Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 38, 2013, 181–191

ON A ONE-PHASE FREE BOUNDARY PROBLEM

Benny Avelin

Uppsala University, Department of Mathematics
S-751 06 Uppsala, Sweden; benny.avelin@math.uu.se

Abstract. In this paper we extend a result regarding the free boundary regularity in a one-phase problem, by De Silva and Jerison [DJ], to non-divergence linear equations of second order. Roughly speaking we prove that the free boundary is given by a Lipschitz graph.

1. Introduction

Recently De Silva and Jerison [DJ] studied the following one-phase free boundary problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega^+(u) := \{x \in \Omega\colon u(x) > 0\}, \\ |\nabla u| = 1, & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega, \end{cases}$$

where $\Omega \subset \mathbf{R}^n$ is a cylinder. Specifically they proved that if we assume that $\Omega^+(u)$ is an NTA domain and that the free boundary $F(u)$ is a graph in the e_n-direction, then $F(u)$ is given by a Lipschitz graph. They prove this by comparing vertical (e_n-direction) translates of the solution. In essence they prove that the change in u in the vertical direction is comparable to the change in u in the direction, normal to each level surface. This is equivalent with level surfaces being Lipschitz, with uniform bound. In this paper we extend this result to non-divergence equations, with the matrix $\{a_{ij}\}$ independent of the e_n-direction. This allows us to compare solutions with their vertical translates. Specifically we consider the following problem

$$\begin{cases} Lu = 0, & \text{in } \Omega^+(u), \\ |\nabla u| = 1, & \text{on } F(u), \end{cases} \tag{1.1}$$

for operators L of the form

$$Lu = \sum_{i,j=1}^n a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0, \tag{1.2}$$

where the matrix a_{ij} is uniformly elliptic, i.e.

$$\lambda|\xi|^2 \le \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \le \Lambda|\xi|^2, \tag{1.3}$$

and that $\frac{\partial a_{ij}}{\partial x_n} = 0$, $a_{ij} \in C^{0,1}(\mathbf{R}^n)$. We will be concerned with the question wether the free boundary is Lipschitz, assuming that $\Omega^+(u)$ is an NTA domain and $F(u)$ a graph in the e_n, direction. To properly state our results we need to introduce some notation. Points in Euclidean n-space $\mathbf{R}^n$ are denoted by $x = (x_1, \dots, x_n)$ or (x', x_n)

doi:10.5186/aasfm.2013.3815
2010 Mathematics Subject Classification: Primary 35J25.
Key words: One-phase, free boundary, NTA, non-divergence, linear.

where $x' = (x_1, \dots, x_{n-1}) \in \mathbf{R}^{n-1}$. Let $\bar{E}$, ∂E, $\operatorname{diam} E$ be the closure, boundary, and diameter of E. Let $\cdot$ denote the standard inner product on $\mathbf{R}^n$, $|x| = (x \cdot x)^{1/2}$, the Euclidean norm of x, and let dx be Lebesgue n-measure on $\mathbf{R}^n$. Given $x \in \mathbf{R}^n$ and $r > 0$, let $B_r(x) = \{y \in \mathbf{R}^n \colon |x - y| < r\}$, $B'_r(x) = \{y \in \mathbf{R}^{n-1} \colon |x - y| < r\}$, for short we write $B_r := B_r(0)$, and $B'_r := B'_r(0)$. Given $E, F \subset \mathbf{R}^n$, let $d(E, F)$ be the Euclidean distance from E to F. In case $E = \{y\}$, we write $d(y, F)$.

If $O \subset \mathbf{R}^n$ is open and $1 \le q \le \infty$, then by $W^{1,q}(O)$, we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_n})$, both of which are q-th power integrable on O. Let

$$\|f\|_{W^{1,q}(O)} = \|f\|_{L^q(O)} + \||\nabla f|\|_{L^q(O)},$$

be the norm in $W^{1,q}(O)$ where $\| \cdot \|_{L^q(O)}$ denotes the usual Lebesgue q-norm in O. Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O. By $\nabla\cdot$ we denote the divergence operator. Finally, given $n \ge 1$ we let H^k, for $k \in \{1, \dots, n\}$, denote the k-dimensional Hausdorff measure on $\mathbf{R}^n$.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and let $\Delta(w, r) = \partial\Omega \cap B(w, r)$ whenever $w \in \partial\Omega$, $0 < r$. Given Ω we will in the following always let σ denote the restriction of the $(n-1)$-dimensional Hausdorff measure to $\partial\Omega$.

Given a matrix $A = \{a_{ij}\} \colon \mathbf{R}^n \to \mathbf{R}^n \times \mathbf{R}^n$ and a domain $\Omega \in \mathbf{R}^n$, we define ω_∇^x as the elliptic measure at x relative Ω for the operator $\nabla \cdot A\nabla\cdot$, and ω_L^x as the elliptic measure at x relative Ω for the operator $L\cdot$, also denoted as the L-elliptic measure.

Let

$$\mathcal{C}(R, K) := \{|x'| < R\} \times (-K, K) \subset \mathbf{R}^n, \quad \mathcal{C}_K := \mathcal{C}(1, K). \tag{1.4}$$

Definition 1.1. Let $\Omega \subset \mathbf{R}^n$ be a domain, then we say that a function $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$ for some $1 < p < \infty$ is a strong solution to $Lu = f$, if $Lu = f$ holds a.e. in Ω.

Note that if $f \in L^\infty_{\mathrm{loc}}(\Omega)$ then a strong solution lies in $W^{2,p}_{\mathrm{loc}}(\Omega)$ for any $1 < p < \infty$.

As in [DJ], we define solutions to the one-phase problem (1.1), with the modification of strong solutions instead of viscosity solutions.

Definition 1.2. Let u be a non-negative function in $C(\Omega) \cap W^{2,p}_{\mathrm{loc}}(\Omega^+(u))$ for all $1 < p < \infty$. We say that u is a solution to (1.1) in Ω if and only if the following conditions are satisfied:

(1) If u is a strong solution to $Lu = 0$ in $\Omega^+(u)$;
(2) If $x_0 \in F(u)$ and $F(u)$ has at x_0 a tangent ball B_ε from either the positive or the zero side, then for ν the unit radial direction of ∂B_ε at x_0 into $\Omega^+(u)$,

$$u(x) = \langle x - x_0, \nu\rangle^+ + o(|x - x_0|), \quad \text{as } x \to x_0.$$

Definition 1.3. A solution u to (1.1) with $\Omega = B(0, 1)$, is non-degenerate if there is a constant $c > 0$ such that $u(x) > cd(x)$, for all $x \in B(0, 1)^+(u)$.

Lemma 1.4. [GT, Theorem 9.19] *Let $u \in W^{2,p}_{\mathrm{loc}}(\Omega)$, be a solution to $Lu = f$ a.e. in Ω, where the coefficients of L belong to $C^{k-1,1}(\Omega)$, $f \in W^{k,q}_{\mathrm{loc}}(\Omega)$, with $1 < p, q < \infty$, $k \ge 0$. Then $u \in W^{k+2,q}_{\mathrm{loc}}(\Omega)$.*

We remark that in general a strong solution is different from a viscosity solution, however since our coefficients are Lipschitz continuous, we see from the above lemma together with Sobolev embedding, that they are equivalent. See e.g. [C]. In this paper we prove the following generalization of Theorem 1.3 in [DJ].

Theorem 1.5. *Let u be a solution to* (1.1) *as in Definition 1.2, in the cylinder $\mathcal{C}_K$, for some $K > 0$. Suppose that u is monotone in the vertical direction,*

$$\frac{\partial u}{\partial x_n} > 0, \quad \text{on } \overline{\mathcal{C}_K^+(u)},$$

and its free boundary is given as the graph of a continuous function ϕ, i.e. $F(u) = \{(x', x_n) \colon x' \in B_1' \colon x_n = \phi(x')\}$. Suppose that the oscillation of ϕ is bounded,

$$\max_{x' \in B_1'} |\phi(x')| \leq K - 1,$$

and finally, that there is a non-tangentially accessible (NTA) domain D such that

$$\mathcal{C}(9/10, K - 1/2) \cap \mathcal{C}_K^+(u) \subset D \subset \mathcal{C}_K^+(u).$$

Then

$$\sup_{x' \in B_{1/2}'} |\nabla \phi(x')| \leq C,$$

for a constant C depending only on K, λ, Λ, the NTA constants M, r_0 and n.

2. Basic estimates

Definition 2.1. A bounded domain Ω is called non-tangentially accessible (NTA) if there exist $M \geq 1$ and r_0 such that the following are fulfilled:

(i) corkscrew condition: for any $w \in \partial\Omega$, $0 < r < r_0$, there exists $a_r(w) \in \Omega$ satisfying $M^{-1}r < |a_r(w) - w| < r$, $d(a_r(w), \partial\Omega) > M^{-1}r$,

(ii) $\mathbf{R}^n \setminus \Omega$ satisfies the corkscrew condition,

(iii) uniform condition: if $w \in \partial\Omega$, $0 < r < r_0$, and $w_1, w_2 \in B(w, r) \cap \Omega$, then there exists a rectifiable curve $\gamma \colon [0,1] \to \Omega$ with $\gamma(0) = w_1, \gamma(1) = w_2$, and such that
  (a) $H^1(\gamma) \leq M|w_1 - w_2|$,
  (b) $\min\{H^1(\gamma([0,t])), H^1(\gamma([t,1]))\} \leq M d(\gamma(t), \partial\Omega)$.

In this section we consider $\Omega \subset \mathbf{R}^n$ to be an NTA domain with constants M, r_0. We also consider the operator L as in (1.2),(1.3). We need some preliminary lemmas.

Lemma 2.2. *If Ω is an NTA domain, then for any $w \in \partial\Omega$ and $r < r_0$, there exists an NTA domain $D \subset \Omega$ such that*

$$B_{M^{-1}r}(w) \cap \Omega \subset D \subset B_{Mr}(w) \cap \Omega.$$

Furthermore, the constant M in the NTA definition for D is independent of w, r.

Proof. See [J]. □

Lemma 2.3. *Suppose that u, v are positive solutions to $L\hat{u} = 0$ in Ω, where $r < r_0$ and $w \in \partial\Omega$ which vanish continuously on $B_r \cap \partial\Omega$. Then for $r' < r$*

$$c^{-1}\frac{u(a_r(w))}{v(a_r(w))} \leq \frac{u(x)}{v(x)} \leq c\frac{u(a_r(w))}{v(a_r(w))},$$

for all $x \in B_{r'}(w) \cap \Omega$, where $c = c(r - r') \geq 1$.

Proof. See [K]. □

Lemma 2.4. *Let $\Omega \subset \mathbf{R}^n$ satisfy the exterior corkscrew condition, for some M, r_0, and let $a_{ij} \in C(\Omega)$. Let $\hat{L}$ be the operator*

$$\hat{L} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial_i \partial_j} + \sum_{i=1}^{n} b_i \frac{\partial}{\partial_i},$$

with

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \forall x \in \Omega,\ \xi \in \mathbf{R}^n,$$

$$d(x,\partial\Omega) \sum_{i=1}^{n} |b_i(x)| = o(1) \quad \text{as } d(x,\partial\Omega) \to 0,\ x \in \Omega.$$

Then the Dirichlet problem

$$\begin{aligned} \hat{L}u &= f \quad \text{almost everywhere in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with f locally bounded and satisfying

$$|f(x)| \le d(x,\partial\Omega)^{2-\beta},$$

for some $\beta \in (0,1)$ near $\partial\Omega$, admits a unique solution $u \in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ for any p, $1 < p < \infty$. Moreover, the following estimate holds

$$\sup_{\Omega} |d(x,\partial\Omega)^{-\beta} u(x)| \le C_\beta \sup_{\Omega} |d(x,\partial\Omega)^{2-\beta} f(x)|.$$

Proof. See [A]. □

Lemma 2.5. *Let $\Omega \subset \mathbf{R}^n$ be a bounded NTA-domain with constants M, r_0. Then the L-elliptic measure at x with respect to Ω, called ω_L^x is mutually absolutely continuous with respect to the elliptic measure ω_∇^x at x with respect to $\nabla \cdot (A\nabla \cdot)$, and Ω.*

Proof. Let $x \in \Omega$, and let $E \subset \partial\Omega$ be such that $\omega_\nabla^x(E) = 0$. Set

$$u(x) = \int_{\partial\Omega} \chi_E(y)\, d\omega_L^x(y),$$

then we see that

$$\nabla \cdot (A\nabla u) = \sum_{i,j=1}^{n} [a_{ij}u_{ij} + (\partial_i a_{ij})u_j] = \sum_{i,j=1}^{n} (\partial_i a_{ij})u_j =: B\nabla u =: f,$$

with $B_i = (\partial_i a_{ij})$, $1 \le i \le n$. From interior Schauder estimates we see that $\hat{L} = L + B\nabla$, f, Ω satisfies Lemma 2.4, hence we can find a solution v to $\hat{L}v = f$, $v = 0$ on $\partial\Omega$ which is obviously a weak solution to $\nabla \cdot (A\nabla v) = f$. Then $\nabla \cdot (A\nabla(u - v)) = 0$, hence if

$$(u - v)(x) = \int (u - v)(y)\, d\omega_\nabla^x(y) = \int u(y)\, d\omega_\nabla^x(y) = 0,$$

telling us that if $\omega_\nabla^x(E) = 0$ then $\omega_L^x(E) = 0$. To prove the other way around, let $E \subset \partial\Omega$ be such that $\omega_L^x(E) = 0$ for some $x \in \Omega$, take

$$u(x) = \int_{\partial\Omega} \chi_E(y)\, d\omega_\nabla^x(y).$$

Then we see that $Lu = -B\nabla u =: f$. Let now v be the solution according to Lemma 2.4 to the equation $Lv = f$, with $v = 0$ on $\partial\Omega$. Since $L(u - v) = 0$ we see that

$$(u-v)(x) = \int_{\partial\Omega} (u-v)(y)\, d\omega_L^x(y) = \int_{\partial\Omega} \chi_E(y)\, d\omega_L^x(y) = 0. \qquad \square$$

This next lemma replaces the fact that for harmonic function, the absolute value of the gradient is a subsolution. In the variable coefficients case we have to add a correction term $|x|^2$ to obtain a subsolution.

Lemma 2.6. *Let $\Omega \subset \mathbf{R}^n$ be a domain. Let $w \geq 0$ be a positive strong solution to $Lw = 0$ in Ω, assume also that there exists constants $C_A, C_w \geq 0$ such that $|\nabla w| < C_w$ and $|\partial a_{jk}/\partial x_i| < C_A$ for all $1 \leq i, j, k \leq n$. Then there exists a constant $C_2 = C_2(C_w, C_A, n, 1/\lambda)$ such that $u = |\nabla w|^2 + C_2|x|^2$ is a subsolution.*

Proof. Let $A = \{a_{ij}\}$, then let us look at $v = |\nabla w|^2$, where w is such that $Lw = 0$. Then we see that

$$\begin{aligned} Lv &= \sum_{i=1}^n Lw_i^2 = \sum_{i=1}^n [2w_i Lw_i + A\nabla w_i \cdot \nabla w_i] \\ &= \sum_{i=1}^n \left[A\nabla w_i \cdot \nabla w_i - 2w_i \operatorname{Tr}\left(\frac{\partial A}{\partial x_i} \nabla^2 w \right) \right] \geq \sum_{i=1}^n \lambda |\nabla w_i|^2 - 2C_w C_A \sum_{k,l=1}^n |w_{ij}|. \end{aligned}$$

In order to bound the above right-hand side, we estimate the following

$$\left|\left|\nabla^2 w\right|\right| = \sup_{|\xi|=1} |(\nabla^2 w)\xi| = \sup_{|\xi|=1} \big(\sum_{j=1}^n |\nabla w_j \cdot \xi|^2 \big)^{1/2} \leq \big(\sum_{j=1}^n |\nabla w_j|^2 \big)^{1/2}.$$

Hence $\left|\left|\nabla^2 w\right|\right|^2 \leq \sum_{j=1}^n |\nabla w_j|^2$. Let $\xi_k = e_k$, then

$$n^2 \left|\left|\nabla^2 w\right|\right|^2 \geq n \sum_{j,k=1}^n |\nabla w_j \cdot \xi_k|^2 = n \sum_{j,k=1}^n |w_{jk}|^2 \geq \big(\sum_{j,k=1}^n |w_{jk}| \big)^2.$$

This yields

$$Lv \geq \sum_{i=1}^n \lambda |\nabla w_i|^2 - 2C_w C_A \sum_{k,l=1}^n |w_{ij}| \geq \lambda \left|\left|\nabla^2 w\right|\right|^2 - 2C_w C_A n \left|\left|\nabla^2 w\right|\right|.$$

Hence if at $x \in \Omega$

$$\left|\left|\nabla^2 w\right|\right| \geq 2C_w C_A n/\lambda. \tag{2.1}$$

Then $|\nabla w|^2$ is a subsolution at x. Else if at $x \in \Omega$, (2.1) does not hold,

$$\begin{aligned} Lv &\geq \left|\left|\nabla^2 w\right|\right| \big(\lambda \left|\left|\nabla^2 w\right|\right| - 2C_w C_A n \big) \\ &\geq C_w C_A n(-2C_w C_A n)/\lambda = -C_1 = -C_1(C_w, C_A, n, 1/\lambda). \end{aligned}$$

In both cases $u = v + \frac{C_1}{\lambda n}|x|^2$ satisfies $Lu \geq 0$. $\square$

3. Refined estimates for solutions to (1.1) in $\mathcal{C}_K$

In this section we prove estimates needed for us to be able to use the proof designed by De Silva and Jerison in [DJ] to prove Theorem 1.5. We begin with a Lipschitz bound.

Lemma 3.1. *Let u be a solution to* (1.1) *as in Theorem 1.5. Then there exist a constant C, such that*
(1) $|\nabla u| \leq C$ *on* $\mathcal{C}(1/2, K/2)^+(u)$,
(2) $u \leq Cd$ *on* $\mathcal{C}(1/2, K/2)^+(u)$.

Proof. The proof follows closely that of Lemma 2.1 in [DS]. We start by proving that if $x_0 \in \mathcal{C}(1/2, K/2)^+(u)$ and $d = d(x_0, F(u))$, then

$$u(x_0) \leq Cd. \tag{3.1}$$

Let $x_0 \in C^+_{1/2}(u)$. Then $v(x) = \frac{1}{d}u(x_0 + dx)$ is a solution to a non-divergence form equation of the same type as in (1.2), (1.3), i.e. $\hat{L}v = 0$ in B_1. By Harnack's inequality we see that

$$v(x) \geq cv(0) \quad \text{in } B_{1/2}(0).$$

Let us choose $\beta < 0$ such that the radially symmetric function

$$g(x) = \frac{cv(0)}{2^{-\beta} - 1}(|x|^\beta - 1),$$

satisfies $\hat{L}g \geq 0$ in the annulus $B_1 \setminus \overline{B}_{1/2}$. Consider $\hat{g} = |x|^\beta$, and $x \in B_1 \setminus \overline{B}_{1/2}$. Then

$$\begin{aligned}
&\hat{g}_i(x) = \beta x_i |x|^{\beta-2},\\
&\hat{g}_{ij}(x) = \beta(\beta-2)x_i x_j |x|^{\beta-4} + \beta\delta_{ij}|x|^{\beta-2},\\
&\qquad L\hat{g} \geq [\lambda\beta(\beta-2) + \Lambda n\beta]|x|^{\beta-2} = |x|^{\beta-2}(\lambda\beta^2 + (2\lambda + \Lambda n)\beta) \geq 0,
\end{aligned}$$

if $\beta < -(2\lambda+\Lambda n)$. Choosing $\beta = -2(2\lambda+\Lambda n)$ we see that $Lg \geq 0$, $g = 0$ on ∂B_1 and $g = cv(0)$ on $\partial B_{1/2}$. Then by the maximum principle $g(x) \leq v(x)$. Now let $x_1 \in \partial B_1$ be such that $v(x_1) = 0$. Let ν be the inward normal to ∂B_1 at x_1. Then at x_1,

$$1 = |\nabla v(x_1)| \geq v_\nu \geq g_\nu \geq Cv(0),$$

which yields (3.1) for $x_0 \in C^+_{1/2}(u)$. It now follows from Harnack's inequality and interior Schauder estimates that (1) and (2) hold. □

The next lemma is for us a technical necessity in proving Lemma 3.3.

Lemma 3.2. *Let u be a solution as in Theorem 1.5, then ∇u has non-tangential limits ω_L-almost everywhere on $F(u) \cap D$.*

Lemma 3.3. *Let u be a solution as in Theorem 1.5, u non-degenerate in $B_{3/4}$, and $0 \in F(u)$. Then, $F(u) \cap B_{1/2}$ is smooth almost everywhere with respect to the L-elliptic measure.*

Proof. Using Lemma 3.2, we can proceed as in [DJ, Lemma 2.7]. □

Proof of Lemma 3.2. Let $k = 1, 2, \ldots, n$. Then $\partial_k u = h$ solves the following equation

$$Lh = -\sum_{i,j=1}^{n} (\partial_k a_{ij})\frac{\partial^2 u}{\partial_i \partial_j} =: f.$$

From Lemma 3.1 we see that using standard interior Schauder estimates, L, f, D satisfies the requirements of Lemma 2.4, and hence there exists a solution v to $Lv = f$ and $v = 0$ on ∂D. Let $g = h - v$, then $Lg = 0$. We need to prove that g has non-tangential limits ω_L-a.e., since this implies the same for h. To begin, let us observe that if $\theta \in C_0^\infty(D)$ is a test-function, then by integration by parts

$$0 = \sum_{i,j=1}^{n} \int a_{ij} \frac{\partial^2 g}{\partial x_i \partial x_j} \theta \, dx = - \sum_{i,j=1}^{n} \int \frac{\partial a_{ij}}{\partial x_j} \frac{\partial g}{\partial x_i} \theta \, dx - \sum_{i,j=1}^{n} \int a_{ij} \frac{\partial g}{\partial x_i} \frac{\partial \theta}{\partial x_j} \, dx.$$

Thus we see that g is a weak solution to the equation $\nabla \cdot A \nabla g - B \cdot \nabla g = 0$, where $B_j = \sum_{i=1}^n \frac{\partial a_{ij}}{\partial x_i}$. We immediately see that $B_j \in L^\infty(\overline{D})$, hence there exists a unique, weak solution $\tilde{u}$ to $\nabla \cdot A \nabla \tilde{u} - B \nabla \tilde{u} = 0$, $\tilde{u} = 0$ on ∂D and continuous in $\overline{D}$ (see [GT, Theorem 8.31]). Using a version of Fatou's theorem in [JK], we obtain that $g - \tilde{u}$ has non-tangential limits ω_∇-a.e. on $F(u) \cap D$, and hence we see that g has by Lemma 2.5 non-tangential limits ω_L-almost everywhere. $\square$

Next we prove the existence of a subsolution which is crucial to the proof of Theorem 1.5, inspired by the subsolution in [DJ].

Lemma 3.4. *There exists a function $g(x) = \phi(|x'|^2) e^{A x_n}$, with ϕ a $C^2(\mathbf{R}_+ \cup \{0\})$ function, and a constant $A = A(\lambda, \Lambda, n) > 0$, such that ϕ satisfies*

$$\phi(r) = \begin{cases} 1, & \text{if } r < 1/4, \\ 0, & \text{if } r \geq 7/10, \end{cases}$$

$\phi(r) \geq 0$ for all $r \in \mathbf{R}_+ \cup \{0\}$, and such that g satisfies

$$Lg(x) \geq 0 \quad \text{in } \mathbf{R}^n.$$

Proof. To prove this lemma, we will make an explicit choice of the function ϕ. Let A be a positive constant to be chosen later. Take ϕ as the following function

$$\phi(s) = \begin{cases} 0, & s \geq 7/10, \\ \exp\left(\frac{(s-1/4)^3}{s - 7/10} \right), & 7/10 \geq s \geq 1/4, \\ 1, & s < 1/4, \end{cases}$$

one easily sees that this function is in $C^2(\mathbf{R}_+)$. Let $s = |x'|^2$. Then

$$\nabla g(x) = \left\{ 2x' \phi'(s) \;\middle|\; A \right\} g(x),$$

$$\nabla^2 g(x) = \left\{ \begin{array}{c|c} 4x' \otimes x' \phi''(s) - 2 I_{(n-1)\times(n-1)} \phi'(s) & 2Ax'\phi'(s) \\ \hline 2Ax'\phi'(s) & A^2 \phi(s) \end{array} \right\} e^{A x_n}.$$

We can now calculate Lg

$$Lg = \sum_{i,j=1}^{n-1} a_{ij} g_{ij} + \sum_{i=1}^{n-1} [a_{in} g_{in} + a_{ni} g_{ni}] + a_{nn} g_{nn},$$

where

$$\sum_{i,j=1}^{n-1} a_{ij}\left[4x_i x_j \phi''(s) - 2\delta_{ij}\phi'(s)\right] e^{Ax_n} \geq \left[4\lambda s^2 \phi''(s) - 2\Lambda(n-1)n\phi'(s)\right] e^{Ax_n},$$

and

$$\sum_{i=1}^{n-1}[a_{in}g_{in} + a_{ni}g_{ni}] = -\sum_{i=1}^{n-1} a_{ij} 2Ax_i\phi'(s)e^{Ax_n} \geq -2\Lambda s\phi'(s)e^{Ax_n}.$$

We obtain

$$Lg \geq \left[4\lambda s^2\phi''(s) + (-4n\Lambda(n-1) - 4ns\Lambda)\phi'(s) + \lambda A^2\phi(s)\right] e^{Ax_n}.$$

By inspection we see that the above expression is non-negative for $1/4 \leq s < 7/10$, when $A = A(\lambda, \Lambda, n)$ large enough, i.e. $Lg \geq 0$. When $s < 1/4$, $Lg(x) = A^2 g(x) > 0$, and if $s \geq 7/10$, we have $Lg = 0$. $\square$

4. Proof of Theorem 1.5

To prove Theorem 1.5 we follow the game plan of [DJ].

Step 1. Non-degeneracy and separation of level sets at the top. We first show the non-degeneracy of u, namely that if $B_\rho(x_0) \subset \mathcal{C}_L^+(u), \rho < 1$, then

$$u(x_0) \geq \gamma\rho, \tag{4.1}$$

for a constant $\gamma = \gamma(\lambda, \Lambda, n) > 0$. Let $g(x) = \frac{a_n}{1-2^\beta}(1 - |x|^\beta)$, where $\beta < 0$ is as in the proof of Lemma 3.1, then g is a strict supersolution in $B_2 \setminus B_1$, $g = a_n$ on ∂B_2 and $g = 0$ on ∂B_1, choose $a_n = \frac{1-2^\beta}{2|\beta|}$, then $|\nabla g| < 1$ on ∂B_1.

Notice that since a_{ij} are independent on the x_n direction we have $Lu_{x_n} = 0$. Using this fact we can apply the argument of step1 in [DJ] to obtain (4.1).

Next we prove that the level sets near the top of the cylinder are separated by an appropriate amount. Let $\epsilon > 0$ and define

$$v(x) := u(x - \epsilon e_n).$$

Since u is strictly monotone in the vertical direction, we have $v(x) < u(x)$ on $\mathcal{C}_K^+(u)$. Using Lemma 3.1 together with the argument of step1 in [DJ] we obtain

$$v(x) \leq u(x) - c\epsilon \quad \text{on } B_{9/10}(0) \times \{K - 1/2\}.$$

Step 2. Construction of a family of supersolutions. From the hypotheses of Theorem 1.5, there exists by Lemma 2.2 an NTA domain between any pair $\mathcal{C}(r_1, L - a_1)$ and $\mathcal{C}(r_2, L - a_2)$ for $r_1 < r_2 \leq 9/10$ and $a_1 > a_2 \geq 1/2$. Thus the boundary Harnack inequality, (Lemma 2.3) has the following corollary.

Corollary 4.1. *Let u be as in Theorem 1.5, and let $r_1 < r_2 \leq 9/10$ and $a_1 > a_2 \geq 1/2$. Then there is a constant C depending on $K, M, r_0, r_2 - r_1 > 0$, and $a_1 - a_2 > 0$ such that if h_1 and h_2 are positive solutions to $Lh = 0$ on $\mathcal{C}(r_2, K - a_2) \cap \mathcal{C}_K^+(u)$, vanishing on $\partial D \cap \mathcal{C}(r_2, L - a_2)$, then*

$$\frac{h_1(x)}{h_2(x)} \leq C\frac{h_1(y)}{h_2(y)},$$

for every x and y in $\mathcal{C}(r_1, K - a_1) \cap \mathcal{C}_K^+(u)$.

Let us call $\mathfrak{C}_1 = \mathcal{C}(9/10, K-1/2)$, $\mathfrak{C}_2 = \mathcal{C}(8/10, M_2)$, and $\mathfrak{C}_3 = \mathcal{C}(7/10, M_3)$, $K-1 < M_3 < M_2 < K-1/2$. Let w be a solution to $Lw = 0$ in $\mathfrak{C}_1^+(u)$, satisfying the following boundary conditions

$$\begin{aligned} w &= 0, \quad \text{on } F(u),\\ v < w &\le u, \quad \text{on } \overline{\mathfrak{C}_1^+(u)} \cap \partial\mathfrak{C}_1, \qquad (4.2)\\ v + \frac{c_1}{4}\varepsilon < w &< u - \frac{c_1}{4}\varepsilon, \quad \text{on } \overline{\mathfrak{C}_1} \cap \mathbf{R}^n \times \{K-1/2\}. \end{aligned}$$

By step 1, we have separated our level sets at the top, hence we can achieve (4.2). Since $Lv \ge 0$ and $Lu = Lw = 0$, we have by the maximum principle

$$v < w < u \quad \text{in } \mathfrak{C}_1^+(u).$$

Moreover $\mathfrak{C}_1^+(w) = \mathfrak{C}_1^+(u)$, with coinciding free boundaries inside $\mathfrak{C}_1$. We claim next that in the smaller cylinder $\overline{\mathfrak{C}}_2$,

$$|\nabla w|(x) \le C_1, \quad x \in \overline{\mathfrak{C}}_2.$$

Define $d(x) = d(x, F(u))$. At points $x \in \overline{\mathfrak{C}}_2 \cap \mathfrak{C}_1^+(u)$ such that $d(x) \ge 1/10$, this follows from standard interior Schauder estimates and the fact that w is bounded. On the other hand, at points that are close to $F(u)$, we have that $B_{d(x)}(x) \subset \mathfrak{C}_1^+(u)$ and from Lemma 3.1 we see that

$$w(x) < u(x) < Cd(x).$$

Using interior Schauder estimates we get the claim.

Set $h = u - w$. Then $Lh = 0$, $h > 0$, inside $\mathfrak{C}_1^+(u)$ with coinciding free boundary with u in $\mathfrak{C}_1$. Now let H be a solution to $LH = 0$, inside $B'_{9/10} \times (K-1, K-1/2)$ such that $H = c_1/2$ on the top and vanishing elsewhere on the boundary. Then in view of (4.2) we see that $h \ge \varepsilon H$. Thus, $h(x_1) \ge c_1\varepsilon/4$, at $x_1 = (K - 1/2 - \delta_n)e_n$ for a small constant δ_n depending on the dimension and the ellipticity. Moreover by the Lipschitz continuity of u we get that $h(x_1) < (u-v)(x_1) \le N\varepsilon$. Using non-degeneracy and Lipschitz continuity of u we also have that $b_n \le u(x_1) \le 2KN$. Thus Corollary 4.1 gives

$$\frac{h(x)}{u(x)} \le C\frac{h(x_1)}{u(x_1)} \le C\frac{N\varepsilon}{b_n},$$

and

$$\frac{c_1\varepsilon/4}{2KN} \le \frac{h(x_1)}{u(x_1)} \le C\frac{h(x)}{u(x)}.$$

Hence we see that

$$\frac{1}{C}\varepsilon u(x) \le h(x) \le C\varepsilon u(x),$$

where $C = C(c_1, K, N, b_n)$, on $\overline{\mathfrak{C}_2^+(u)}$. The upper bound on h above implies

$$(1 - C_2\varepsilon)u \le w \le (1 - c_2\varepsilon)u \quad \text{on } \overline{\mathfrak{C}_2^+(u)}, \qquad (4.3)$$

In particular, if $F(u)$ is smooth around a point $x_0 \in \mathfrak{C}_2$ then $|\nabla u|(x_0) = 1$, which combined with (4.3) gives

$$|\nabla w|(x_0) \le 1 - c_2\varepsilon.$$

But according to Lemma 3.3, we have

$$|\nabla w| \le 1 - c_2\varepsilon, \quad \omega_L \text{ a.e. on } F(u) \cap \mathfrak{C}_2. \qquad (4.4)$$

Next we use (4.4) to show that, by restricting to the smaller cylinder $\mathfrak{C}_3$, we have

$$|\nabla w|^2 \le (1-c_2\varepsilon)^2 + C_4\sqrt{u} \quad \text{on } \mathfrak{C}_3^+(u). \tag{4.5}$$

To do this we let f be the solution to $Lf = 0$, $f(x) = C_1^2 + C_2^*|x|^2$ in $\partial\mathfrak{C}_2^+(u) \setminus \partial F(u)$ such that (C_2^* is C_2 from Lemma 2.6), and

$$f(x) = (1-c_2\varepsilon)^2 + C_2^*|x|^2 \quad \text{on } F(u) \cap \mathfrak{C}_2.$$

From this we see that $f \ge |\nabla w|^2 + C_2^*|x|^2$. We can split f into two parts, $f = h + g$, where $Lh, g = 0$, $h = (1-c_2\varepsilon)^2$ on $F(u) \cap \mathfrak{C}_2$ and $h = C_1^2$ on $\partial\mathfrak{C}_2^+(u) \setminus \partial F(u)$, $g = C_2^*|x|^2$ on $\partial\mathfrak{C}_2^+(u)$. Then $h - (1-c_2\varepsilon)^2$ is a positive L-harmonic function, we can use Corollary 4.1 to obtain

$$h - (1-c_2\varepsilon)^2 \le C_3 u,$$

giving us the estimate

$$|\nabla w|^2 \le (1-c_2\varepsilon)^2 + C_3 u + g - C_2^*|x|^2.$$

To estimate $P = g - C_2^*|x|^2$, we note that $P = 0$ on $\partial\mathfrak{C}_2^+(u)$, and $-\lambda C \ge LP \ge -\Lambda C$. It now follows from Lemma 2.4 that for any $\hat\beta \in (0,1)$ we have

$$\sup_{\mathfrak{C}_2^+(u)} \left|\frac{P(x)}{d(x,\partial\Omega)^{\hat\beta}}\right| \le C \sup_{\mathfrak{C}_2^+(u)} \left|d(x,\partial\Omega)^{2-\hat\beta}\Lambda C\right| \le C_3^*(\hat\beta),$$

which tells us that

$$P(x) \le C_3^* d(x,\partial\Omega)^{\hat\beta}, \quad \beta \in (0,1),$$

in $\mathfrak{C}_2^+(u)$. From the non-degeneracy, boundedness of u, and choosing $\hat\beta = 1/2$, we have

$$|\nabla w|^2 \le (1-c_2\varepsilon)^2 + C_3 u + C_3^* u^{\hat\beta} \le (1-c_2\varepsilon)^2 + C_4\sqrt{u},$$

in $\mathfrak{C}_3^+(u)$. Hence (4.5) holds.

Let us now define the following family of supersolutions, for $t \ge 0$,

$$w_t(x) = w(x) - tg(x), \quad x \in \mathfrak{C}_1,$$

with $g(x)$ as in Lemma 3.4. Thus w_t is a L-supersolution on $\mathfrak{C}_1^+(w_t)$. Moreover (4.5) together with (4.3) we get

$$|\nabla w_t| \;\le\; |\nabla w| + t|\nabla g| \le \sqrt{(1-c_2\varepsilon)^2 + C_5\sqrt{w}} + t|\nabla g|$$

in $\mathfrak{C}_3^+(u)$. In particular on $F(w_t) \cap \mathfrak{C}_3$ we have $w = tg$ and hence

$$|\nabla w_t| \le 1 - \frac{c_2\varepsilon}{2}$$

provided $0 < t < c_3\varepsilon$, with c_3 small enough depending on c_2, C_5 and the constant A from Lemma 3.4.

Step 3. Arguing in the same way as in Step 3 [DJ] we obtain the conclusion of Theorem 1.5.

5. Concluding remarks

Regarding what is done in [DJ], we are at this point in time not able to prove that a general one-phase free boundary is NTA and a graph. This is due to the lack of a monotonicity formula, as in [ACF], if we had such a formula then we could probably proceed as in [DS2] to prove that the free boundary of a monotone solution as in Theorem 1.5 is NTA and a graph in the e_n-direction. Regarding the assumption that the coefficients in our equation where invariant with respect to the e_n direction, we see that this is essential to the argument developed by De Silva and Jerison [DJ]. However, it would be interesting to see if this could be done without this assumption, with some additional continuity assumption on the coefficients, and using an estimate for the difference between solutions, whose coefficients are close.

References

[A] Ancona, A.: On strong barriers and an inequality of Hardy for domains in $\mathbf{R}^n$. - J. Lond. Math. Soc. (2) 34:2, 1986, 274–290.

[ACF] Alt, H. W., L. A. Caffarelli, and A. Friedman: Variational problems with two phases and their free boundaries. - Trans. Amer. Math. Soc. 282:2, 1984, 431–461.

[C] Caffarelli, L.: Interior a priori estimates for solutions of fully non-linear equations. - Ann. of Math. (2) 130:1, 1989, 189–213.

[DS] De Silva, D.: Bernstein-type techniques for 2D free boundary graphs. - Math. Z. 260, 2008, 47–60.

[DS2] De Silva, D.: Existence and regularity of monotone solutions to a free boundary problem. - Amer. J. Math. 131:2, 2009, 351–378.

[DJ] De Silva, D., and D. S. Jerison: A gradient bound for free boundary graphs. - Comm. Pure Appl. Math. 64:4, 2011, 538–555.

[GT] Gilbarg, D., and N. S. Trudinger: Elliptic partial differential equations of second order. Second edition. - Grundlehren Math. Wiss. 224, Springer-Verlag, Berlin, 1983.

[JK] Jerison, D. S., and C. E. Kenig: Boundary behavior of harmonic functions in non-tangentially accessible domains. - Adv. Math. 46, 1982, 80–147.

[J] Jones, P. W.: Quasiconformal mappings and extendability of functions in Sobolev spaces. - Acta Math. 147:1-2, 1981, 71–88.

[K] Kenig, C. E.: Potential theory of non-divergence form elliptic equations. - In: Dirichlet forms (Varenna, Italy, 1992), Lecture Notes in Math. 1563, Springer-Verlag, Berlin 1993, 89–128.

Received 14 February 2012 • Accepted 15 October 2012