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FACTORING DERIVATIVES OF FUNCTIONS IN THE NEVANLINNA AND SMIRNOV CLASSES

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Abstract. We prove that, given a function f in the Nevanlinna class $\mathcal{N}$ and a positive integer n, there exist $g \in \mathcal{N}$ and $h \in \mathrm{BMOA}$ such that $f^{(n)} = gh^{(n)}$. We may choose g to be zero-free, so it follows that the zero sets for the class $\mathcal{N}^{(n)} := \{f^{(n)} \colon f \in \mathcal{N}\}$ are the same as those for $\mathrm{BMOA}^{(n)}$. Furthermore, while the set of all products $gh^{(n)}$ (with g and h as above) is strictly larger than $\mathcal{N}^{(n)}$, we show that the gap is not too large, at least when $n = 1$. Precisely speaking, the class $\{gh' \colon g \in \mathcal{N},\, h \in \mathrm{BMOA}\}$ turns out to be the smallest ideal space containing $\{f' \colon f \in \mathcal{N}\}$, where "ideal" means invariant under multiplication by H^∞ functions. Similar results are established for the Smirnov class $\mathcal{N}^+$.

1. Introduction and results

Let $\mathcal{H}(\mathbf{D})$ stand for the set of holomorphic functions on the disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$. Given a class $X \subset \mathcal{H}(\mathbf{D})$ and an integer $n \in \mathbf{N} := \{1, 2, \dots\}$, we write

$$X^{(n)} := \{f^{(n)} \colon f \in X\},$$

where $f^{(n)}$ is the nth derivative of f. When $n = 1$, we also use the notation X' instead of $X^{(1)}$. Further, we denote by $\mathcal{Z}(X)$ the collection of zero sets for X; a (discrete) subset E of $\mathbf{D}$ will thus belong to $\mathcal{Z}(X)$ if and only if $E = \{z \in \mathbf{D} \colon f(z) = 0\}$ for some non-null function $f \in X$. Now, if X and Y are subclasses of $\mathcal{H}(\mathbf{D})$, we put

$$X \cdot Y := \{fg \colon f \in X,\, g \in Y\}.$$

Finally, a vector space X contained in $\mathcal{H}(\mathbf{D})$ is said to be *ideal* if

$$H^\infty \cdot X \subset X,$$

where, as usual, H^∞ is the space of bounded holomorphic functions on $\mathbf{D}$.

Our starting point is a result of Cohn and Verbitsky [3] which asserts, or rather implies, that

$$(H^p)^{(n)} = H^p \cdot \mathrm{BMOA}^{(n)} \tag{1.1}$$

whenever $0 < p < \infty$ and $n \in \mathbf{N}$. Here, we write H^p for the classical (holomorphic) *Hardy spaces* on the disk, and BMOA for the "analytic subspace" of $\mathrm{BMO} = \mathrm{BMO}(\mathbf{T})$, the space of functions with *bounded mean oscillation* on the circle $\mathbf{T} := \partial\mathbf{D}$. Precisely speaking, BMOA can be defined as $H^1 \cap \mathrm{BMO}$; as to the definitions of (and background information on) H^p and BMO, the reader will find these standard matters in [5, Chapters II and VI].

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For $n = 1$, identity (1.1) was first obtained in Cohn's earlier paper [2]. It was then extended in [3] to higher order (possibly fractional) derivatives and still further; indeed, more general factorization theorems involving tent spaces—and Triebel spaces—were actually established there. It was also shown in [3] that, when factoring $f^{(n)}$ for $f \in H^p$ in the sense of (1.1), one may choose the H^p factor on the right to be an outer function. As a consequence, one sees that

$$\mathcal{Z}\left((H^p)^{(n)}\right) = \mathcal{Z}\left(\mathrm{BMOA}^{(n)}\right). \tag{1.2}$$

In particular, for any fixed n, the zero sets for $(H^p)^{(n)}$ are the same for all $p \in (0, \infty)$. This last fact was contrasted in [3] with the Bergman space situation, where different A^p spaces happen to have different zero sets; see [7]. We wish to add, in this connection, that a similar Bergman-type phenomenon (different zero sets for different p's) is also encountered in certain "small" H^p-related spaces; namely, it occurs [4] for the *star-invariant subspaces* $H^p \cap \theta\,\overline{H_0^p}$ associated with an inner function θ.

Also related to (1.1), in the case $n = 1$, is Aleksandrov and Peller's work from [1]. There, for a given $f \in H^p$, a *weak factorization* $f' = \sum_{j=1}^m g_j h_j'$ was constructed with suitable $g_j \in H^p$ and $h_j \in H^\infty$. This was done with $m = 2$ for $1 < p < \infty$, with $m = 4$ for $p = 1$, and with a certain larger m for $0 < p < 1$. Yet another weak factorization theorem from [1], which establishes a connection between BMOA' and $(H^\infty)'$, will be employed in Section 4 below.

The purpose of this paper is to find out whether—and/or to which extent—the (strong) factorization theorem (1.1) carries over to the *Nevanlinna class* $\mathcal{N}$, or the *Smirnov class* $\mathcal{N}^+$, in place of H^p.

Let us recall that $\mathcal{N}$ is defined as the set of functions $f \in \mathcal{H}(\mathbf{D})$ with

$$\sup_{0<r<1} \int_{\mathbf{T}} \log^+ |f(r\zeta)|\,|d\zeta| < \infty,$$

while $\mathcal{N}^+$ is formed by those $f \in \mathcal{N}$ which satisfy

$$\lim_{r \to 1^-} \int_{\mathbf{T}} \log^+ |f(r\zeta)|\,|d\zeta| = \int_{\mathbf{T}} \log^+ |f(\zeta)|\,|d\zeta|.$$

Equivalently, the elements of $\mathcal{N}$ (resp., $\mathcal{N}^+$) are precisely the ratios u/v, with $u, v \in H^\infty$ and with v nonvanishing (resp., outer) on $\mathbf{D}$; for this and other characterizations of the two classes, see [5, Chapter II].

As far as factorization theorems of the form (1.1) are concerned, we can hardly expect the behavior of $\mathcal{N}$ or $\mathcal{N}^+$ to mimic that of H^p too closely. In fact, as we shall soon explain, it is the "easy" part of (1.1), i.e., the inclusion

$$(H^p)^{(n)} \supset H^p \cdot \mathrm{BMOA}^{(n)} \tag{1.3}$$

that admits no extension to the Nevanlinna or Smirnov setting. Meanwhile, we remark that (1.3) is indeed easy to deduce, at least for $p = 2$, from the (not so easy, but readily available) descriptions of $(H^p)^{(n)}$ and $\mathrm{BMOA}^{(n)}$ as the appropriate Triebel spaces; see [11]. One of these tells us that, for $\varphi \in \mathcal{H}(\mathbf{D})$,

$$\varphi \in (H^p)^{(n)} \iff \int_{\mathbf{T}} \left(\int_0^1 |\varphi(r\zeta)|^2 (1-r)^{2n-1} dr \right)^{p/2} |d\zeta| < \infty$$

for all $n \in \mathbf{N}$ and $0 < p < \infty$, a fact that has no counterpart for $\mathcal{N}$ or $\mathcal{N}^+$. The other, which involves a Carleson measure characterization of BMOA, will be mentioned in Section 2 below.

Now, to see that the $\mathcal{N}$ and $\mathcal{N}^+$ versions of (1.3) actually break down, already for $n = 1$, one may recall results of Hayman [6] and Yanagihara [12] saying that neither $\mathcal{N}$ nor $\mathcal{N}^+$ is invariant with respect to integration. More precisely, Hayman gave an example of a function $f \in \mathcal{N}$ whose antiderivative $F(z) := \int_0^z f(\zeta)\, d\zeta$ is not in $\mathcal{N}$, and Yanagihara refined this by showing that F need not be in $\mathcal{N}$ even for $f \in \mathcal{N}^+$. Consequently, $\mathcal{N}^+$ is not contained in $\mathcal{N}'$, whence *a fortiori*

$$\mathcal{N} \not\subset \mathcal{N}' \quad \text{and} \quad \mathcal{N}^+ \not\subset (\mathcal{N}^+)'. \tag{1.4}$$

Since $\mathcal{N} \cdot \mathrm{BMOA}'$ (resp., $\mathcal{N}^+ \cdot \mathrm{BMOA}'$) contains $\mathcal{N}$ (resp., $\mathcal{N}^+$), we readily deduce from (1.4) that

$$\mathcal{N} \cdot \mathrm{BMOA}' \not\subset \mathcal{N}' \quad \text{and} \quad \mathcal{N}^+ \cdot \mathrm{BMOA}' \not\subset (\mathcal{N}^+)'. \tag{1.5}$$

A similar conclusion holds for higher order derivatives as well.

We prove, however, that the "difficult" part of (1.1), i.e., the inclusion

$$(H^p)^{(n)} \subset H^p \cdot \mathrm{BMOA}^{(n)} \tag{1.6}$$

does remain valid with either $\mathcal{N}$ or $\mathcal{N}^+$ in place of H^p.

Theorem 1.1. *For each $n \in \mathbf{N}$, we have*

$$\mathcal{N}^{(n)} \subset \mathcal{N} \cdot \mathrm{BMOA}^{(n)} \quad \textit{and} \quad (\mathcal{N}^+)^{(n)} \subset \mathcal{N}^+ \cdot \mathrm{BMOA}^{(n)}.$$

Moreover, given $f \in \mathcal{N}$ (resp., $f \in \mathcal{N}^+$), one can find a zero-free function $g \in \mathcal{N}$ (resp., an outer function $g \in \mathcal{N}^+$) and an $h \in \mathrm{BMOA}$ such that $f^{(n)} = gh^{(n)}$.

It should be mentioned that our method also applies to the meromorphic Nevanlinna class $\mathcal{N}_{\mathrm{mer}}$, defined as the set of quotients u/v, where $u, v \in H^\infty$ and v is merely required to be non-null. In fact, a glance at our proof of Theorem 1.1 will reveal that if the original function f is of the form F/I, with $F \in \mathcal{N}^+$ and I inner, then we may take $g = G/I^{n+1}$, with G outer. And again, just as in the H^p setting, our factorization theorem yields information on the zero sets.

Corollary 1.2. *We have*

$$\mathcal{Z}\left(\mathcal{N}^{(n)}\right) = \mathcal{Z}\left(\mathrm{BMOA}^{(n)}\right), \quad n \in \mathbf{N}.$$

Indeed, Theorem 1.1 shows that every zero set for $\mathcal{N}^{(n)}$ is a zero set for $\mathrm{BMOA}^{(n)}$, while the converse is immediate from the fact that $\mathrm{BMOA} \subset \mathcal{N}$. Furthermore, since $\mathcal{N}^+$ lies between BMOA and $\mathcal{N}$, as does every H^p with $0 < p < \infty$, Corollary 1.2 obviously implies the identity

$$\mathcal{Z}\left((\mathcal{N}^+)^{(n)}\right) = \mathcal{Z}\left(\mathrm{BMOA}^{(n)}\right)$$

and also (1.2).

Finally, restricting ourselves to the case $n = 1$, we wish to take a closer look at the inclusion

$$\mathcal{N}' \subset \mathcal{N} \cdot \mathrm{BMOA}'$$

from Theorem 1.1, along with its $\mathcal{N}^+$ counterpart. We know from (1.5) that the inclusion is proper, and we now stress an important point of distinction between the two sides. Namely, the right-hand side, $\mathcal{N} \cdot \mathrm{BMOA}'$, is ideal (i.e., invariant under multiplication by H^∞ functions), whereas the left-hand side, $\mathcal{N}'$, is not. Moreover, the space $\mathcal{N}'$ is *highly nonideal* in the sense that even the identity function z is not a multiplier thereof! (Otherwise, the formula

$$g = (zg)' - zg', \quad g \in \mathcal{N},$$

would imply that $\mathcal{N}$ is contained in $\mathcal{N}'$, which we know is false.) A similar remark applies to $(\mathcal{N}^+)'$.

Our last result states, then, that $\mathcal{N} \cdot \mathrm{BMOA}'$ is actually the smallest ideal space containing $\mathcal{N}'$, and that the same is true in the $\mathcal{N}^+$ setting.

Theorem 1.3. (a) *The class $\mathcal{N} \cdot \mathrm{BMOA}'$ is the ideal hull of $\mathcal{N}'$. In other words, $\mathcal{N} \cdot \mathrm{BMOA}'$ is an ideal vector space that contains $\mathcal{N}'$ and is contained in every ideal space X with $\mathcal{N}' \subset X$.*

(b) *Similarly, $\mathcal{N}^+ \cdot \mathrm{BMOA}'$ is the ideal hull of $(\mathcal{N}^+)'$.*

Now let us turn to the proofs.

2. Preliminaries

A couple of lemmas will be needed.

Lemma 2.1. *Let $k \ge 0$ and $l \ge 1$ be integers. If $\varphi \in \mathrm{BMOA}^{(l)}$ and ψ is a function in $\mathcal{H}(\mathbf{D})$ satisfying*

$$\psi(z) = O((1-|z|)^{-k}), \quad z \in \mathbf{D}, \tag{2.1}$$

then $\varphi\psi \in \mathrm{BMOA}^{(k+l)}$.

Proof. It is known (see, e.g., [8, 10, 11]) that a function $F \in \mathcal{H}(\mathbf{D})$ will be in $\mathrm{BMOA}^{(n)}$, with $n \in \mathbf{N}$, if and only if the measure $|F(z)|^2(1-|z|)^{2n-1}\,dx\,dy$ (where $z = x+iy$) is a Carleson measure. The required result follows from this immediately, since (2.1) yields

$$|\varphi(z)\psi(z)|^2(1-|z|)^{2(k+l)-1} \le \mathrm{const} \cdot |\varphi(z)|^2(1-|z|)^{2l-1}$$

for all $z \in \mathbf{D}$. □

When $k = 0$, the above lemma reduces to saying that

$$H^\infty \cdot \mathrm{BMOA}^{(n)} \subset \mathrm{BMOA}^{(n)} \tag{2.2}$$

for all $n \in \mathbf{N}$; in other words, $\mathrm{BMOA}^{(n)}$ is an ideal space. This in turn leads to the next observation.

Lemma 2.2. *For each $n \in \mathbf{N}$, the sets $\mathcal{N} \cdot \mathrm{BMOA}^{(n)}$ and $\mathcal{N}^+ \cdot \mathrm{BMOA}^{(n)}$ are ideal vector spaces.*

Proof. It is clear that the two sets are invariant under multiplication by H^∞ functions, but maybe not quite obvious that they are vector spaces. It is the linearity property

$$f_1, f_2 \in \mathcal{N} \cdot \mathrm{BMOA}^{(n)} \implies f_1 + f_2 \in \mathcal{N} \cdot \mathrm{BMOA}^{(n)}$$

(and a similar fact with $\mathcal{N}^+$ in place of $\mathcal{N}$) that should be verified. To this end, we write

$$f_j = \frac{u_j}{v_j} \cdot w_j^{(n)} \quad (j = 1, 2),$$

where $u_j, v_j \in H^\infty$ and $w_j \in \mathrm{BMOA}$, and where v_j is zero-free (resp., outer if the f_j's are from $\mathcal{N}^+ \cdot \mathrm{BMOA}^{(n)}$). Note that

$$f_1 + f_2 = \frac{1}{v_1 v_2} \cdot \left(u_1 v_2 w_1^{(n)} + u_2 v_1 w_2^{(n)} \right).$$

The two terms in brackets, and hence their sum, will be in $\mathrm{BMOA}^{(n)}$ by virtue of (2.2), while the factor $1/(v_1 v_2)$ will be in $\mathcal{N}$ (resp., in $\mathcal{N}^+$). □

3. Proof of Theorem 1.1

We treat the case of $\mathcal{N}$ first. Take $f \in \mathcal{N}$ and write $f = u/v$, where $u, v \in H^\infty$ and v has no zeros in $\mathbf{D}$. We have then

$$f^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} (1/v)^{(k)}. \tag{3.1}$$

For each $k \in \{0, \dots, n\}$, Faà di Bruno's formula (see [9, Chapter 3]) yields

$$\left(\frac{1}{v}\right)^{(k)} = \sum C(m_1, \dots, m_k)\, v^{-m_1 - \dots - m_k - 1} \prod_{j=1}^{k} \left(v^{(j)}\right)^{m_j}, \tag{3.2}$$

where the sum is over the k-tuples $(m_1, \dots, m_k)$ of nonnegative integers satisfying

$$m_1 + 2m_2 + \dots + k m_k = k \tag{3.3}$$

and where

$$C(m_1, \dots, m_k) = (-1)^{m_1 + \dots + m_k} \frac{(m_1 + \dots + m_k)!}{m_1! \dots m_k!} \frac{k!}{1!^{m_1} \dots k!^{m_k}}.$$

For any fixed multiindex $(m_1, \dots, m_k)$ as above, we clearly have

$$v^{-m_1 - \dots - m_k - 1} = v^{-n-1} \cdot v^{n - m_1 - \dots - m_k}, \tag{3.4}$$

the last factor on the right being bounded. Indeed,

$$v^{n - m_1 - \dots - m_k} \in H^\infty, \tag{3.5}$$

since it follows from (3.3) that $n - m_1 - \dots - m_k \ge 0$. We further observe that, for $j \in \mathbf{N}$,

$$v^{(j)}(z) = O((1 - |z|)^{-j}), \quad z \in \mathbf{D} \tag{3.6}$$

(because $v \in H^\infty$), and this implies together with (3.3) that

$$\prod_{j=1}^{k} \left[v^{(j)}(z)\right]^{m_j} = O((1 - |z|)^{-k}), \quad z \in \mathbf{D}. \tag{3.7}$$

Combining (3.2) and (3.4), we see that the kth summand in (3.1) takes the form $v^{-n-1} w_k$, where

$$w_k := \binom{n}{k} \sum C(m_1, \dots, m_k)\, u^{(n-k)} v^{n - m_1 - \dots - m_k} \prod_{j=1}^{k} \left(v^{(j)}\right)^{m_j}; \tag{3.8}$$

the sum is understood as in (3.2). We want to show that $w_k \in \mathrm{BMOA}^{(n)}$, and our plan is to check the corresponding inclusion for each individual term in (3.8). Thus, we claim that the function

$$\Phi_{m_1,\dots,m_k} := u^{(n-k)} v^{n-m_1-\dots-m_k} \prod_{j=1}^{k} \left(v^{(j)}\right)^{m_j}$$

satisfies

$$\Phi_{m_1,\dots,m_k} \in \mathrm{BMOA}^{(n)} \tag{3.9}$$

whenever $0 \le k \le n$ and the m_j's are related by (3.3).

First let us verify (3.9) in the case $k \le n-1$. To this end, we notice that

$$u^{(n-k)} \in (H^\infty)^{(n-k)} \subset \mathrm{BMOA}^{(n-k)},$$

where $n-k \ge 1$, while

$$[v(z)]^{n-m_1-\dots-m_k} \prod_{j=1}^{k} \left[v^{(j)}(z)\right]^{m_j} = O((1-|z|)^{-k}), \quad z \in \mathbf{D},$$

by virtue of (3.5) and (3.7). The validity of (3.9) is then guaranteed by Lemma 2.1.

Now if $k = n$, then the multiindices involved are of the form $(m_1, \dots, m_n)$ with $\sum_{j=1}^{n} j m_j = n$. For any such multiindex, at least one of the m_j's (say, m_l with an $l \in \{1, \dots, n\}$) must be nonzero, so that $m_l \ge 1$ and

$$l(m_l - 1) + \sum_{1 \le j \le n,\, j \ne l} j m_j = n - l. \tag{3.10}$$

Consider the factorization

$$\Phi_{m_1,\dots,m_n} = v^{(l)} \cdot \left\{ u v^{n-m_1-\dots-m_n} \left(v^{(l)}\right)^{m_l-1} \prod_{1 \le j \le n,\, j \ne l} \left(v^{(j)}\right)^{m_j} \right\}.$$

The first factor, $v^{(l)}$, is then in $(H^\infty)^{(l)}$ and hence in $\mathrm{BMOA}^{(l)}$, while the second factor (the one in curly brackets) is $O((1-|z|)^{-n+l})$. The latter estimate is due to (3.6) and (3.10), coupled with the fact that u and v are in H^∞. Applying Lemma 2.1 to the current factorization, we arrive at (3.9), this time with $k = n$.

Now that (3.9) is known to be true, we infer that the functions w_k from (3.8) are all in $\mathrm{BMOA}^{(n)}$, whence obviously $\sum_{k=0}^{n} w_k \in \mathrm{BMOA}^{(n)}$. Recalling that

$$f^{(n)} = v^{-n-1} \sum_{k=0}^{n} w_k,$$

we finally conclude that $f^{(n)}$ can be written as $gh^{(n)}$, where $g := v^{-n-1} \in \mathcal{N}$ and h is a BMOA function satisfying $h^{(n)} = \sum_{k=0}^{n} w_k$.

The case of $\mathcal{N}^+$ is similar. This time, v is taken to be an *outer* function in H^∞, so $g = v^{-n-1}$ will be an outer function in $\mathcal{N}^+$. □

4. Proof of Theorem 1.3

We shall only prove (a), the proof of (b) being similar. We know from Lemma 2.2 that $\mathcal{N} \cdot \mathrm{BMOA}'$ is an ideal space. Furthermore, Theorem 1.1 tells us that $\mathcal{N} \cdot \mathrm{BMOA}'$ contains $\mathcal{N}'$. It remains to verify that, whenever X is an ideal space with $\mathcal{N}' \subset X$, we necessarily have

$$\mathcal{N} \cdot \mathrm{BMOA}' \subset X. \tag{4.1}$$

Take any $g \in \mathcal{N}$ and $h \in H^\infty$. Note that

$$gh' = (gh)' - g'h, \tag{4.2}$$

where both terms on the right are in X. Indeed, $(gh)'$ is obviously in $\mathcal{N}'$ and hence in X, while the inclusion $g'h \in X$ is due to the facts that $g' \in \mathcal{N}' \subset X$ and $hX \subset X$ (recall that X is ideal). It now follows from (4.2) that $gh' \in X$, and we have thereby checked that

$$\mathcal{N} \cdot (H^\infty)' \subset X. \tag{4.3}$$

Finally, given $\eta \in \mathrm{BMOA}$, we invoke a result of Aleksandrov and Peller [1, Theorem 3.4] to find functions $\varphi_j, \psi_j \in H^\infty$ $(j = 1, 2)$ such that $\eta' = \varphi_1\psi_1' + \varphi_2\psi_2'$. Letting $g \in \mathcal{N}$ as before, we get

$$g\eta' = g\varphi_1\psi_1' + g\varphi_2\psi_2'. \tag{4.4}$$

Here, the two terms of the form $g\varphi_j\psi_j'$ are in $\mathcal{N} \cdot (H^\infty)'$, so we infer from (4.3) that they are also in X. The right-hand side of (4.4) is therefore in X, and so is the left-hand side, $g\eta'$. Thus we conclude that $g\eta' \in X$ for all $g \in \mathcal{N}$ and $\eta \in \mathrm{BMOA}$. This establishes (4.1) and completes the proof. □

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