a|^2}{|1-\overline{a}\zeta|^2}\,|d\zeta| \\
&\lesssim \int_{\partial\mathbf{D}} |f(\zeta)-f_{I_a}|^2 \frac{1-|a|^2}{|1-\overline{a}\zeta|^2}\,|d\zeta| \\
&\lesssim \left(\int_{J_0} + \sum_{n=0}^{N-1}\int_{J_{n+1}\setminus J_n}\right) |f(\zeta)-f_{I_a}|^2 \frac{1-|a|^2}{|1-\overline{a}\zeta|^2}\,|d\zeta|
\end{aligned}$$

$$\lesssim \frac{1}{|I_a|}\left(\int_{J_0}+\sum_{n=0}^{N-1}2^{-2n}\int_{J_{n+1}\setminus J_n}\right)|f(\zeta)-f_{I_a}|^2\,|d\zeta|$$
$$\lesssim \frac{1}{|I_a|}\int_{J_0}|f(\zeta)-f_{J_0}|^2\,|d\zeta|+\sum_{n=0}^{N-1}\frac{2^{-n}}{|J_{n+1}|}\int_{J_{n+1}\setminus J_n}|f(\zeta)-f_{J_0}|^2\,|d\zeta|.$$

By the Cauchy–Schwarz inequality and Lemma 3.2,

$$\begin{aligned}|f_{J_{n+1}}-f_{J_n}| &\le \frac{2}{|J_{n+1}|}\int_{J_{n+1}}|f(\zeta)-f_{J_{n+1}}|\,|d\zeta|\\ &\le \left(\frac{2}{|J_{n+1}|}\int_{J_{n+1}}|f(\zeta)-f_{J_{n+1}}|^2\,|d\zeta|\right)^{1/2}\\ &\le \sqrt{\frac{K(|J_{n+1}|)}{|J_{n+1}|}}\,\|f\|_{H^2_K}\lesssim 2^{n(p-1)/2}\sqrt{\frac{K(1-|a|)}{1-|a|^2}}\,\|f\|_{H^2_K}.\end{aligned}$$

Therefore

$$|f_{J_{n+1}}-f_{J_0}|\le |f_{J_{n+1}}-f_{J_n}|+\cdots+|f_{J_1}-f_{J_0}|\lesssim C(n,p)\sqrt{\frac{K(1-|a|)}{1-|a|^2}}\|f\|_{H^2_K},$$

where $C(n,p)=(1-2^{(p-1)/2})^{-1}$ for $0<p<1$, $C(n,p)=n$ for $p=1$ and $C(n,p)=2^{n(p-1)/2}$ for $1<p<2$. On the other hand, the Minkowski inequality gives

$$\begin{aligned}&\frac{1}{|J_{n+1}|}\int_{J_{n+1}}|f(\zeta)-f_{J_0}|^2\,|d\zeta|\\ &\le\left(\left(\frac{1}{|J_{n+1}|}\int_{J_{n+1}}|f(\zeta)-f_{J_{n+1}}|\,|d\zeta|\right)^{1/2}+|f_{J_n+1}-f_{J_0}|\right)^2\\ &\lesssim (C(n,p))^2\frac{K(1-|a|)}{1-|a|^2}\,\|f\|^2_{H^2_K}.\end{aligned}$$

Since $\sum_{n=0}^{\infty}\frac{(C(n,p))^2}{2^n}$ is convergent, the above estimates show that

$$|\widehat{f-f(a)}|^2(a)\lesssim\frac{K(1-|a|)}{1-|a|^2}\,\|f\|^2_{H^2_K}\sum_{n=0}^{\infty}\frac{(C(n,p))^2}{2^n}\lesssim\frac{K(1-|a|)}{1-|a|^2}\,\|f\|^2_{H^2_K}.$$

Hence, (1) ⇒ (5) holds.

Let (5) hold. For any given $I\subset\partial\mathbf{D}$, we choose $a_I\in\mathbf{D}$ such that $a_I/|a_I|$ is the center of $I$ and $|a_I|=1-|I|$. Then

$$\begin{aligned}\frac{1}{K(|I|)}\int_I|f(\zeta)-f_I|^2\,|d\zeta| &= \frac{1}{K(|I|)}\int_I|(f(\zeta)-f(a_I))-(f_I-f(a_I))|^2\,|d\zeta|\\ &\le\frac{4}{K(|I|)}\int_I|f(\zeta)-f(a_I)|^2\,|d\zeta|\\ &\lesssim\frac{1-|a_I|^2}{K(1-|a_I|)}\int_I|f(\zeta)-f(a_I)|^2\frac{1-|a_I|^2}{|1-\bar{\zeta}a_I|^2}\,|d\zeta|\\ &\lesssim\frac{1-|a_I|^2}{K(1-|a_I|)}|\widehat{f-f(a_I)}|^2(a_I).\end{aligned}$$

The above estimate shows that (5) $\Rightarrow$ (1) holds.

Now we will prove that (2) $\Leftrightarrow$ (3). For given $I \subset \partial\mathbf{D}$, let $a_I/|a_I|$ be the midpoint of $I$ and $1-|a_I| = |I|$. Note that

$$|1-\overline{a_I}z| \approx |I|, \quad z \in S(I).$$

Then

$$\begin{aligned}
&\frac{1}{K(|I|)}\int_{S(I)} |f'(z)|^2(1-|z|^2)\,dA(z)\\
&\approx \frac{1-|a_I|^2}{K(1-|a_I|)}\int_{S(I)} |f'(z)|^2(1-|\varphi_{a_I}(z)|^2)\,dA(z)\\
&\le \sup_{a_I\in\mathbf{D}} \frac{1-|a_I|^2}{K(1-|a_I|)}\int_{\mathbf{D}} |f'(z)|^2(1-|\varphi_{a_I}(z)|^2)\,dA(z),
\end{aligned}$$

which shows that (3) $\Rightarrow$ (2).

Conversely, suppose (2) holds. There exists a constant $M$ such that

$$\mu(S(I)) = \int_{S(I)} d\mu(z) \le MK(|I|)$$

for any $I \subset \partial\mathbf{D}$, where $d\mu(z) = |f'(z)|^2(1-|z|^2)\,dA(z)$. For any given nonzero $a \in \mathbf{D}$, let $I_a$ be the subarc of $\partial\mathbf{D}$ with the mid-pointer $a/|a|$ and length $1-|a|$. By (3.5), (3.6) and Lemma 3.2 we have

$$\begin{aligned}
(1-|a|^2)\int_{\mathbf{D}} \frac{1-|a|^2}{|1-\bar{a}z|^2}\,d\mu(z) &\le \sum_{n=1}^{\infty}\frac{1}{2^{2n}}\int_{S(2^{n+1}I_a)\setminus S(2^nI_a)} d\mu(z)\\
&\le \sum_{n=1}^{\infty}\frac{1}{2^{2n}}\mu(S(2^{n+1}I_a)) \le M\sum_{n=1}^{\infty}\frac{1}{2^{2n}}K(2^{n+1}|I_a|)\\
&\lesssim \sum_{n=1}^{\infty}\frac{1}{2^{(2-p)n}}K(1-|a|) \approx K(1-|a|).
\end{aligned}$$

Taking the supremum over $a \in \mathbf{D}$, we have that (2) $\Rightarrow$ (3).

By the Littlewood–Paley identity ([6], p. 236)

$$\int_{\mathbf{D}} |f'(z)|^2(1-|z|^2)\,dA(z) \approx \int_{\mathbf{D}} |f'(z)|^2\log\frac{1}{|z|^2}\,dA(z) \approx \int_{\partial\mathbf{D}} |f(\zeta)-f(0)|^2\,|d\zeta|,$$

we can figure out (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). The proof is complete. □

We conclude this paper by proving a connection between $Q_K$ and $H^2_K$ spaces.

**Theorem 3.2.** *Let $K$ satisfy the conditions (1.3) and (1.4).*
*(1) If $f \in Q_K$, there exists a $q$, $0<q<p$, such that $f^{(\frac{1-q}{2})} \in H^2_K$.*
*(2) If $f \in H^2_K$, there exists a $q$, $0<q<p$, such that $f^{(\frac{q-1}{2})} \in Q_K$.*

*Proof.* We note that $(f^{(\alpha)})' = f^{(\alpha+1)}$. In fact, we consider the Taylor series of $f$: $f(z) = \sum_{j=0}^{\infty} a_j z^j$. Then $f^{(\alpha)}(z) = \sum_{j=0}^{\infty} a_{j,\alpha} z^j$, where $a_{j,\alpha}$ is defined as in (2.5).

Since $[\alpha-1]+1=[\alpha]$, we have

$$
\begin{aligned}
(f^{(\alpha)})'(z) &= \sum_{j=0}^{\infty} a_{j+1,\alpha}(j+1)z^j \\
&= \sum_{j=0}^{\infty} a_{j+[\alpha-1]+2}(j+1)\left(\frac{\Gamma(b+j+1+\alpha)\Gamma(j+[\alpha-1]+3)}{\Gamma(b+j+[\alpha-1]+2)\Gamma(j+2)}\right) z^j \\
&= \sum_{j=0}^{\infty} a_{j+[\alpha]+1}\left(\frac{\Gamma(b+j+1+\alpha)\Gamma(j+[\alpha]+2)}{\Gamma(b+j+[\alpha]+1)\Gamma(j+1)}\right) z^j = f^{(\alpha+1)}(z).
\end{aligned}
$$

We now prove (1). If $f \in Q_K$, then $d\mu = |f^{(\frac{3-q}{2})}(z)|^2(1-|z|^2)^{1-q}\, dA(z)$ is $K-$Carleson measure. For any given $I \subset \partial\mathbf{D}$, We have

$$
\begin{aligned}
&\frac{1}{K(|I|)}\int_{S(I)} |f^{(\frac{3-q}{2})}(z)|^2(1-|z|^2)\, dA(z) \\
&\lesssim \frac{|I|^q}{K(|I|)}\int_{S(I)} |f^{(\frac{3-q}{2})}(z)|^2(1-|z|^2)^{1-q}K\left(\frac{1-|z|}{|I|}\right) dA(z) \\
&\le \frac{1}{K(1)}\sup_{I\subset\partial\mathbf{D}}\int_{S(I)} |f^{(\frac{3-q}{2})}(z)|^2(1-|z|^2)^{1-q}K\left(\frac{1-|z|}{|I|}\right) dA(z) \lesssim \|\mu\|_K^2.
\end{aligned}
$$

Here we used Lemma 3.1, which shows that there exists a $q$, $0<q<p$, such that $K(t)/t^q$ is non-increasing. Thus, we obtain that $f^{(\frac{1-q}{2})} \in H_K^2$ by Theorem 3.1; that is, (1) holds.

By Lemma 2.1 in [4], there exists a $q$, $0<q<p$, such that $K(t)/t^q$ is nondecreasing. For any $I \subset \partial\mathbf{D}$, we have

$$
\begin{aligned}
\int_{S(I)} |f'(z)|^2(1-|z|^2)^{1-q}K\left(\frac{1-|z|}{|I|}\right) dA(z) &\lesssim \frac{1}{|I|^q}\int_{S(I)} |f'(z)|^2(1-|z|^2)\, dA(z) \\
&\lesssim \frac{1}{K(|I|)}\int_{S(I)} |f'(z)|^2(1-|z|^2)\, dA(z)
\end{aligned}
$$

Suppose $f \in H_K^2$. By Theorem 3.1 we obtain that $|f'(z)|^2(1-|z|^2)^{1-q}$ is a $K$-Carleson measure. We note that

$$
f^{(\frac{q+1}{2})}(z) = \frac{\Gamma(b+\frac{q+1}{2})}{\Gamma(b)}\int_{\mathbf{D}} \frac{(1-|w|^2)^{b+\frac{q-3}{2}}}{(1-\overline{w}z)^{b+\frac{q+1}{2}}}\, \overline{w}^{[\frac{q-1}{2}]}(1-|w|^2)^{1-\frac{q+1}{2}} f'(w)\, dA(w).
$$

Lemma 2.3 implies that $|f^{(\frac{q+1}{2})}(z)|^2\, dA(z)$ is a $K$-Carleson measure. Since $f^{(\frac{q+1}{2})}(z) = (f^{(\frac{q-1}{2})})'(z)$, we have $f^{(\frac{q-1}{2})} \in Q_K$ by Theorem 2.1. Now (2) follows. □

**Remark.** Carefully checking the proof of Theorem 3.2, we find that we need a non-increasing function $K(t)/t^{q_1}$ for $q_1 \in (0,p)$ in the proof of (1) and a nondecreasing function $K(t)/t^{q_2}$ in the proof of (2) for $q_2 \in (0,p)$. Generally, $q_1 \neq q_2$ unless $K(t)=t^q$. In this case $Q_K$ coincides with $Q_q$. Therefore, we have the following result about $Q_q$ which appeared in [10].

**Corollary 3.1.** *Let $0 < q < 2$ and $f \in H^2$. Then $f \in Q_q$ if and only if $f^{(\frac{1-q}{2})} \in H^{2,q}$, where*

$$H^{2,q} = \left\{ f \in H^2 \colon \sup_{I \subset \partial \mathbf{D}} \frac{1}{|I|^q} \int_I |f(\zeta) - f_I|^2 \, |d\zeta| < \infty \right\}.$$

*Consequently, $f \in H^{2,q}$ if and only if $f^{(\frac{q-1}{2})} \in Q_q$.*

## 4. Final remark

The space $Q_{K,0}$ consists of analytic functions $f$ in $\mathbf{D}$ with the property that

$$\lim_{|a| \to 1^-} \int_{\mathbf{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) \, dA(z) = 0.$$

It can be checked that $Q_{K,0}$ is a closed subspace in $Q_K$.

A positive Borel measure $\mu$ on $\mathbf{D}$ is called a vanishing $K$-Carleson measure if

$$\lim_{|I| \to 0} \int_{S(I)} K\left( \frac{1 - |z|}{|I|} \right) d\mu(z) = 0.$$

Carefully checking the proofs of Theorem 2.1 and several lemmas in Section 2 and Theorem 3.1 in [4], we see that the little oh version of Theorem 2.1 holds as well, from which we obtain the following.

**Theorem 4.1.** *Let $K$ satisfy the conditions (1.3) and (1.4). If $\alpha > 1/2$, then $f \in Q_{K,0}$ if and only if $|f^{(\alpha)}(z)|^2 (1 - |z|^2)^{2(\alpha - 1)} \, dA(z)$ is a vanishing $K$-Carleson measure.*

We are also able to give the little oh versions of Theorems 3.1 and 3.2. Here we omit the details about them.

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