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ALGEBRAS OF FUNCTIONS WITH PRESCRIBED RADII OF BOUNDEDNESS AND THE SPECTRA OF $\mathcal{H}(U)$

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Abstract. We study the spectra of algebras of holomorphic functions with prescribed radii of boundedness, and use these results to study the τ_ω and τ_δ spectra of $\mathcal{H}(U)$, where U is an open subset of a non-separable Banach space. We construct τ_δ-continuous characters on $\mathcal{H}(U)$ which are not evaluations at points of U. We also discuss subsets of ℓ_∞ which are bounding for $\mathcal{H}(U)$, $U \subset \ell_\infty$.

Introduction

In the beginning of the 1970's algebras of holomorphic functions with prescribed radii of boundedness were defined and studied by Coeuré [8] and by Matos [18]. It is well-known today that each holomorphic function can be extended to some open subset of the bidual of its original domain, making possible evaluation at points in the bidual. The trick which enables this extension is now an important tool in infinite-dimensional holomorphy known as the Aron–Berner extension. However, when the above cited papers were written, this tool was still years away [1], so a detailed study of the spectrum of such an algebra was not possible.

In the 70's and 80's Mujica ([19] [20], [21]) studied the spectra of algebras $\mathcal{H}(U)$ of analytic functions on an open subset U of a Fréchet space E, both with the τ_ω and the τ_δ topologies, obtaining important results. Mujica characterized homomorphisms of these algebras and proved that if U is pseudoconvex and E has the approximation property, the τ_ω spectrum identifies with evaluations on the set U, and also that if E is separable and has the bounded approximation property, then the τ_δ spectrum identifies with evaluations on U.

In this paper we first study the spectrum of algebras of holomorphic functions with prescribed radii of boundedness, and then apply our results to the study of the spectra of $\mathcal{H}(U)$. We restrict our attention, however, to convex and balanced open subsets of Banach spaces. We obtain that in the non-separable setting, τ_δ-continuous characters in general are not evaluations at points of U, even if E has the bounded approximation property. The prime example of a non-separable Banach space with the bounded approximation property is ℓ_∞; thus we are naturally led to

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consider bounding subsets of ℓ_∞ and of its unit ball, and we characterize sets which are bounding with respect to holomorphic functions on the unit ball of ℓ_∞.

The algebras $\mathcal{H}_t(U)$ of holomorphic functions with prescribed radii of boundedness are introduced in section 1, and their spectra studied in detail in section 2, together with the spectra of $(\mathcal{H}(U),\tau_\omega)$ and $(\mathcal{H}(U),\tau_\delta)$. Section 3 is devoted to the study of bounding subsets of ℓ_∞.

We shall use standard notation and notions from infinite-dimensional holomorphy theory as presented, e.g., in [12] or [20].

1. Holomorphic functions with prescribed radii of boundedness

Throughout, U will denote an open subset of a Banach space E, and

$$t\colon U \to (0,\infty]$$

a function such that $t(a) \le d(a,\partial U)$. Define the algebra of holomorphic functions with t-radii on U as

$$\mathcal{H}_t(U) = \{f \in \mathcal{H}(U)\colon r_f(a) \ge t(a) \text{ for all } a \in U\},$$

where $r_f(a)$ denotes the radius of boundedness of f at a; i.e., f is bounded on all balls centered at a and with radii strictly less than $r_f(a)$. On this space introduce the locally convex topology τ_t, induced by the following family of seminorms

$$p_{a,r}(f) = \sup_{B(a,r)} |f|, \quad \text{where } a \in U \text{ and } r < t(a),$$

where $B(a,r)$ denotes the open ball of E centered at a and of radius $r>0$.

Proposition 1.1. *$\mathcal{H}_t(U)$ is a complete locally convex algebra.*

Proof. The submultiplicativity of the seminorms is clear. We prove completeness: say (f_i) is a τ_t-Cauchy net. Then for any $a \in U$ and $r < t(a)$, if $x \in B(a,r)$; given $\varepsilon > 0$ there is an i such that if $j,k \ge i$,

$$|f_j(x) - f_k(x)| \le p_{a,r}(f_j - f_k) < \varepsilon.$$

Thus $(f_i(x))$ is a Cauchy net, so we may define $f\colon U \to \mathbf{C}$ by $f(x) = \lim_i f_i(x)$. Note that $f_i \to f$ uniformly on each $B(a,r)$, so $f \in \mathcal{H}(U)$. Also,

$$r_f(a) = \lim_i r_{f_i}(a) \ge t(a) \text{ for all } a \in U. \qquad \square$$

Note that when $t \le t'$ we have a continuous inclusion $\mathcal{H}_{t'}(U) \to \mathcal{H}_t(U)$. One may consider on $\mathcal{H}_t(U)$ the topologies τ_0, τ_ω and τ_δ (the compact-open topology, the Nachbin or compactly-ported topology, and the countable covers topology [12]). It is easy to see that τ_t is finer than τ_ω: indeed, if p is ported by a compact subset K of U, from the open cover $\{B(a,r)\colon a \in K,\ r < t(a)\}$ extract a finite subcover

$$K \subset \bigcup_{i=1}^n B(a_i,r_i);$$

since p is ported by K, there is a $c>0$ such that

$$p(f) \le c \max_i p_{a_i,r_i}(f) \text{ for all } f \in \mathcal{H}_t(U).$$

Although we will be mainly interested in the non-separable case, it should be mentioned here that when E is a separable space, the $\mathcal{H}_t(U)$'s are Fréchet algebras (take only a's in a countable dense subset D of U and rational r's). Also, τ_t is finer than the τ_δ topology: given p a τ_δ-continuous seminorm, consider the countable cover $\{B(a,r)\colon a \in D,\ r < t(a),\ r \in \mathbf{Q}\}$ to obtain an inequality as above. In fact, in the separable case, one may check that

$$\lim_{\rightarrow} \mathcal{H}_t(U) = (\mathcal{H}(U), \tau_\delta),$$

and that this inductive limit is regular.

Note that the bounded subsets of $\mathcal{H}_t(U)$ are the subsets B such that for all $a \in U$ and $r < t(a)$, there is a constant $C_{a,r}$ such that

$$p_{a,r}(f) \le C_{a,r} \quad \text{for all } f \in B.$$

Thus τ_t-bounded subsets of $\mathcal{H}_t(U)$ are locally bounded. The inverse does not hold in general.

2. The spectra of $\mathcal{H}_t(U)$, $(\mathcal{H}(U), \tau_\omega)$ and $(\mathcal{H}(U), \tau_\delta)$

Every holomorphic function on an open subset of a Banach space can be evaluated at some points of the bidual of the space. Indeed, any homogeneous continuous polynomial on the space can be extended in a canonical way to the bidual [1], a fact which used in conjunction with Taylor series expansions produces local extensions agreeing on intersections. We will use this fact and the fact that the norm of a continuous polynomial is the same as that of its extension [9].

Denote by $B''(a, t(a))$ the open ball of E'' centered at a and of radius $t(a)$. We define

$$U_t = \bigcup_{a \in U} B''(a, t(a)).$$

Clearly [1], [9] all functions $f \in \mathcal{H}_t(U)$ can be extended to $\overline{f}\colon U_t \to \mathbf{C}$.

Proposition 2.1. *For each $z \in U_t$, the evaluation $e_z(f) = \overline{f}(z)$ is a continuous character of $\mathcal{H}_t(U)$.*

Proof. That the Aron–Berner extension is a homomorphism was proved in [23]. For the continuity, say $\|z - a\| < r < t(a)$, and $\varepsilon > 0$. Then for large enough n, and all $y \in B''(a, r)$,

$$\left| \overline{f}(y) - \sum_{k=0}^{n} \overline{P_{ka}}(y-a) \right| < \varepsilon/2,$$

where $\sum_{k=0}^{\infty} P_{ka}$ is the Taylor series expansion of f at a. By [9, Theorem 1], one may find $x \in B(a,r)$ such that $|\sum_{k=0}^{n} \overline{P_{ka}}(z-a) - \sum_{k=0}^{n} P_{ka}(x-a)| < \varepsilon/2$, so

$$\begin{aligned} |e_z(f)| = |\overline{f}(z)| = \left| \sum_{k=0}^{\infty} \overline{P_{ka}}(z-a) \right| &\le \left| \sum_{k=0}^{n} \overline{P_{ka}}(z-a) \right| + \varepsilon/2 \\ &\le \left| \sum_{k=0}^{n} P_{ka}(x-a) \right| + \varepsilon \le |f(x)| + \frac{3}{2}\,\varepsilon. \end{aligned}$$

Thus $|e_z(f)| \le p_{a,r}(f)$. □

Definition 1. We will denote the spectrum of $\mathcal{H}_t(U)$ by $M_t(U)$. Also,

$$\pi\colon M_t(U) \to E''$$

will indicate the restriction mapping $\pi(\varphi) = \varphi|_{E'}$.

We have just proved that $U_t \subset \pi(M_t(U))$. For every $\varphi \in M_t(U)$ there are $a_i \in U$ and $0 < r_i < t(a_i)$, $i = 1, \dots, n$, such that with $p(f) = \sup\{|f(x)| : x \in \bigcup_{i=1}^n B(a_i, r_i)\}$ we have

$$|\varphi(f)| \le p(f) \text{ for all } f \in \mathcal{H}_t(U).$$

In other words, the constant c in $|\varphi| \le cp$ can be taken to be one by the usual algebraic trick of considering f^n, taking n-th roots and letting $n \to \infty$. We will use the notation $p \sim \binom{a_1,\dots,a_n}{r_1,\dots,r_n}$ for the above seminorm. We will also write P_{ka} for the k-homogeneous continuous polynomial in the Taylor expansion of f at a. We adapt the following from the study of the bounded case $H_b(U)$ [3].

Proposition 2.2. *Say $\varphi \in M_t(U)$ is p-continuous, with $p \sim \binom{a_1,\dots,a_n}{r_1,\dots,r_n}$. Then for each $w \in E''$ with $\|w\| < \min_{1\le j\le n}\{t(a_j) - r_j\}$,*

$$\varphi^w(f) = \sum_{k=0}^{\infty} \varphi\left(\overline{P_{k(\cdot)}(w)}\right)$$

defines a p'-continuous character (for $p' \sim \binom{a_1,\dots,a_n}{r'_1,\dots,r'_n}$) whenever $r_j < r'_j < t(a_j)$ for $j = 1, \dots, n$.

Proof. Let $V = \bigcup_{j=1}^n B(a_j, r_j)$. Note that if $r_j < r'_j < t(a_j)$ for $j = 1, \dots, n$, then given $x \in V$, if $x \in B(a_j, r_j)$, then $\overline{B(x, r'_j - r_j)} \subset B(a_j, r'_j) \subset U$, and we may apply the Cauchy inequality

$$\|P_{kx}\| \le \frac{1}{(r'_j - r_j)^k} \sup_{B(x, r'_j - r_j)} |f| \le \frac{1}{(r'_j - r_j)^k} \sup_{V'} |f|,$$

where $V' = \bigcup_{j=1}^n B(a_j, r'_j)$.

Taking $w \in E''$ with $\|w\| < \min_{1\le j\le n}\{t(a_j) - r_j\}$, we have for each k, that $\overline{P_{k(\cdot)}(w)} \in \mathcal{H}_t(U)$: indeed, $x \mapsto \overline{P_{kx}(w)}$ is holomorphic, and if $a \in U$ and $r < t(a)$, taking $r < r' < t(a)$ and $x \in B(a, r)$, $\overline{B(x, r' - r)} \subset B(a, r') \subset U$ and

$$|\overline{P_{kx}(w)}| \le \|\overline{P_{kx}}\| \|w\|^k = \|P_{kx}\| \|w\|^k \le \frac{\|w\|^k}{(r' - r)^k} \sup_{B(x, r'-r)} |f| \le \frac{\|w\|^k}{(r' - r)^k} p_{ar'}(f) < \infty.$$

Thus $p_{ar}\left(\overline{P_{k(\cdot)}(w)}\right) < \infty$, and $\overline{P_{k(\cdot)}(w)} \in \mathcal{H}_t(U)$. Hence

$$\left|\varphi\left(\overline{P_{k(\cdot)}(w)}\right)\right| \le p\left(\overline{P_{k(\cdot)}(w)}\right).$$

Now for $r_j < r'_j < t(a_j)$,

$$\begin{aligned}\sum_{k=0}^{\infty} \left|\varphi\left(\overline{P_{k(\cdot)}(w)}\right)\right| &\le \sum_{k=0}^{\infty} p\left(\overline{P_{k(\cdot)}(w)}\right) \le \sum_{k=0}^{\infty} \max_{1\le j\le n} p_{a_j r_j}\left(\overline{P_{k(\cdot)}(w)}\right) \\ &\le \sum_{k=0}^{\infty} \max_{1\le j\le n} \left[\frac{\|w\|}{(r'_j - r_j)}\right]^k p_{a_j r'_j}(f) = p'(f) \sum_{k=0}^{\infty} \left(\frac{\|w\|}{\rho}\right)^k < \infty,\end{aligned}$$

if $\|w\| < \rho = \min_j\{r'_j - r_j\}$. Thus for such w

$$\varphi^w(f) = \sum_{k=0}^{\infty} \varphi\left(\overline{P_{k(\cdot)}(w)}\right)$$

is defined and p'-continuous with $p' \sim \binom{a_1,\dots,a_n}{r'_1,\dots,r'_n}$. □

Observe that $\pi(\varphi^w) = \pi(\varphi) + w$: indeed, if $\gamma \in E'$, $\gamma(x) = \gamma(a) + \gamma(x-a)$, so $P_{0a} = \gamma(a)$ and $P_{1a} = \gamma$. Thus $\overline{P_{0a}}(w) = \gamma(a)$ and $\overline{P_{1a}}(w) = w(\gamma)$. Then

$$\varphi^w(\gamma) = \varphi(\gamma) + \varphi(w(\gamma)) = \varphi(\gamma) + w(\gamma) = (\pi(\varphi) + w)(\gamma).$$

Recall that in the bounded case ($H_b(E)$), $\varphi^w(f) = \varphi(\overline{f} \circ T_w)$, where T_w is the translation in w. Such an equality is not possible here, even if $U = E$: indeed, $\overline{f} \circ T_w$ may be undefined (consider $t(x) = e^{-\|x\|^2}$), or even if defined, may not be an element of $\mathcal{H}_t(E)$ (consider $t \equiv 1$).

Now, continuing as in [3], given $\varphi \in M_t(U)$ p-continuous for $p \sim \binom{a_1,\dots,a_n}{r_1,\dots,r_n}$ we may define

$$V_{\varphi p} = \{\varphi^w \colon \|w\| < \min_{1\le j\le n} \{t(a_j) - r_j\}\},$$

and when E is a symmetrically regular Banach space $(V_{\varphi p})_{\varphi,p}$ form a basis for a Hausdorff topology on $M_t(U)$ making $\pi\colon M_t(U) \to E''$ a local homeomorphism. In fact, one obtains the following Proposition. The proof is as in [3] and we omit it.

Proposition 2.3. *If E is symmetrically regular, $M_t(U)$ admits an analytic structure over E''.*

Recall that if $t \le t'$ we have a continuous inclusion $\mathcal{H}_{t'}(U) \to \mathcal{H}_t(U)$. By transposition we also have the mappings

$$M_t(U) \to M_{t'}(U).$$

We prove now that these are continuous for the topology defined above on the $M_t(U)$'s.

Proposition 2.4. *For the topology defined above, the map $M_t(U) \to M_{t'}(U)$ is continuous whenever $t \le t'$.*

Proof. Let $\varphi \in M_t(U)$, and call its image φ'. Say φ' is p-continuous, where the seminorm p is $p \sim \binom{a_1,\dots,a_n}{r'_1,\dots,r'_n}$, and consider the neighborhood

$$W_{\varphi' p} = \{\varphi'^w \colon \|w\| < \min_{1\le j\le n} \{t'(a_j) - r'_j\}\}.$$

Since $r'_j - t'(a_j) < 0$, $t(a_j) + (r'_j - t'(a_j)) < t(a_j)$. Now, for each j, take $r_j > 0$ such that

$$t(a_j) + (r'_j - t'(a_j)) < r_j < t(a_j),$$

and take the seminorm $q \sim \binom{a_1,\dots,a_n}{r_1,\dots,r_n}$. Note that we have

$$t(a_j) - r_j < t'(a_j) - r'_j \text{ for each } j.$$

Now consider the neighborhood of $\varphi \in M_t(U)$:

$$V_{\varphi q} = \{\varphi^w \colon \|w\| < \min_{1\le j\le n} \{t(a_j) - r_j\}\}.$$

Since $\|w\| < \min_{1\le j\le n}\{t(a_j) - r_j\} < \min_{1\le j\le n}\{t'(a_j) - r'_j\}$, it is easily seen that $V_{\varphi q}$ maps into $W_{\varphi' p}$. □

Note that for every t we have the mappings

$$H_b(U) \to \mathcal{H}_t(U) \to (\mathcal{H}(U), \tau_\omega).$$

Considering the corresponding spectra and transposing we have

$$\begin{array}{ccccccc} M_\omega(U) & \longrightarrow & M_t(U) & \longrightarrow & M_b(U) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi(M_\omega(U)) & \longrightarrow & \pi(M_t(U)) & \longrightarrow & \pi(M_b(U)) & \longrightarrow & E'' \end{array}$$

and for separable E,

$$\begin{array}{ccccccc} M_\delta(U) & \longrightarrow & M_t(U) & \longrightarrow & M_b(U) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi(M_\delta(U)) & \longrightarrow & \pi(M_t(U)) & \longrightarrow & \pi(M_b(U)) & \longrightarrow & E'' \end{array}$$

We wish to determine the sets on the lower rows. For this we now suppose that U is convex.

Proposition 2.5. *If U is convex and t concave, we have $\pi(M_t(U)) = U_t$.*

Proof. We have seen $\supset$, so we now consider $\subset$. First, note that U_t is convex. Indeed, take z_1 and z_0 in U_t, and say $z_i \in B(a_i, t(a_i))$. Now for any $\alpha \in [0,1]$,

$$\begin{aligned} &\|(\alpha z_1 + (1-\alpha)z_0) - (\alpha a_1 + (1-\alpha)a_0)\| = \|\alpha(z_1 - a_1) + (1-\alpha)(z_0 - a_0)\| \\ &\le \alpha\|z_1 - a_1\| + (1-\alpha)\|z_0 - a_0\| < \alpha t(a_1) + (1-\alpha)t(a_0) \le t(\alpha a_1 + (1-\alpha)a_0), \end{aligned}$$

so $\alpha z_1 + (1-\alpha)z_0 \in U_t$.

Now consider the set of complex-affine functions over E'',

$$A(E'') = \{L + c\colon L \in E''', c \in \mathbf{C}\}.$$

Set $\varphi \in M_t(U)$ and $z = \pi(\varphi)$. If φ is p-continuous, with $p \sim \binom{a_1,\dots,a_n}{r_1,\dots,r_n}$, let

$$V = \bigcup_{j=1}^{n} B''(a_j, r_j).$$

We want to see that for any $L + c \in A(E'')$, $|L(z) + c| \le \sup_V |L + c| = p(L + c)$. Suppose not, that is, say there are $L \in E'''$ (which we may suppose of norm one) and $c \in \mathbf{C}$ such that

$$|L(z) + c| > \sup_V |L + c|.$$

By Goldstine's theorem, we may take $\gamma \in E'$ of norm one such that $\overline{\gamma}$ almost coincides with L on the finite set $\{z, a_1, \dots, a_n\}$. Now consider each ball $B''(a,r)$ (we drop the index j for a moment). Since L has norm 1,

$$(L + c)B''(a,r) = L(a) + c + r\Delta,$$

where Δ is the unit disk of $\mathbf{C}$. Similarly

$$\overline{\gamma}B''(a,r) = \overline{\gamma}(a) + c + r\Delta.$$

Since $\overline{\gamma}$ almost coincides with L on a,

$$\sup_{B''(a,r)} |L + c| \simeq \sup_{B''(a,r)} |\overline{\gamma} + c|,$$

and so, the same over V. But we had

$$|\overline{\gamma}(z) + c| \simeq |L(z) + c| > \sup_V |L + c| \simeq \sup_V |\overline{\gamma} + c|.$$

Since $z = \pi(\varphi)$, $\pi(\varphi)(\gamma) = z(\gamma) = \overline{\gamma}(z)$, we have

$$|\varphi(\gamma + c)| > \sup_V |\gamma + c|.$$

This contradicts the p-continuity of φ. Thus $|L(z)+c| \leq \sup_V |L+c| = p(L+c)$, for all L and c, so z is in the complex-affine hull of V. By a result of Noverraz [22], this is in the closed convex hull of V, but this is contained in U_t, which we have seen to be convex. □

Proposition 2.6. *If U is convex, there are U_t such that $\pi(M_t(U)) = U_t$ and $\bigcap_t U_t = U$.*

Proof. We show first that the distance function $d\colon U \to (0, \infty]$ defined by $d(x) = d(x, \partial U)$ is concave: let a_1 and a_0 in U, and take $r_i < t(a_i)$ (so that $B(a_i, r_i) \subset U$). For each $\alpha \in [0,1]$ we set $a_\alpha = \alpha a_1 + (1-\alpha)a_0$ and $r_\alpha = \alpha r_1 + (1-\alpha)r_0$. Then $B(a_\alpha, r_\alpha) \subset U$. Indeed, if $x \in B(a_\alpha, r_\alpha)$, set for $i = 0, 1$,

$$x_i = \frac{r_i}{r_\alpha}(x - a_\alpha) + a_i.$$

Each x_i is in $B(a_i, r_i)$, for $\|x_i - a_i\| = \frac{r_i}{r_\alpha}(x - a_\alpha) < r_i$. Thus each x_i is in U. Since U is convex, the following point is also in U:

$$\begin{aligned}\alpha x_1 + (1-\alpha)x_0 &= \frac{\alpha r_1 + (1-\alpha)r_0}{r_\alpha}(x - a_\alpha) + \alpha a_1 + (1-\alpha)a_0 \\ &= \frac{r_\alpha}{r_\alpha}(x - a_\alpha) + a_\alpha = x.\end{aligned}$$

Thus, $d(a_\alpha) \geq r_\alpha = \alpha r_1 + (1-\alpha)r_0$ for all $r_1 < t(a_1)$ and $r_0 < t(a_0)$. Hence $d(a_\alpha) \geq \alpha d(a_1) + (1-\alpha)d(a_0)$; d is concave.

Now for each $\varepsilon > 0$, let

$$t_\varepsilon(a) = \min\{\varepsilon, \varepsilon d(a, \partial U)\}.$$

Being the minimum of two concave functions, t_ε is concave, thus U_{t_ε} is convex for all ε, so $\pi(M_{t_\varepsilon}(U)) = U_{t_\varepsilon}$. Also,

$$\bigcap_{\varepsilon>0} U_{t_\varepsilon} = U.$$

Indeed, if $z \notin E$, since E is a closed subspace of E'', one may take $d(z, E) > \varepsilon > 0$; if $z \in E$ but is not in U, for each $a \in U$ $d(a, z) \geq d(a, \partial U) \geq t(a)$ thus z does not belong to $B(a, t(a))$ for any t. □

As an immediate consequence we have the following Corollary, where we denote with $M_\omega(U)$ the spectrum of $(\mathcal{H}(U), \tau_\omega)$ and with $M_\delta(U)$ the spectrum of $(\mathcal{H}(U), \tau_\delta)$.

Corollary 2.7. *If U is convex, then*

i) $\pi(M_\omega(U)) = U$, *and*
ii) *when E is separable,* $\pi(M_\delta(U)) = U$.

We comment that if we put $t = d$ the algebras $\mathcal{H}_t(U)$ obtained are those considered in [13]. If U is either the whole space E or a ball, then $\mathcal{H}_d(U) = H_b(U)$, and we also have $\pi(M_b(U)) = U_d$.

Isidro [15] and Mujica [19] studied $M_\omega(U)$ in the 70's. Mujica shows that if U is pseudoconvex, and E has the approximation property then $M_\omega(U) = U$. He later shows [21] that if E is separable and has the bounded approximation property, $M_\delta(U) = U$.

We have just seen, for convex open U that if E is separable, $\pi(M_\delta(U)) = U$, without approximation property. We will see later that if E is non-separable, then one may have $\pi(M_\delta(U)) \neq U$, even with the bounded approximation property. Our examples will be in ℓ_∞, and will take us to study bounding subsets for $\mathcal{H}(\ell_\infty)$ and for $\mathcal{H}(B_{\ell_\infty})$.

We will need the following definition.

Definition 2. A subset A of U will be called U-bounding if all $f \in \mathcal{H}(U)$ are bounded on A. We will write $\|f\|_A = \sup_A |f|$.

For completeness, we present now our results on spectra, using the following characterization, Theorem 3.3,—which we will prove in the next section—of sets which are bounding with respect to $\mathcal{H}(B_{\ell_\infty})$.

A subset $A \subset B_{\ell_\infty}$ is B_{ℓ_∞}-bounding if and only if the following two conditions hold:

i) *there is an $0 < r < 1$ such that $A \subset rB_{\ell_\infty}$, and*
ii) *A is a bounding set for $\mathcal{H}(\ell_\infty)$.*

We then have the following.

Proposition 2.8. i) *$M_\delta(\ell_\infty) \neq \ell_\infty$; in fact, $\pi(M_\delta(\ell_\infty)) \neq \ell_\infty$, and*
ii) *$M_\delta(B_{\ell_\infty}) \neq B_{\ell_\infty}$; in fact, $\pi(M_\delta(B_{\ell_\infty})) \neq B_{\ell_\infty}$.*

Proof. We are going to construct a τ_δ-continuous character φ of $\mathcal{H}(B_{\ell_\infty})$ such that $\pi(\varphi)$ is not an evaluation on ℓ_∞.

Since any bounded set in c_0 is bounding for ℓ_∞ by Josefson's general result [16, Corollary 2], if we take $v_n = \sum_{i=1}^n re_i$ with $0 < r < 1$, then $\{v_n : n \in \mathbf{N}\}$ is bounding for $\mathcal{H}(B_{\ell_\infty})$ by Theorem 3.3 below. Take $\mathcal{U}$ an ultrafilter on $\mathbf{N}$ containing $\{\{n, n+1, \dots\}, n = 1, 2, \dots\}$. Then we can define

$$\varphi(f) = \lim_{\mathcal{U}} f(v_n),$$

for $f \in \mathcal{H}(B_{\ell_\infty})$.

First, we check that $\pi(\varphi)$ is not in c_0: consider $e_k \in \ell_1 \subset \ell'_\infty$. We have

$$\pi(\varphi)_k = \pi(\varphi)(e_k) = \varphi(e_k) = \lim_{\mathcal{U}} e_k(v_n) = r, \quad \text{for all } k.$$

But $\pi(\varphi)$ is not $(r, r, r, \dots)$ either: consider an element $\gamma \in \ell'_\infty$ (this is a holomorphic function on ℓ_∞ and on its unit ball) such that $\gamma((r, r, r, \dots)) = 1$, but $\gamma \equiv 0$ when restricted to c_0. Then

$$\pi(\varphi)(\gamma) = \varphi(\gamma) = \lim_{\mathcal{U}} \gamma(v_n) = \lim_{\mathcal{U}} 0 = 0 \neq 1 = \gamma((r, r, r, \dots)).$$

Hence $\pi(\varphi) \notin \ell_\infty$. □

3. On U-bounding subsets of ℓ_∞

Dineen devoted a deep paper [11] to the study of ℓ_∞-bounding sets, that is, bounding sets for entire functions on ℓ_∞. His main result [11, Theorem 1] is that the canonical basis $A = \{e_n\}_{n \in \mathbf{N}}$ of c_0 is an ℓ_∞-bounding set. He also proved that any bounded sequence whose terms have disjoint supports is bounding.

We characterize RB_{ℓ_∞}-bounding subsets. We first need a series of results.

In the next Proposition we follow Dineen ([11, Proposition 2], see also [12, Example 3.20 (c)]).

Proposition 3.1. *Let U be a balanced subset of a Banach space E. A subset A of U is U-bounding if and only if for each $f \in \mathcal{H}(U)$,*

$$\lim_{n\to\infty} n^2\|P_n\|_A = 0 \ ,$$

where P_n is the n-homogeneous term in the Taylor series of f about 0.

Proof. $\Rightarrow$) Consider $\beta_n = n^3$. Note that $\lim_n(\beta_n)^{\frac{1}{n}} = 1$. Then

$$f_\beta(x) = \sum_{n=0}^{\infty} \beta_n P_n(x)$$

defines a holomorphic function on U by [10, Proposition 3.15]. Also, for each compact subset K of U, there is a constant $c_{\beta,K}$ such that

$$|\beta_n|\|P_n\|_K = \left\|\frac{1}{n!}\widehat{D^n f_\beta(0)}\right\|_K \le c_{\beta,K} < \infty.$$

Thus $\{\beta_n\, P_n \colon n \in \mathbf{N}\}$ is bounded for the τ_0 topology. But since A is U-bounding, $\|f\|_A$ is a τ_δ-continuous seminorm (see Proposition 3.18 and the proof of Example 3.20 (c) in [12]). Since the τ_δ topology has the same bounded sets as the τ_0 topology, $\{\beta_n\, P_n \colon n \in \mathbf{N}\}$ is bounded for the τ_δ topology. Hence, for some $c_{\beta,A}$,

$$|\beta_n|\|P_n\|_A = nn^2\|P_n\|_A \le c_{\beta,A} < \infty \ \text{ for all } n,$$

so $n^2\,\|P_n\|_A \le \frac{c_{\beta,A}}{n}$ for all n, and letting $n \to \infty$,

$$\lim_{n\to\infty} n^2\|P_n\|_A = 0.$$

$\Leftarrow$) For any $x \in A$,

$$|f(x)| \le \sum_{n=0}^{\infty} |P_n(x)| \le \sum_{n=0}^{\infty} \|P_n\|_A \le M \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \qquad \square$$

Lemma 3.2. *Let $A \subset B_{\ell_\infty}$ such that* $\sup\{||x|| \colon x \in A\} = 1$. *Then A is not a bounding set for $\mathcal{H}(B_{\ell_\infty})$.*

Proof. We prove first that if $f = \sum_n P_n \in \mathcal{H}(B_{\ell_\infty})$ and $C \subset B_{\ell_\infty}$ is a bounding set, then

$$\sup\{\|g\|_C \colon g \in F_f\} < \infty, \ \text{where } F_f = \left\{\sum_n \beta_n P_n \colon \beta \in \overline{B_{\ell_\infty}}\right\}.$$

The proof is a trivial consequence of Proposition 3.1. Since C is bounding, then

$$\lim_n n^2\|P_n\|_C = 0 \ .$$

In our case, $\|\beta_n P_n\|_C \le \|P_n\|_C < \infty$ for all n. Thus

$$\left\|\sum_n \beta_n P_n\right\|_C \le \sum_n |\beta_n|\|P_n\|_C \le \sum_n \|P_n\|_C \le M \sum_n \frac{1}{n^2} < \infty,$$

where $M = \sup_n n^2\|P_n\|_C < \infty$. In other words, if we consider in $\mathcal{H}(B_{\ell_\infty})$ the locally convex topology of uniform convergence on bounding sets, the family F_f is bounded.

By hypothesis we can find a sequence $(x_n) \subset A \subset B_{\ell_\infty}$ such that

$$1 \geq ||x_n|| > (\frac{n}{n+1})^{1/n},$$

for every $n \in \mathbf{N}$. We write

$$x_n = (x_{1,n}, x_{2,n}, \ldots, x_{k,n}, \ldots) \text{ for } n \geq 1.$$

For each n we can find a coordinate k_n such that

$$|x_{k_n,n}| > (\frac{n}{n+1})^{1/n}.$$

We have now two possibilities. The set $K = \{k_n : n \in \mathbf{N}\}$ is bounded or not. i) If K is a bounded set, we can find $s \in K$ and a subsequence $(x_{n_l})_{l=1}^{\infty}$ such that $k_{n_l} = s$ for every l. Thus, by taking again a subsequence if necessary, there is $a \in \mathbf{C}$ such that $|a| = 1$ and the sequence $(x_{s,n_l})_{l=1}^{\infty}$ converges to a; then the function $f(z) = \frac{1}{z_s - a}$, which is in $\mathcal{H}(B_{\ell_\infty})$ is unbounded on A, a contradiction, or

ii) K is not a bounded set. In this case, taking in account that the sequence $(\frac{n}{n+1})^{1/n}$ is strictly increasing, we can assume passing to an appropriate subsequence that $k_{n+1} > k_n$ for every n. Now let

$$f(z) = \sum_{h=1}^{\infty} h z_{k_h}^h.$$

Then $f \in H_b(B_{\ell_\infty})$. If we assume that A is bounding then

$$\sup\{\|g\|_A : g \in F_f\} < \infty.$$

But if, for each $(h, n) \in \mathbf{N}^2$, we take $\lambda_{h,n} \in \mathbf{C}$ such that $|\lambda_{h,n}| = 1$ and $\lambda_{h,n}(x_{k_h,n})^h = |x_{k_h,n}|^h$ and, for each n, we define $g_n \in F_f$, by

$$g_n(z) = \sum_{h=1}^{\infty} \lambda_{h,n} h z_{k_h}^h,$$

we have,

$$g_n(x_n) = \sum_{h=1}^{\infty} h|x_{k_h,n}|^h \geq n|x_{k_n,n}|^n > n\left(\frac{n}{n+1}\right)^{n/n} = \frac{n^2}{n+1},$$

so $\sup_{g \in F_f} \|g\| \geq \sup_{n \in \mathbf{N}} \frac{n^2}{n+1} = \infty$, a contradiction. Thus A is not bounding. □

We have therefore the following characterization of B_{ℓ_∞}-bounding sets.

Theorem 3.3. *A subset $A \subset RB_{\ell_\infty}$ is RB_{ℓ_∞}-bounding if and only if the following two conditions hold:*

i) *there is an $0 < r < 1$ such that $A \subset rRB_{\ell_\infty}$, and*

ii) *A is a bounding set for $\mathcal{H}(\ell_\infty)$.*

Proof. It is clearly enough to consider $R = 1$. Recall [16, Theorem 1] that a bounded set $D \subset \ell_\infty$ is a bounding set for $\mathcal{H}(\ell_\infty)$ if and only if it is strongly bounding, i.e., D is bounding for $\mathcal{H}(cB_{\ell_\infty})$ for every $c > \sup_{z \in D} \|z\|$.

Now, if $A \subset B_{\ell_\infty}$ is $\mathcal{H}(B_{\ell_\infty})$-bounding, by the Lemma above for some $0 < r < 1$ we will have $A \subset rB_{\ell_\infty}$. A is trivially ℓ_∞-bounding.

If A is ℓ_∞-bounding, and $\|z\| < r$ for all $z \in A$, since $r < 1$, A is B_{ℓ_∞}-bounding by Josefson's Theorem. □

Note, for example, that if $0 < r_k < 1$ with $\lim_k r_k = 1$, then $\{r_k e_k : k \in \mathbf{N}\}$ is not B_{ℓ_∞}-bounding. In fact, it is not even bounding with regard to functions in $H_b(B_{\ell_\infty})$. Indeed, since $\lim_s r_s^k = 1$, we can find $1 \leq n_1 < n_2 < \ldots < n_k < \ldots$ such that $r_{n_k}^k > 1/2$ for all k. Consider

$$g(x) = \sum_k k x_{n_k}^k.$$

If $\|x\| \leq t < 1$, then $|k x_{n_k}^k| \leq k t^k$, and $\sum_k k x_{n_k}^k$ converges uniformly on tB_{ℓ_∞} for each $t < 1$. Thus $g \in H_b(B_{\ell_\infty})$. But

$$g(r_{n_k} e_{n_k}) = k r_{n_k}^k > k/2,$$

so $\{r_k e_k : k \in \mathbf{N}\}$ is not a bounding set with regard to functions in $H_b(B_{\ell_\infty})$.

We remark that Theorem 3.3 is not valid for general U, bounded convex and balanced open subsets of ℓ_∞.

Indeed, Lempert [17] constructed a bounded convex and balanced open subset U of ℓ_∞ containing $A = \{e_n : n \in \mathbf{N}\}$ and such that A is not U-bounding. He defined a norm $|||\cdot|||$ on ℓ_∞ by

$$|||z||| = \frac{2}{3} \sup_{j_1 < j_2 < j_3} |z_{j_1}| + |z_{j_2}| + |z_{j_3}|, \quad z = (z_1, z_2, \ldots) \in \ell_\infty.$$

This norm is equivalent to the sup norm, in fact if U is its open unit ball,

$$\frac{1}{2} B_{\ell_\infty} \subset U \subset \frac{3}{2} B_{\ell_\infty}.$$

Observe that $|||e_n||| = \frac{2}{3}$ for all $n \in \mathbf{N}$, so

$$A \subset \frac{2}{3}\overline{U} \subset rU \quad \text{for} \quad \frac{2}{3} < r < 1,$$

however, Lempert showed that the function $f(z) = \sum_{j=1}^\infty j z_j^j$ is holomorphic on U and $\sup_n |f(e_n)| = \infty$.

Recall that an entire function f on c_0 admits a holomorphic extension to ℓ_∞ if and only if f is of bounded type on c_0. Also, by our previous results, a function $f \in \mathcal{H}(B_{c_0})$ admits a holomorphic extension to B_{ℓ_∞} if and only if f is of bounded type on B_{c_0}. In contrast, consider $U_0 = U \cap c_0$, and Lempert's function $f(x) = \sum_{j=1}^\infty j x_j^j$ on U_0. Then f is not of bounded type on U_0, but can be extended to U. In fact, the $f(z)$ written above is its Aron–Berner extension.

The set $A = \{e_n\}_{n \in \mathbf{N}}$ is RB_{ℓ_∞}-bounding, for every $R > 1$ [16]. Thus, one could construct continuous characters on $(\mathcal{H}(RB_{\ell_\infty}), \tau_\delta)$ using ultrafilters, as we did in Proposition 2.8, i.e,

$$\varphi(f) = \lim_{\mathcal{U}} f(e_n).$$

But this approach will always produce the evaluation at 0. This is a consequence of the well-known fact that every holomorphic function on RB_{ℓ_∞} is weakly continuous when restricted to rB_{c_0} for $0 < r < R$. That can be obtained, for example, by applying that rB_{c_0} is a bounding set for $\mathcal{H}(RB_{\ell_\infty})$ (1978 Josefson [16]), that every continuous polynomial on c_0 is weakly continuous when restricted to bounded sets (1957 Bogdanowicz [12, Proposition 1.59]) and [12, Lemma 4.50]. This can also be obtained, at least for $H(\ell_\infty)$, as follows. By Josefson's result every $f \in H(\ell_\infty)$ is bounded on limited sets; and by Carrión et al. [6, Proposition 3.3], f is weakly continuous on such sets. Since $\{e_n\}$ is weakly null, $\varphi(f) = \lim f(e_n) = f(0)$.

We will prove here a stronger result. Namely that the sequence $(f(e_n) - f(0))$ belongs to the space of summable sequences ℓ_1 for every $f \in \mathcal{H}(RB_{\ell_\infty})$. Aron and Globevnik proved this claim for the case of homogeneous continuous polynomials on c_0 in [4]. Lempert showed us that this statement is true for the case of an entire function f on ℓ_∞ during a visit he made to Valencia in 2006. We present here a slight improvement of that result. For that we need two lemmata.

Lemma 3.4. *Let E be a Banach space, $x_1, \dots, x_n \in E$ and $S > 1$. If f is an entire function on E such that $f(0) = 0$, then there exist $\theta_1, \dots, \theta_n \in [0, 2\pi]$ such that*

$$|f(x_1)| + \cdots + |f(x_n)| \leq \frac{1}{S-1} |f(S(e^{i\theta_1} x_1 + \cdots + e^{i\theta_n} x_n))|$$

.

Proof. For each $m \geq 1$, let $(r_{m,j})_{j=1}^{\infty}$ be the generalized Rademacher functions such that

$$\mathbf{E}(r_{m,j_1} \dots r_{m,j_m}) = \begin{cases} 1 & \text{if } j_1 = \cdots = j_m, \\ 0 & \text{otherwise,} \end{cases}$$

$$|f(x_1)| + \cdots + |f(x_n)| = \sum_{k=1}^{n} |\sum_{m=1}^{\infty} P_m(x_k)| \leq \sum_{m=1}^{\infty} \sum_{k=1}^{n} |P_m(x_k)|$$

$$= \sum_{m=1}^{\infty} \sum_{k=1}^{n} P_m(\alpha_{m,k} x_k) = \sum_{m=1}^{\infty} \int_{\Omega} P_m(r_{m,1}\alpha_{m,1} x_1 + \cdots + r_{m,n}\alpha_{m,n} x_n) d\mu,$$

for suitable $(\alpha_{m,k})$ complex numbers of absolute value one. Then

$$|f(x_1)| + \cdots + |f(x_n)| \leq \sum_{m=1}^{\infty} \|P_m\|_A = \sum_{m=1}^{\infty} \frac{1}{S^m} \|P_m\|_{SA} \leq \frac{1}{S-1} \|f\|_{SA},$$

where $A = \{\lambda_1 x_1 + \cdots + \lambda_n x_n \colon |\lambda_j| \leq 1\}$. Observe that SA is a compact convex and balanced subset of X. The function $g \colon \mathbf{C}^n \to \mathbf{C}$ defined by

$$g(\lambda_1, \dots, \lambda_n) = f(\lambda_1 x_1 + \cdots + \lambda_n x_n),$$

is clearly an entire function on $\mathbf{C}^n$ and

$$\|g\|_B = \|f\|_{SA}$$

where B is the closed polydisk centered at 0 and of radius $(S, \dots, S)$. The conclusion now follows from the maximum modulus theorem for the polydisk. □

Lemma 3.5. *Let $x_1, \dots, x_n \in B_{\ell_\infty}$ with disjoint supports and $S > 1$ such that $S\|x_j\| < 1$ for $j = 1, \dots, n$. If f is a holomorphic function on B_{ℓ_∞} such that $f(0) = 0$, then there exist $\theta_1, \dots, \theta_n \in [0, 2\pi]$ such that*

$$|f(x_1)| + \cdots + |f(x_n)| \leq \frac{1}{S-1} |f(S(e^{i\theta_1} x_1 + \cdots + e^{i\theta_n} x_n))|.$$

Proof. Observe that $\{\lambda_1 x_1 + \cdots + \lambda_n x_n \colon |\lambda_j| \leq S\}$ is a compact subset of B_{ℓ_∞}. Denote $Q_m = \sum_{k=1}^{m} P_k$, for each $m \geq 1$. By Lemma 3.4 there exist $\theta_1^m, \dots, \theta_n^m \in [0, 2\pi]$ such that

$$|Q_m(x_1)| + \cdots + |Q_m(x_n)| \leq \frac{1}{S-1} |Q_m(S(e^{i\theta_1^m} x_1 + \cdots + e^{i\theta_n^m} x_n))|,$$

for all m. Considering a subsequence (m_h) such that $\theta_j^{m_h}$ is convergent to some $\theta_j \in [0, 2\pi]$, for $j = 1, \dots, n$, by taking limits in both sides of above inequality we get

$$|f(x_1)| + \cdots + |f(x_n)| \leq \frac{1}{S-1}|f(S(e^{i\theta_1}x_1 + \cdots + e^{i\theta_n}x_n))|. \qquad \square$$

Theorem 3.6. *If (x_n) is a bounded sequence in ℓ_∞ with disjoint supports and $R > \sup_n \|x_n\|$, then $(f(x_n) - f(0)) \in \ell_1$ for every holomorphic function f on RB_{ℓ_∞}.*

Proof. Clearly it is enough to prove that $(f(x_n) - f(0)) \in \ell_1$ when f is a holomorphic function on B_{ℓ_∞} such that $f(0) = 0$ and (x_n) is a sequence in B_{ℓ_∞} with disjoint supports and $\sup_n \|x_n\| < 1$. If this is not true, then there exist a holomorphic function f on B_{ℓ_∞} with $f(0) = 0$ and a sequence (x_n) in ℓ_∞ with disjoint supports and $\sup_n \|x_n\| < 1$ such that $\sum_{n=1}^{\infty} |f(x_n)|$ diverges. Then we can find a subsequence (p_j) with $p_1 = 1$, such that

$$\sum_{n=p_j}^{p_{j+1}-1} |f(x_n)| > j.$$

We take $S > 1$ such that $M := S \sup_n \|x_n\| < 1$. By Lemma 3.5 there exists for each j a finite sequence $\lambda_{n,j} \in \mathbf{C}$ with $n = p_j, \dots, p_{j+1} - 1$ such that $|\lambda_{n,j}| = S$ for all n and j and such that

$$j < \sum_{n=p_j}^{p_{j+1}-1} |f(x_n)| \leq \frac{1}{S-1}|f(u_j)|,$$

where

$$u_j = \sum_{n=p_j}^{p_{j+1}-1} \lambda_{n,j} x_n \in B_{\ell_\infty}$$

for all j. That shows that the sequence (u_j) contained in MB_{ℓ_∞} and of disjoint supports in not B_{ℓ_∞}-bounding. This is a contradiction by Theorem 3.3, and Dineen's result regarding ℓ_∞-bounding sets ([11], Theorem 1 and Comment (1)). $\square$

In the following result for B_{ℓ_∞} we follow closely Dineen's proof of the analogous result for ℓ_∞ [11]. Thus we present a direct proof which does not require the use of our characterization (Theorem 3.3) or Josefson's results.

Theorem 3.7. *If $(v_n) \subset B_{\ell_\infty}$ is a sequence of elements with disjoint supports and $\sup_n \|v_n\| < 1$, then $A = \{v_n\}$ is B_{ℓ_∞}-bounding.*

Proof. We consider first $v_n = r_n e_n$, where $0 < r_n < 1$ with $\sup_n r_n < 1$. If $A = \{r_n e_n\}$ were not bounding, by Proposition 3.1 there would exist $f \in \mathcal{H}(B_{\ell_\infty})$, a subsequence (n_j) of the positive integers and $\varepsilon > 0$ with

$$n_j^2 \|P_{n_j}\|_A > \varepsilon,$$

(where the P_k's are the homogeneous continuous polynomials in the Taylor series expansion of f at zero) and thus for each $j \in \mathbf{N}$, there are n_j and m_j such that

$$n_j^2 |P_{n_j}(r_{m_j} e_{m_j})| > \varepsilon,$$

so $\frac{n_j^2}{\varepsilon}|P_{n_j}(r_{m_j} e_{m_j})| > 1$ for all j. For each j set

$$\beta_{n_j} = \frac{n_j^2}{\varepsilon} \frac{\overline{P_{n_j}(r_{m_j} e_{m_j})}}{|P_{n_j}(r_{m_j} e_{m_j})|}.$$

Note that $|\beta_{n_j}|^{\frac{1}{n_j}} = (\frac{n_j^2}{\varepsilon})^{\frac{1}{n_j}} \to 1$ as $j \to \infty$ we may define $f_\beta \in \mathcal{H}(B_{\ell_\infty})$ by setting

$$f_\beta(x) = \sum_{j=1}^{\infty} \beta_{n_j} P_{n_j}(x), \quad \text{for all } x \in B_{\ell_\infty}.$$

We have, for each j,

$$\frac{1}{n_j!}\widehat{D^{n_j} f_\beta(0)}(r_{m_j} e_{m_j}) = \beta_{n_j} P_{n_j}(r_{m_j} e_{m_j}) = \frac{n_j^2}{\varepsilon}|P_{n_j}(r_{m_j} e_{m_j})| > 1.$$

The proof now follows very closely Dineen's original proof ([11, Theorem 1]). We give it only for the sake of completeness.

If S is a subset of the positive integers $\mathbf{N}$, $\ell_\infty(S)$ denotes the subspace of ℓ_∞ consisting of those elements whose support lies in S. If S is infinite then $\ell_\infty(S)$ is a closed complemented subspace of ℓ_∞ isomorphic to ℓ_∞. In order to simplify notation we write

$$\|f\|_S = \sup_{x\in\ell_\infty(S),\ \|x\|\leq 1} |f(x)|$$

for any f defined on ℓ_∞.

Since we can replace ℓ_∞ by $\ell_\infty(S)$ we may assume without loss of generality that we have an element $f_\beta(x) = \sum_{j=1}^{\infty} \beta_{n_j} P_{n_j}(x)$ of $\mathcal{H}(B_{\ell_\infty})$ where $(n_j)_{j=1}^{\infty}$ is a subsequence of the positive integers and $\frac{1}{n_j!}\widehat{D^{n_j} f_\beta(0)}(r_j e_j) > 1$, for all j.

For each n_j let A_{n_j} be the symmetric n_j-linear mapping on ℓ_∞ such that $\hat{A}_{n_j} = \frac{1}{n_j!}\widehat{D^{n_j} f_\beta(0)}$.

Thus $\hat{A}_{n_1}(r_1 e_1) > 1$ and $\hat{A}_{n_1} \in \mathcal{P}(^{n_1}\ell_\infty)$. Let $k_1 = 1$ and by [11, Proposition 4] choose S_1 infinite such that $k_1 \notin S_1$ and

$$\sup_{|\lambda|\leq 1,\ \lambda\in\mathbf{C}} \sum_{0<r\leq n_1} \binom{n_1}{r} \|A_{n_1}(\lambda r_1 e_1)^{n_1 - r}\|_{S_1} \leq \frac{1}{n_1!}.$$

Suppose k_i and S_i have been chosen for $1 \leq i \leq l-1$. Choose $k_l \in S_{l-1}$ and $C_l = \{k_1, \ldots, k_l\}$. Take $S_l \subset S_{l-1}$ such that S_l does not contain $1, 2, \ldots, k_l$ and

$$\sup_{y\in\ell_\infty(C_l),\ \|y\|\leq 1} \sum_{r>0} \binom{n_{k_l}}{r} \|A_{n_{k_l}}(y)^{n_{k_l} - r}\|_{S_l} \leq \frac{1}{n_{k_l}!}, \tag{1}$$

since C_l is finite and we are taking the supremum over a compact set of a finite sum of continuous functions each one of them can be made small by [11, Proposition 4]. By restricting everything to $\ell_\infty(\mathfrak{S})$ where $\mathfrak{S} = \bigcup_{l=1}^{\infty}\{k_l\}$ we can suppose $k_l = l$. Let $C_l^\perp = \bigcup_{n=l+1}^{\infty}\{k_n\}$ then $C_l \bigcup C_l^\perp = \mathfrak{S}$. Let R_{n_l} denote the restriction of $\hat{A}_{n_l}$ to $\ell_\infty(C_l)$ and define $T_{n_l} \in \mathcal{P}(^{n_l}\ell_\infty(\mathfrak{S}))$ by

$$T_{n_l}(x+y) = R_{n_l}(x) \quad \text{for all } \ x \in \ell_\infty(C_l) \ \text{ and } \ y \in \ell_\infty(C_l^\perp)\ .$$

Now

$$\begin{aligned}\|\hat{A}_{n_l} - T_{n_l}\|_{\mathfrak{S}} &\leq \sup_{x\in\ell_\infty(C_l),\ y\in\ell_\infty(C_l^\perp),\ \|x\|=\|y\|=1} |A_{n_l}(x+y)^{n_l} - A_{n_l}(x)^{n_l}| \\ &\leq \sup_{x\in\ell_\infty(C_l),\ \|x\|\leq 1} \sum_{r>0} \binom{n_l}{r} \|A_{n_l}(y)^{n_l - r}\|_{S_l} \leq \frac{1}{n_l!},\end{aligned}$$

since $C_l^\perp \subset S_l$ and (1). Hence $\sum_{l=1}^{\infty} T_{n_l} \in \mathcal{H}(B_{\ell_\infty})$.

If $v_l = r_l e_l$, then

$$T_{n_l}(v_l) = R_{n_l}(v_l) = \hat{A}_{n_l}(v_l) = \hat{A}_{n_{k_l}}(r_{k_l} e_{k_l}) > 1.$$

For $\lambda \in \mathbf{C}$ and $y \in C_2^\perp$

$$T_{n_2}(v_1 + \lambda v_2 + y) = T_{n_2}(v_1 + \lambda v_2) = \sum_{i=0}^{n_2-1} \alpha_i(T_{n_2})\lambda^i + T_{n_2}(v_2)\lambda^{n_2}$$

where $\alpha_i(T_{n_2})$ are complex numbers independent of λ. By Cauchy's inequalities there exists $\lambda_2 \in \mathbf{C}$ $|\lambda_2| = 1$ such that for all $y \in C_2^\perp$

$$|T_{n_2}(v_1 + \lambda_2 v_2)| = |T_{n_2}(v_1 + \lambda_2 v_2 + y)| \geq 1.$$

Suppose $\lambda_2, \ldots, \lambda_k \in \mathbf{C}$ have been chosen such that $|\lambda_i| \leq 1$ and

$$|T_{n_i}(v_1 + \sum_{j=2}^{i} \lambda_j v_j + y)| \geq 1.$$

for all $y \in C_i^\perp$ and $i = 2, \ldots, k$. Then

$$T_{n_{k+1}}(v_1 + \sum_{j=2}^{k} \lambda_j v_j + \lambda v_{k+1} + y) = T_{n_{k+1}}(v_1 + \sum_{j=2}^{k} \lambda_j v_j + \lambda v_{k+1})$$
$$= \sum_{i=0}^{n_{k+1}-1} \alpha_i(T_{n_{k+1}})\lambda^i + T_{n_{k+1}}(v_{k+1})\lambda^{n_{k+1}},$$

where $\alpha_i(T_{n_{k+1}})$ are complex numbers independent of λ. By Cauchy's inequalities there exists $\lambda_{k+1} \in \mathbf{C}$, $|\lambda_{k+1}| \leq 1$, such that for all $y \in C_{k+1}^\perp$

$$|T_{n_{k+1}}(v_1 + \sum_{j=2}^{k+1} \lambda_j v_j + y)| \geq 1.$$

If $x_0 = (r_1, \lambda_2 r_2, \lambda_3 r_3, \ldots, \lambda_n r_n, \ldots)$, then $\|x_0\| \leq \sup_n r_n < 1$ and $|T_{n_{k+1}}(x_0)| \geq 1$, hence the sequence $(T_{n_{k+1}}(x_0))_k$ is not a null sequence which contradicts the fact $\sum_{k=1}^\infty T_{n_k} \in \mathcal{H}(B_{\ell_\infty})$.

For the general case, if the v_n's have disjoint supports, set $r_n = \|v_n\|$. Then $\{\|v_n\| e_n\}$ is bounding. Define $L\colon B_{\ell_\infty} \to B_{\ell_\infty}$ by

$$L(a) = \sum_{k=1}^{\infty} a_k \frac{v_k}{\|v_k\|}.$$

Now for any $f \in \mathcal{H}(B_{\ell_\infty})$,

$$|f(v_n)| = |f(L(\|v_n\| e_n))| = |(f \circ L)(\|v_n\| e_n)| \leq c \text{ for all } n. \qquad \square$$

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