MODULUS OF CONTINUITY OF p-DIRICHLET SOLUTIONS IN A METRIC MEASURE SPACE

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Abstract. Let $1 < p < \infty$ and let X be a metric measure space with a doubling measure and a $(1,p)$-Poincaré inequality. Let Ω be a bounded domain in X. For a function f on $\partial\Omega$ we denote by $\mathcal{P}_\Omega f$ the p-Dirichlet solution of f over Ω. It is well known that if Ω is p-regular and $f \in C(\partial\Omega)$, then $\mathcal{P}_\Omega f$ is p-harmonic in Ω and continuous in $\overline{\Omega}$. We characterize the family of domains Ω such that improved continuity of boundary functions f ensures improved continuity of $\mathcal{P}_\Omega f$. We specify such improved continuity if X is Ahlfors regular and $X \setminus \Omega$ is uniformly p-fat.

1. Introduction

Let $X = (X, d, \mu)$ be a complete connected metric measure space endowed with a metric d and a positive complete Borel measure μ such that $0 < \mu(U) < \infty$ for all non-empty bounded open sets U.

By the symbol C we denote an absolute positive constant whose value is unimportant and may change from line to line. Let $B(x, r) = \{y \in X \colon d(x, y) < r\}$ denote the open ball centered at x with radius r. We assume that μ is doubling, i.e., there is a constant $C \geq 1$ such that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for every $x \in X$ and $r > 0$. Let $1 < p < \infty$. We assume that X admits a $(1, p)$-Poincaré inequality.

We denote by Cap_p the p-capacity defined on X (Definition 2.5). Let $\Omega \subset X$ be a bounded domain with $\mathrm{Cap}_p(X \setminus \Omega) > 0$. For a function f on $\partial\Omega$ we donate by $\mathcal{P}_\Omega f$ the p-Perron solution of f over Ω. A point $\xi \in \partial\Omega$ is said to be a *p-regular point* (with respect to the p-Dirichlet problem) if

$$\lim_{\Omega \ni x \to \xi} \mathcal{P}_\Omega f(x) = f(\xi)$$

for every $f \in C(\partial\Omega)$. If every boundary point is a p-regular point, then Ω is called *p-regular*. It is well known that if Ω is p-regular and $f \in C(\partial\Omega)$, then $\mathcal{P}_\Omega f$ is p-harmonic in Ω and continuous in $\overline{\Omega}$. It is natural to raise the following question:

Question 1.1. Does improved continuity of a boundary function f guarantee improved continuity of $\mathcal{P}_\Omega f$?

Aikawa and Shanmugalingam [3] studied this question in the context of Hölder continuity. Aikawa [2] investigated this question in the context of general modulus of continuity for the classical setting, i.e., for harmonic functions in a Euclidean domain.

The purpose of this paper is to study this question in the context of general modulus of continuity in a metric measure space.

Let $\mathcal{M}$ be the family of all positive nondecreasing concave functions ψ on $(0, \infty)$ with $\psi(0) = \lim_{t \to 0} \psi(t) = 0$. We say that f is ψ-*Hölder continuous* if $|f(x) - f(y)| \leq C\psi(d(x, y))$. The modulus of continuity of a uniformly continuous function on any geodesic space is comparable to a certain concave function. See [5, Chapter 2 §6] and Propositions 2.13 and 2.14. The author would like to thank Kuroda for drawing his attention to [5]. Therefore, we have only to check ψ-Hölder continuity for $\psi \in \mathcal{M}$ to study Question 1.1 in the context of modulus of continuity.

As a typical example of $\psi \in \mathcal{M}$ we consider $\psi_{\alpha\beta}$ defined by

$$\psi_{\alpha\beta}(t) = \begin{cases} t^\alpha(-\log t)^{-\beta} & \text{for } 0 < t < t_0, \\ t_0^\alpha(-\log t_0)^{-\beta} & \text{for } t \geq t_0. \end{cases}$$

where either $0 < \alpha < 1$ and $\beta \in \mathbf{R}$ or $\alpha = 0$ and $\beta > 0$; and t_0 is so small that $\psi_{\alpha\beta} \in \mathcal{M}$. In particular, we write $\varphi_\alpha = \psi_{\alpha 0}$, and we say that f is α-Hölder continuous if f is φ_α-continuous.

Let $\psi \in \mathcal{M}$ and $E \subset X$. We consider the family $\Lambda_\psi(E)$ of all bounded continuous functions f on E with norm

$$\|f\|_{\psi,E} = \sup_{x \in E} |f(x)| + \sup_{\substack{x,y \in E \\ x \neq y}} \frac{|f(x) - f(y)|}{\psi(d(x, y))} < \infty.$$

We define the operator norm

$$\|\mathcal{P}_\Omega\|_\psi = \sup_{\substack{f \in \Lambda_\psi(\partial\Omega) \\ \|f\|_{\psi,\partial\Omega} \neq 0}} \frac{\|\mathcal{P}_\Omega f\|_{\psi,\Omega}}{\|f\|_{\psi,\partial\Omega}}.$$

Observe that ψ-Hölder continuity of a boundary function f ensures ψ-Hölder continuity of $\mathcal{P}_\Omega f$ if and only if $\|\mathcal{P}_\Omega\|_\psi < \infty$.

Aikawa [2] characterized the family of Euclidean domains Ω such that $\|\mathcal{P}_\Omega\|_\psi < \infty$ for $\psi \in \mathcal{M}$ in context of harmonic functions. We consider the same problem in the context of p-harmonic functions in a metric measure space. It is known that there exists $\alpha_0 \in (0, 1]$ such that every p-harmonic function in any domain Ω is locally α_0-Hölder continuous in Ω (see [10]). Hence, $\|\mathcal{P}_\Omega\|_\psi < \infty$ can hold only for $\psi \in \mathcal{M}$, in some sense, bigger than the function $\varphi_{\alpha_0}(t) = t^{\alpha_0}$.

Let $\psi, \varphi \in \mathcal{M}$. We say that $\varphi \precsim \psi$ if there are $r_0 > 0$ and $C > 0$ such that

$$\frac{\varphi(s)}{\varphi(r)} \leq C\frac{\psi(s)}{\psi(r)} \quad \text{for } 0 < s < r < r_0.$$

Let $\mathcal{M}_0$ be the family of all $\psi \in \mathcal{M}$ with $t^{\alpha_0} \precsim \psi(t)$. For example, if either $0 < \alpha < \alpha_0$ and $\beta \in \mathbf{R}$ or $\alpha = 0$ and $\beta > 0$, then $\psi_{\alpha\beta} \in \mathcal{M}_0$. But if $\alpha = \alpha_0$ and $\beta < 0$, then $\psi_{\alpha_0\beta} \notin \mathcal{M}_0$. Hence we see that $\mathcal{M}_0 \subsetneq \mathcal{M}$. Our results will be given for $\psi \in \mathcal{M}_0$.

Let U be an open set in X and let E be a Borel set in ∂U. We denote by $\omega_p(x, E, U)$ the p-harmonic measure evaluated at x of E in U. Note that the p-harmonic *measure* is not a measure, i.e., the p-harmonic measure is not additive. We define two decay properties for p-harmonic measures. We say that Ω enjoys the *Local Harmonic Measure Decay property* with ψ (abbreviated to the LHMD(ψ) property)

if there are positive constants C_1 and r_0 depending only Ω and ψ such that

$$(1.1) \qquad \omega_p(x, \Omega \cap \partial B(a, r), \Omega \cap B(a, r)) \leq C_1 \frac{\psi(d(x, a))}{\psi(r)} \quad \text{for } x \in \Omega \cap B(a, r),$$

whenever $a \in \partial\Omega$ and $0 < r < r_0$. We say that Ω enjoys the *Global Harmonic Measure Decay property* with ψ (abbreviated to the GHMD(ψ) property) if there are positive constants C_2 and r_0 depending only Ω and ψ such that

$$(1.2) \qquad \omega_p(x, \partial\Omega \setminus B(a, r), \Omega) \leq C_2 \frac{\psi(d(x, a))}{\psi(r)} \quad \text{for } x \in \Omega \cap B(a, r),$$

whenever $a \in \partial\Omega$ and $0 < r < r_0$. By the comparison principle (see [9, Theorem 7.2]) it is easy to see that (1.1) implies (1.2).

Without loss of generality, we may assume that Ω is a bounded p-regular domain (see [3, Proposition 2.1]). For $a \in \partial\Omega$ we define a test function $\tau_{a,\psi}$ on $\partial\Omega$ by

$$\tau_{a,\psi}(\xi) = \psi(d(a, \xi)) \quad \text{for } \xi \in \partial\Omega.$$

Then we have the following theorem.

Theorem 1.2. *Let $\psi \in \mathcal{M}_0$ and let Ω be a bounded p-regular domain. Consider the following conditions:*

(i) $\|\mathcal{P}_\Omega\|_\psi < \infty$.
(ii) *There is a constant C such that*

$$\mathcal{P}_\Omega \tau_{a,\psi}(x) \leq C\psi(d(x, a)) \quad \text{for } x \in \Omega,$$

  whenever $a \in \partial\Omega$.
(iii) *Ω satisfies the GHMD(ψ) property.*
(iv) *Ω satisfies the LHMD(ψ) property.*

Then we have

$$\text{(i)} \Longleftrightarrow \text{(ii)} \Longrightarrow \text{(iii)} \Longleftarrow \text{(iv)}.$$

The remaining implications in Theorem 1.2 are of interest. Theorem 4.1 in Section 4 will give the equivalence (iii) $\Longleftrightarrow$ (iv) under additional assumptions on X and $\psi \in \mathcal{M}_0$. As was observed in [3, Remark 2.4], the implication (iv) $\Longrightarrow$ (i) does not hold. However, we prove that a condition slightly stronger than (iv) implies (i).

Theorem 1.3. *Let $\psi, \psi_1 \in \mathcal{M}_0$. Let $\psi_2 = \psi_1/\psi$. Suppose that $\lim_{r\to 0} \psi_2(r) = 0$ and there are constants $0 < C_3 < 1$ and $r_0 > 0$ such that ψ_2 is increasing on $(0, r_0)$ and*

$$(1.3) \qquad \sup_{0 < \rho < r \leq r_0} \left\{ \frac{\psi(r)}{\psi(\rho)} : \frac{\psi_2(\rho)}{\psi_2(r)} = C_3 \right\} < \infty.$$

If Ω satisfies the LHMD(ψ_1) property, then $\|\mathcal{P}_\Omega\|_\psi < \infty$.

Condition (1.3) looks rather complicated. We have a simple condition.

Corollary 1.4. *Let $\psi, \psi_1 \in \mathcal{M}_0$. Let $\psi_2 = \psi_1/\psi$. Suppose that there are constants $0 < C_4 < 1$ and $r_0 > 0$ such that ψ is increasing on $(0, r_0)$ and*

$$(1.4) \qquad \inf_{0 < r \leq r_0} \frac{\psi_2(r)}{\psi_2(C_4 r)} > 1.$$

If Ω satisfies the LHMD(ψ_1) property, then $\|\mathcal{P}_\Omega\|_\psi < \infty$.

Theorem 1.3 and Corollary 1.4 are main results of this paper. They give several corollaries for $\psi_{\alpha\beta}$.

Corollary 1.5. *Let Ω be a bounded p-regular domain. Consider the following conditions:*

(i) $0 < \alpha < \alpha' < \alpha_0$ *and* $\beta, \beta' \in \mathbf{R}$.
(ii) $0 = \alpha < \alpha' < \alpha_0$ *and* $\beta > 0, \beta' \in \mathbf{R}$.
(iii) $\alpha = \alpha' = 0$ *and* $0 < \beta < \beta'$.

Assume that either (i), (ii), *or* (iii) *holds. If Ω satisfies the LHMD($\psi_{\alpha'\beta'}$) property, then $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$.*

We say that $E \subset X$ is *uniformly p-fat* or satisfies the *p-capacity density condition* if there are constants $C > 0$ and $r_0 > 0$ such that

$$(1.5) \qquad \frac{\mathrm{Cap}_p(E \cap B(a,r), B(a,2r))}{\mathrm{Cap}_p(B(a,r), B(a,2r))} \geq C,$$

whenever $a \in E$ and $0 < r < r_0$. The uniform p-fatness of the complement of a domain Ω is closely related to the condition $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$. For $\alpha > 0$ we obtain the following corollary.

Corollary 1.6. *Let Ω be a bounded p-regular domain. If $X \setminus \Omega$ is uniformly p-fat, then there is a constant $0 < \alpha_1 \leq \alpha_0$ such that $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for $0 < \alpha < \alpha_1$ and $\beta \in \mathbf{R}$. Conversely, if $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for some $0 < \alpha < \alpha_0$ and $\beta \in \mathbf{R}$, then $X \setminus \Omega$ is uniformly p-fat, provided that there is a constant $Q \geq p$ such that X is Ahlfors Q-regular, i.e.,*

$$C^{-1} r^Q \leq \mu(B(x,r)) \leq C r^Q$$

for every $x \in X$ and $r > 0$.

Aikawa and Shanmugalingam [3] showed the case $\beta = 0$ of Corollary 1.6. For $\alpha = 0$ we obtain the following corollary.

Corollary 1.7. *If $X \setminus \Omega$ is uniformly p-fat, then $\|\mathcal{P}_\Omega\|_{\psi_{0\beta}} < \infty$ for every $\beta > 0$.*

The plan of this paper is as follows. In the next section we shall define notions of p-harmonicity, p-Dirichlet problem, p-capacity, and p-harmonic measure, and we shall observe some properties for $\mathcal{M}$. In Section 3 we shall show Theorem 1.2. In Section 4 we shall prove that Ω satisfies the LHMD(ψ) property if and only if Ω satisfies the GHMD(ψ) property under certain additional assumptions. The proof of Theorem 1.3 and Corollary 1.4 will be given in Section 5. Finally, we shall give the proof of Corollaries 1.5, 1.6, and 1.7.

2. Preliminaries

In this section we introduce notions of p-harmonicity, p-Dirichlet problem, p-capacity, and p-harmonic measure; for details we refer to [3], and we observe some properties for $\mathcal{M}$.

The integral mean of u over a measurable set E is denoted by

$$\frac{1}{\mu(E)} \int_E u \, d\mu = \fint_E u \, d\mu = u_E.$$

Definition 2.1. We say that a Borel function g on X is an *upper gradient* of a real-valued function u on X if

$$(2.1) \qquad |u(x) - u(y)| \leq \int_\gamma g \, ds$$

for any $x, y \in X$ and all compact rectifiable curves γ joining x and y. If (2.1) fails only for a curve family with zero p-modulus (see [7, Definition 2.1]), then g is said to be a *p-weak upper gradient* of u. We say that g is a *minimal p-weak upper gradient* of u if $g \leq g'$ μ-almost everywhere for another p-weak upper gradients g' of u. We denote by g_u a minimal p-weak upper gradient of u.

Definition 2.2. Let $u \in L^p(X)$. We define the seminorm

$$\|u\|_{N^{1,p}(X)} = \|u\|_{L^p} + \inf_g \|g\|_{L^p},$$

where the infimum is taken over all p-weak upper gradients g of u. The *Newtonian space* on X is the quotient space

$$N^{1,p}(X) = \{u \in L^p(X) \colon \|u\|_{N^{1,p}(X)} < \infty\}/\sim,$$

where $u \sim v$ if and only if $\|u - v\|_{N^{1,p}(X)} = 0$.

Remark 2.3. The Newtonian space $N^{1,p}(X)$ with the norm $\|\cdot\|_{N^{1,p}(X)}$ is a Banach space. Every function $u \in N^{1,p}(X)$ has the minimal p-weak upper gradient g_u.

Definition 2.4. We say that X admits a $(1,p)$-*Poincaré inequality* if there are constants $\kappa \geq 1$ and $C \geq 1$ such that for all balls $B(x,r) \subset X$, all measurable functions u on X, and all p-weak upper gradients g of u we have

$$(2.2) \qquad \fint_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq Cr \left(\fint_{B(x,\kappa r)} g^p \, d\mu \right)^{1/p}.$$

A consequence of the $(1,p)$-Poincaré inequality is the following p-Sobolev inequality (see [10, Lemma 2.1]): if $0 < \gamma < 1$ and $\mu(\{z \in B(x,R) \colon |u(z)| > 0\}) \leq \gamma \mu(B(x,R))$, then there exists a positive constant C_γ depending only on γ such that

$$(2.3) \qquad \left(\fint_{B(x,R)} |u|^p \, d\mu \right)^{1/p} \leq C_\gamma R \left(\fint_{B(x,\kappa R)} g_u^p \, d\mu \right)^{1/p}.$$

If X admits a $(1,p)$-Poincaré inequality, then X admits a $(1,q)$-Poincaré inequality for every $q \geq p$ by Hölder's inequality. Keith and Zhong [8] showed that if X is proper (that is, closed and bounded subsets of X are compact) and X admits a $(1,p)$-Poincaré inequality, then there exists $q < p$ such that X admits a $(1,q)$-Poincaré inequality. Because X is a complete metric space equipped with a doubling measure, X is proper. Therefore we can use their result.

Definition 2.5. The *p-capacity* of a subset $E \subset X$ is defined by

$$\mathrm{Cap}_p(E) = \inf_u \|u\|^p_{N^{1,p}(X)},$$

where the infimum is taken over all $u \in N^{1,p}(X)$ such that $u \geq 1$ on E.

We say that a property holds p-*quasieverywhere* (p-q.e.) if the set of points for which the property fails to hold has p-capacity zero. We let

$$N_0^{1,p}(\Omega) = \{u \in N^{1,p}(X) \colon u = 0 \ p\text{-q.e. on } X \setminus \Omega\}.$$

We say that $u \in N_{loc}^{1,p}(\Omega)$ if for every $x \in \Omega$ there is r_x such that $f|_{B(x,r_x)} \in N^{1,p}(B(x,r_x))$. This is clearly equivalent to saying that $f \in N^{1,p}(V)$ for every relatively compact subset V of Ω. We now introduce the notion of p-harmonicity.

Definition 2.6. We call a function $u \in N_{loc}^{1,p}(\Omega)$ a *p-harmonic function* in Ω if u is continuous and

$$(2.4) \qquad \int_U g_u^p \, d\mu \leq \int_U g_{u+\varphi}^p \, d\mu.$$

for all relatively compact subsets U of Ω and all functions $\varphi \in N_0^{1,p}(U)$. A function $u \in N_{loc}^{1,p}(\Omega)$ is said to be a *p-superminimizer* in Ω if (2.4) holds for all relatively compact subsets U of Ω and all nonnegative functions $\varphi \in N_0^{1,p}(U)$. We call a function $u \in N_{loc}^{1,p}(\Omega)$ a *p-subminimizer* in Ω if (2.4) holds for all relatively compact subsets U of Ω and all nonpositive functions $\varphi \in N_0^{1,p}(U)$.

Let u and v be p-harmonic functions and let α, $\beta \in \mathbf{R}$. Then $\alpha u + \beta$ is p-harmonic. But in general $u + v$ is not p-harmonic. Kinnunen and Shanmugalingam [10, Theorem 5.2] showed the following local Hölder continuity of p-harmonic functions. Here, we denote by $\mathrm{osc}_E\, u$ the oscillation $\sup_E u - \inf_E u$.

Theorem 2.7. *Suppose a function u is p-harmonic on $B(x, 2\kappa R)$. Then there are constants $0 < \alpha_0 \leq 1$ and $C \geq 1$ such that*

$$\operatorname*{osc}_{B(x,\kappa r)} u \leq C \left(\frac{r}{R} \right)^{\alpha_0} \operatorname*{osc}_{B(x,\kappa R)} u \quad \text{for } 0 < r \leq R.$$

The constants α_0 and C are independent of u, x, and R.

Next we define p-Dirichlet solutions over Ω. For a function $f \in N^{1,p}(\Omega)$ we denote by $\mathcal{H}_\Omega f$ the Dirichlet solution of f over Ω, i.e., $\mathcal{H}_\Omega f$ is a function on $\overline{\Omega}$ that is p-harmonic in Ω with $f - \mathcal{H}_\Omega f \in N_0^{1,p}(\Omega)$. For $E \subset X$ we denote by $\mathrm{Lip}(E)$ the family of all Lipschitz continuous functions on E. For every $f \in \mathrm{Lip}(\partial\Omega)$ there is a function $Ef \in \mathrm{Lip}(\overline{\Omega})$ such that $f = Ef$ on $\partial\Omega$. Therefore we can define $\mathcal{H}_\Omega f$ by the function $\mathcal{H}_\Omega Ef$; this is independent of the extension Ef. We say that a lower semicontinuous function u on Ω is a *p-superharmonic function* in Ω if

    (i) $-\infty < u \leq \infty$;
    (ii) u is not identically ∞ in Ω;
    (iii) $\mathcal{H}_{\Omega'} v \leq u$ in Ω' for every relatively compact subset Ω' of Ω and all functions $v \in \mathrm{Lip}(\partial\Omega')$ such that $v \leq u$ on $\partial\Omega'$.

If $-u$ is p-superharmonic, then we say that u is *p-subharmonic*.

The following comparison principle is very useful in nonlinear potential theory (see [9, Theorem 7.2]).

Theorem 2.8. *Let u be a p-superharmonic function on Ω and let v be a p-subharmonic function on Ω. If*

$$(2.5) \qquad \limsup_{\Omega \ni x \to \xi} v(x) \leq \liminf_{\Omega \ni x \to \xi} u(x)$$

for every $\xi \in \partial\Omega$, and if both sides of (2.5) are not simultaneously ∞ or $-\infty$, then $v \leq u$ in Ω.

Definition 2.9. Let f be a function on $\partial\Omega$. Let $\mathcal{U}_f$ be the set of all p-superharmonic functions u on Ω bounded below such that $\liminf_{\Omega \ni x \to \xi} u(x) \geq f(\xi)$ for each $\xi \in \partial\Omega$.

The *upper Perron solution* of f is defined by

$$\overline{\mathcal{P}}_\Omega f(x) = \inf_{u \in \mathcal{U}_f} u(x) \quad \text{for } x \in \Omega.$$

Similarly, we define the *lower Perron solution* of f by

$$\underline{\mathcal{P}}_\Omega f(x) = \sup_{s \in \mathcal{L}_f} s(x) \quad \text{for } x \in \Omega,$$

where $\mathcal{L}_f = -\mathcal{U}_{-f}$ is the set of all p-subharmonic functions s on Ω bounded above such that $\limsup_{\Omega \ni x \to \xi} s(x) \le f(\xi)$ for each $\xi \in \partial\Omega$. If $\overline{\mathcal{P}}_\Omega f = \underline{\mathcal{P}}_\Omega f$, then we write $\mathcal{P}_\Omega f = \overline{\mathcal{P}}_\Omega f$, and we say that f is *resolutive*. We call $\mathcal{P}_\Omega f$ the *Perron solution* of f.

A. Björn, J. Björn and Shanmugalingam [4, Theorem 6.1] showed that if $f \in C(\partial\Omega)$, then f is resolutive. Moreover, if $f \in N^{1,p}(X)$, then f is resolutive and $\mathcal{P}_\Omega f = \mathcal{H}_\Omega f$, by [4, Theorem 5.1]. We define the p-harmonic measure as follows.

Definition 2.10. Let U be an open subset of X and let E be a Borel set in ∂U. The *p-harmonic measure* evaluated at x of E in U is defined by

$$\omega_p(x, E, U) = \overline{\mathcal{P}}_U \chi_E(x) \quad \text{for } x \in U.$$

The p-harmonic measure is not additive because of the non-linear nature of p-harmonic functions. Therefore the p-harmonic measure is not a measure.

Definition 2.11. Let $E \subset U \subset X$. We define the *relative p-capacity* of E in U by

$$\text{Cap}_p(E, U) = \inf_u \int_U g_u^p \, d\mu,$$

where the infimum is taken over all $u \in N_0^{1,p}(U)$ such that $u \ge 1$ on E.

Finally, we observe some properties for $\mathcal{M}$. The following proposition shows an elementary property for $\mathcal{M}$ (see [2, Lemma 2.2]).

Proposition 2.12. *Let $\psi \in \mathcal{M}$. If $c > 1$ and $0 < s \le t \le cs$, then $\psi(s) \le \psi(t) \le c\psi(s)$.*

In Section 1 we have assumed that $\psi \in \mathcal{M}$ is concave. The relevance of concavity of $\psi \in \mathcal{M}$ follows from the following propositions.

Proposition 2.13. *Let φ be a nondecreasing subadditive function on $(0, \infty)$, i.e., if $t_1, t_2 > 0$, then $\varphi(t_1 + t_2) \le \varphi(t_1) + \varphi(t_2)$. Suppose that $\lim_{t \to 0} \varphi(t) = \varphi(0) = 0$. Then there is a function $\psi \in \mathcal{M}$ satisfying*

$$\frac{1}{2}\psi(t) \le \varphi(t) \le \psi(t) \quad \text{for } t \ge 0.$$

Proposition 2.14. *Let (A, d_A) be a geodesic space and let f be a uniformly continuous function on A. Then*

$$\varphi(t) = \varphi(f, t) = \sup_{\substack{d_A(x,y) \le t \\ x,y \in A}} |f(x) - f(y)| \quad \text{for } t \ge 0.$$

is a subadditive function on $(0, \infty)$.

See [2, Section 5], [5, Chapter 2 §6], and [11, Section 3] for these accounts.

3. Proof of Theorem 1.2

To prove Theorem 1.2 we recall the following geometric property (see [6, Proposition 4.4]).

Lemma 3.1. *The space X is quasiconvex, i.e., there exists a constant $C_5 \geq 1$ such that every pair of points $x, y \in X$ can be joined by a curve of length at most $C_5 d(x, y)$. Hence if $x \in E \subsetneq X$, then*

$$\operatorname{dist}(x, X \setminus E) \leq \operatorname{dist}(x, \partial E) \leq C_5 \operatorname{dist}(x, X \setminus E).$$

Proof of Theorem 1.2. Since the LHMD(ψ) property implies the GHMD(ψ) property, it is sufficient to show that Condition (ii) implies Condition (iii) and that Condition (i) is equivalent to Condition (ii).

(ii) $\Longrightarrow$ (iii). Suppose (ii) holds. Let $a \in \partial\Omega$ and $r > 0$. Then

$$\psi(r)\chi_{\partial\Omega \setminus B(a,r)}(\xi) \leq \tau_{a,\psi}(\xi) \quad \text{for } \xi \in \partial\Omega.$$

The comparison principle yields

$$\psi(r)\omega_p(x, \partial\Omega \setminus B(a,r), \Omega) \leq \mathcal{P}_\Omega \tau_{a,\psi}(x) \quad \text{for } x \in \Omega.$$

Hence, (ii) implies that

$$\psi(r)\omega_p(x, \partial\Omega \setminus B(a,r), \Omega) \leq C\psi(d(x,a)) \quad \text{for } x \in \Omega.$$

Thus (iii) follows.

(i) $\Longrightarrow$ (ii). Suppose $\|\mathcal{P}_\Omega\|_\psi < \infty$. Since $\tau_{a,\psi} \in \Lambda_\psi(\partial\Omega)$, we have

$$\|\mathcal{P}_\Omega \tau_{a,\psi}\|_{\psi,\Omega} \leq \|\mathcal{P}_\Omega\|_\psi \|\tau_{a,\psi}\|_{\psi,\partial\Omega} < \infty.$$

By definition

$$|\mathcal{P}_\Omega \tau_{a,\psi}(x) - \mathcal{P}_\Omega \tau_{a,\psi}(y)| \leq \|\mathcal{P}_\Omega \tau_{a,\psi}\|_{\psi,\Omega}\psi(d(x,y)) \quad \text{for } x, y \in \Omega.$$

Letting $y \to a$, we see that $\mathcal{P}_\Omega \tau_{a,\psi}(x) \leq \|\mathcal{P}_\Omega \tau_{a,\psi}\|_{\psi,\Omega}\psi(d(x,a))$. Thus (ii) follows with $C = \|\mathcal{P}_\Omega \tau_{a,\psi}\|_{\psi,\Omega}$.

(ii) $\Longrightarrow$ (i). Suppose (ii) holds. Let $f \in \Lambda_\psi(\partial\Omega)$. Since $|\mathcal{P}_\Omega f|$ is bounded by the supremum of $|f|$ over $\partial\Omega$, it is sufficient to show that

$$(3.1) \qquad |\mathcal{P}_\Omega f(x) - \mathcal{P}_\Omega f(y)| \leq C\|f\|_{\psi,\partial\Omega}\psi(d(x,y)) \quad \text{for } x, y \in \Omega.$$

Let $x, y \in \Omega$. Without loss of generality, we may assume that $\operatorname{dist}(x, X \setminus \Omega) \geq \operatorname{dist}(y, X \setminus \Omega)$. Let $R = \operatorname{dist}(x, X \setminus \Omega)/2\kappa$. Since $\partial\Omega$ is compact, we can take $x^* \in \partial\Omega$ such that $d(x, x^*) = \operatorname{dist}(x, \partial\Omega)$. Then Lemma 3.1 gives

$$(3.2) \qquad 2\kappa R \leq d(x, x^*) \leq 2\kappa C_5 R.$$

Let $f_0(\xi) = f(\xi) - f(x^*)$. By definition

$$|f_0(\xi)| \leq \|f\|_{\psi,\partial\Omega}\tau_{x^*,\psi}(\xi) \quad \text{for } \xi \in \partial\Omega.$$

Hence, by the comparison principle and (ii), we obtain

$$(3.3) \qquad |\mathcal{P}_\Omega f_0(z)| \leq C\|f\|_{\psi,\partial\Omega}\psi(d(z, x^*)) \quad \text{for } z \in \Omega.$$

Let us consider two cases.

Case 1: $d(x, y) \leq d(x, x^*)/(2\kappa C_5)$. Let $r = d(x, y)$. Then $r \leq R$. Since $\mathcal{P}_\Omega f_0$ is p-harmonic, Theorem 2.7 gives

$$\operatorname*{osc}_{B(x,\kappa r)} \mathcal{P}_\Omega f_0 \leq C\left(\frac{r}{R}\right)^{\alpha_0} \operatorname*{osc}_{B(x,\kappa R)} \mathcal{P}_\Omega f_0.$$

We obtain from (3.2) that

$$d(z, x^*) \le d(x, z) + d(x, x^*) \le (1 + 2C_5)\kappa R \quad \text{for } z \in B(x, \kappa R).$$

By Proposition 2.12 we have

$$\psi(d(z, x^*)) \le \psi((1 + 2C_5)\kappa R) \le (1 + 2C_5)\kappa \psi(R).$$

Thus by (3.3) we obtain

$$\operatorname*{osc}_{B(x, \kappa R)} \mathcal{P}_\Omega f_0 \le 2 \sup_{B(x, \kappa R)} |\mathcal{P}_\Omega f_0| \le C\|f\|_{\psi, \partial\Omega} \psi(R).$$

Hence

$$(3.4) \qquad |\mathcal{P}_\Omega f(x) - \mathcal{P}_\Omega f(y)| = |\mathcal{P}_\Omega f_0(x) - \mathcal{P}_\Omega f_0(y)| \le C\left(\frac{r}{R}\right)^{\alpha_0} \|f\|_{\psi, \partial\Omega} \psi(R).$$

Since $\psi \in \mathcal{M}_0$, there is a constant $C > 0$ such that

$$\left(\frac{s}{r}\right)^{\alpha_0} \le C\frac{\psi(s)}{\psi(r)} \quad \text{for } 0 < s < r < 2\kappa \operatorname{diam}(\Omega).$$

Hence by (3.4), we have

$$|\mathcal{P}_\Omega f(x) - \mathcal{P}_\Omega f(y)| \le C\|f\|_{\psi, \partial\Omega} \psi(d(x, y)).$$

Case 2: $d(x, y) \ge d(x, x^*)/(2\kappa C_5)$. We have

$$d(y, x^*) \le d(x, y) + d(x, x^*) \le (1 + 2\kappa C_5)d(x, y).$$

It follows from Proposition 2.12 and (3.3) that

$$\begin{aligned} |\mathcal{P}_\Omega f(x) - \mathcal{P}_\Omega f(y)| = |\mathcal{P}_\Omega f_0(x) - \mathcal{P}_\Omega f_0(y)| &\le |\mathcal{P}_\Omega f_0(x)| + |\mathcal{P}_\Omega f_0(y)| \\ &\le C\|f\|_{\psi, \partial\Omega}(\psi(d(x, x^*)) + \psi(d(y, x^*))) \\ &\le C\|f\|_{\psi, \partial\Omega} \psi(d(x, y)). \end{aligned}$$

Combining the above two cases, we obtain (3.1). Thus (i) follows. $\qquad \square$

4. Equivalence between GHMD(ψ) and LHMD(ψ)

If $\psi = \varphi_\alpha$, then the GHMD(ψ) property and the LHMD(ψ) property are equivalent for Euclidean domains (see [1]) and for a metric measure space (see [3]). If $\psi \ne \varphi_\alpha$, it is not known whether this equivalence holds or not. In this section we show that the equivalence holds under certain additional assumptions.

Let $S(x, r) = \{y \in X : d(x, y) = r\}$ be the sphere with center at x and radius r and let $A(x, r, R)$ be the annulus $B(x, R) \setminus B(x, r)$ with center at x and radii r and R. We say that X is *linearly locally connected* (abbreviated to LLC) if there are constants $C_6 > 1$ and $r_0 > 0$ such that for every $a \in X$ and $0 < r < r_0$ each pair of points $x, y \in S(a, r)$ can be connected by a curve lying in $A(a, r/C_6, C_6 r)$.

Theorem 4.1. *Let Ω be a bounded regular domain. Assume that X is LLC and there is a constant $C > 0$ such that*

$$(4.1) \qquad \frac{\mu(B(a, r))}{\mu(B(a, R))} \le C\left(\frac{r}{R}\right)^p$$

whenever $a \in \partial\Omega$ and $0 < r \leq R < \mathrm{diam}(X)$. Let $\psi \in \mathcal{M}_0$. Suppose that there exist constants $0 < C < 1$ and $r_0 > 0$ such that

$$(4.2) \qquad \inf_{0 < r < r_0} \frac{\psi(r)}{\psi(Cr)} > 1.$$

Then Ω satisfies the LHMD(ψ) property if and only if Ω satisfies the GHMD(ψ) property.

Theorem 4.1 is new, even for the classical setting, i.e., for harmonic functions in Euclidean domains.

The proof is decomposed mainly into two steps. First, we show that the GHMD(ψ) property implies that the uniform perfectness of the boundary (Lemma 4.3). Second, with the aid of the uniform perfectness and a chain property, we will complete proof of Theorem 4.1. See [3, Lemmas 5.1 and 5.2] for Hölder continuity.

Definition 4.2. Let E be a subset of X. We say that E is *uniformly perfect* if there are constants $0 < C_7 < 1$ and $r_0 > 0$ such that $A(x, C_7 r, r) \cap E \neq \emptyset$ for every $x \in E$ and all $0 < r < r_0$.

Lemma 4.3. *Let Ω be a bounded regular domain. Assume that X is LLC and μ satisfies (4.1). Let $\psi \in \mathcal{M}_0$. Suppose that ψ satisfies (4.2). If Ω satisfies the GHMD(ψ) property, then $\partial\Omega$ is uniformly perfect.*

For the proof we state the following lemma, which is proved in the same way as [3, Lemma 5.3].

Lemma 4.4. *Assume that μ satisfies (4.1). If $0 < 2r \leq R < \mathrm{diam}(\Omega)/2$, then*

$$\frac{\mathrm{Cap}_p(\overline{B(a,r)}, B(a,R))}{\mu(B(a,R))} \leq C \left(\log \frac{R}{r} \right)^{1-p} R^{-p}.$$

Proof of Lemma 4.3. Let $a \in \partial\Omega$ and $0 < \rho_1 < \rho_2 < \mathrm{diam}(\Omega)/2$. Suppose $A(a, \rho_1, \rho_2)$ does not intersect $\partial\Omega$. Then it is sufficient to show that the ratio ρ_1/ρ_2 is bounded below by a positive constant C depending only on Ω and ψ.

Without loss of generality, we may assume that $\rho_1 \leq \rho_2/(2C_6^2)$. By the LLC property we see that $A(a, C_6\rho_1, \rho_2/C_6) \subset \Omega$. For simplicity, we let $r = C_6\rho_1$ and $R = \rho_2/C_6$. Then

$$(4.3) \qquad A(a, r, R) \subset \Omega.$$

Letting ρ_2 be larger if necessary, we may assume that $S(a, C_6 R)$ has a point $b \in \partial\Omega$. Let $K = \overline{B(a,r)} \setminus \Omega$. Observe from (4.3) that $K = B(a, R) \setminus \Omega$. By Lemma 4.4,

$$(4.4) \qquad \frac{\mathrm{Cap}_p(K, \Omega \cup K)}{\mu(B(a,R))} \leq \frac{\mathrm{Cap}_p(\overline{B(a,r)}, B(a,R))}{\mu(B(a,R))} \leq C \left(\log \frac{R}{r} \right)^{1-p} R^{-p}.$$

Let u_K be the p-capacitary potential for the condenser $(K, \Omega \cup K)$, i.e., u_K is p-harmonic on Ω, $u_K = 1$ p-q.e. on K, $u_K = 0$ p-q.e. on $X \setminus (\Omega \cup K)$ and

$$\mathrm{Cap}_p(K, \Omega \cup K) = \int_X g_{u_K}^p \, d\mu.$$

We prove that $u_K \leq 1/3$ p-q.e. on $B(b, \beta R)$ for some $0 < \beta < 1$. Since $r \leq R/2$ and $A(a, r, R) \cap \partial\Omega = \emptyset$, it follows from the comparison principle and the GHMD(ψ)

property that

(4.5)
$$u_K(x) \le C_2 \frac{\psi(d(x,b))}{\psi(R/2)} \quad \text{for } x \in \Omega \cap B(b, R/2).$$

Since ψ satisfies (4.2), there is a constant $0 < C_8 < 1$ such that

$$S = \inf_{0 < r < \text{diam}(\Omega)/2} \frac{\psi(r)}{\psi(C_8 r)} > 1.$$

Therefore, we have

$$\frac{\psi(C_8^{j-1} R/2)}{\psi(C_8^j R/2)} \ge S$$

for every positive integer j. Now multiplying the above inequalities over $j = 1, 2, \ldots, N$, we get

$$\frac{\psi(R/2)}{\psi(C_8^N R/2)} \ge S^N.$$

We can find a positive integer N such that

$$\frac{C_2}{S^N} \le \frac{1}{3}.$$

Let $\beta = C_8^N/2$. By the monotonicity of ψ, if $x \in B(b, \beta R)$, then

$$\psi(d(x,b)) \le \psi(\beta R) \le \psi(R/2)/(3C_2).$$

Hence, by (4.5) we obtain

$$u_K(x) \le \frac{1}{3} \quad \text{for } x \in \Omega \cap B(b, \beta R)$$

Since $u_K = 0$ p-q.e. on $B(b, R/2) \setminus \Omega$, we have $u_K \le 1/3$ p-q.e. on $B(b, \beta R)$.

Next we prove that $u_K \ge 2/3$ p-q.e. on $B(a, \beta R)$. It follows from (4.3) and the comparison principle that

$$u_K(x) = 1 - \omega_p(x, \partial\Omega \setminus B(a, R), \Omega) \quad \text{for } x \in \Omega.$$

By the GHMD(ψ), we have

$$\omega_p(x, \partial\Omega \setminus B(a, R), \Omega) \le C_2 \frac{\psi(d(x,a))}{\psi(R)} \quad \text{for } x \in \Omega \cap B(a, R)$$

Hence (4.2) implies

$$u_K(x) \ge \frac{2}{3} \quad \text{for } x \in \Omega \cap B(a, \beta R)$$

Since $u_K = 1$ p-q.e. on $B(a, \beta R) \setminus \Omega \subset B(a, R) \setminus \Omega$, we obtain $u_K \ge 2/3$ p-q.e. on $B(a, \beta R)$.

Let $v = \max\{u_K, 1/3\} - 1/3 \ge 0$. Then

$$\frac{\mu(\{x \in B(a, 2C_6 R): v(x) = 0\})}{\mu(B(a, 2C_6 R))} \ge \frac{\mu(B(b, \beta R))}{\mu(B(a, 2C_6 R))} \ge \gamma$$

where $\gamma > 0$ depends only on β. Hence the p-Sobolev inequality (2.3) and the doubling property of μ imply

$$\left(\int_{B(a, 2C_6 R)} v^p \, d\mu \right)^{1/p} \le CR \left(\int_{B(a, 2\kappa C_6)} g_v^p \, d\mu \right)^{1/p}.$$

By the doubling property of μ we have

$$\int_{B(a,2C_6R)} v^p \, d\mu \geq \int_{B(a,\beta R)} (1/3)^p \, d\mu \geq C\mu(B(a,R)).$$

Hence, we obtain

$$\mathrm{Cap}_p(K, \Omega \cup K) = \int_X g_{u_K}^p \, d\mu \geq \int_{B(a,2\kappa C_6 R)} g_v^p \, d\mu$$

$$\geq CR^{-p} \int_{B(a,2C_6R)} v^p \, d\mu \geq CR^{-p}\mu(B(a,R)).$$

This, together with (4.4), implies that r/R is bounded below and therefore so is ρ_1/ρ_2. Thus the lemma is proved. $\qquad\square$

To prove Theorem 4.1 we state two lemmas in [3].

Lemma 4.5. *Let $0 < R < \mathrm{diam}(\Omega)/6\kappa$ and let u be a p-subminimizer on $B(z, 2\kappa R)$. Suppose $0 \leq u \leq 1$ on $B(z, 2\kappa R)$ and*

$$\frac{\mu(\{x \in B(z,R)\colon u(x) > 1 - s\})}{\mu(B(z,R))} \leq \gamma < 1$$

for some $0 < s < 1$. Then there exists a constant $t > 0$ such that

$$u \geq 1 - t \quad \text{on } B(z.R/2).$$

The constant t is independent of u, z, and R.

Lemma 4.6. *Let $0 < R < \mathrm{diam}(\Omega)/6\kappa$. Let $B(z_1, R/2) \cap B(z_2, R/2) \neq \emptyset$. Suppose u is a p-subminimizer on $B(z_2, 2\kappa R)$ with $0 \leq u \leq 1$ in $B(z_2, 2\kappa R)$. If $u \leq 1 - \varepsilon_1$ on $B(z_1, R/2)$ for some $\varepsilon_1 > 0$, then there is a positive constant $\varepsilon_2 = \varepsilon_2(\varepsilon_1) < 1$ such that $u \leq 1 - \varepsilon_2$ on $B(z_2, R/2)$.*

Proof of Theorem 4.1. It is sufficient to show that if Ω satisfies the GHMD(ψ) property, then Ω satisfies the LHMD(ψ) property. Since Ω is uniformly perfect by Lemma 4.3, there are constants $0 < C_7 < 1$ and $r_0 > 0$ such that $A(x, C_7r, r) \cap \partial\Omega \neq \emptyset$ for every $x \in \partial\Omega$ and all $0 < r < r_0$. Let $a \in \partial\Omega$ and $0 < r < r_0$. Then we can find ρ such that $S(a, \rho) \cap \partial\Omega \neq \emptyset$ and $C_7r \leq \rho < r$.

Let c be a small positive number to be determined later. By the LLC property and the doubling property of μ we can find finitely many points $z_1, \ldots, z_N \in A(a, \rho/C_6, C_6\rho)$ such that the union $\cup_{j=1}^N B(z_j, cr)$ is a covering of $S(a, \rho)$ that forms a chain, that is, for every $k, l \in \{1, \ldots, N\}$ there is a subcollection of balls $B_{j_1}, \ldots, B_{j_m}$ such that $B_k = B_{j_1}$, $B_l = B_{j_m}$ and $B_{j_i} \cap B_{j_{i+1}} \neq \emptyset$ for $i \in \{1, \ldots, m-1\}$. Observe that

$$(4.6) \quad \bigcup_{j=1}^N B(z_j, 4\kappa cr) \subset A\left(a, \frac{\rho}{C_6} - 4\kappa cr, C_6\rho + 4\kappa cr\right) \subset A\left(a, \left(\frac{C_7}{C_6} - 4\kappa c\right)r, (C_6 + 4\kappa c)r\right).$$

Let $c > 0$ be small enough so that $4\kappa c \leq C_7/(2C_6)$. Let $\eta = C_7/(2C_6)$. Consider

$$u = \begin{cases} \omega_p(\partial\Omega \cap B(a, \eta r), \Omega) & \text{on } \Omega, \\ 0 & \text{on } X \setminus \Omega. \end{cases}$$

Then $0 \leq u \leq 1$ on X and u is a p-subminimizer in $X \setminus \overline{B(a, \eta r)} \supset \cup_{j=1}^N B(z_j, 4\kappa cr)$. Fix $z^* \in \partial\Omega \cap S(a, \rho)$. Without loss of generality, we may assume that $z^* \in B(z_1, cr)$.

Since

$$B(z^*, (4\kappa - 1)cr) \subset B(z_1, 4\kappa cr) \subset X \setminus \overline{B(a, \eta r)},$$

it follows from the comparison principle that

$$u(x) \leq \omega_p(x, \partial\Omega \setminus B(z^*, (4\kappa - 1)cr), \Omega) \quad \text{for } x \in \Omega.$$

Since Ω satisfies the GHMD(ψ) property and ψ satisfies (4.2), we obtain

$$u(x) \leq \frac{1}{2} \quad \text{for } x \in B(z^*, \beta r) \cap \Omega$$

for some $\beta > 0$ independent of a and r. Since $u = 0$ on $X \setminus \Omega$, we have $u \leq 1/2$ on $B(z^*, \beta r)$. Hence Lemma 4.5 with $R = 2cr$ yields that $u \leq 1 - \varepsilon_1$ on $B(z_1, cr)$ for some $\varepsilon_1 > 0$ independent of a and r. Since $\cup_{j=1}^{N} B(z_j, cr)$ is a chain, we find some ball, say $B(z_2, cr)$, intersecting $B(z_1, cr)$. Then by Lemma 4.6 we have $u \leq 1 - \varepsilon_2$ on $B(z_2, cr)$ for some $\varepsilon_2 > 0$. We may repeat this argument finitely many times until, by the finiteness of the cover and its chain property, we eventually obtain $u \leq 1 - \varepsilon_0$ on $\cup_{j=1}^{N} B(z_j, cr)$ for some $\varepsilon_0 > 0$ that is independent of a and r. In particular, $u \leq 1 - \varepsilon_0$ on $S(a, \rho)$. Since

$$\omega_p(\partial\Omega \cap B(a, \eta r), \Omega) = 1 - \omega_p(\partial\Omega \setminus B(a, \eta r), \Omega) \quad \text{on } \Omega,$$

it follows that $\omega_p(\partial\Omega \setminus B(a, \eta r), \Omega) \geq \varepsilon_0$ on $\Omega \cap S(a, \rho)$. By the comparison principle we have

$$\frac{1}{\varepsilon_0} \omega_p(\partial\Omega \setminus B(a, \eta r), \Omega) \geq \omega_p(\Omega \cap \partial B(a, \rho), \Omega \cap B(a, \rho)) \quad \text{on } \Omega \cap B(a, \rho).$$

Hence the GHMD(ψ) property and Proposition 2.12 yield

$$\omega_p(x, \Omega \cap \partial B(a, r), \Omega \cap B(a, r)) \leq \omega_p(x, \Omega \cap \partial B(a, \rho), \Omega \cap B(a, \rho))$$
$$\leq \frac{C_2}{\varepsilon_0} \frac{\psi(d(x, a))}{\psi(\eta r)} \leq \frac{C_2}{\varepsilon_0 \eta} \frac{\psi(d(x, a))}{\psi(r)}$$

for all $x \in \Omega \cap B(a, \rho)$. Because $\rho \geq C_7 r$, we obtain $d(x, a) \geq C_7 r$ for all $x \in \Omega \cap B(a, r) \setminus B(a, \rho)$. Proposition 2.12 yields

$$\omega_p(x, \Omega \cap \partial B(a, r), \Omega \cap B(a, r)) \leq 1 \leq \frac{\psi(d(x, a))}{\psi(C_7 r)} \leq \frac{1}{C_7} \frac{\psi(d(x, a))}{\psi(r)}$$

for all $x \in \Omega \cap B(a, r) \setminus B(a, \rho)$. Thus Ω satisfies the LHMD(ψ) property. $\qquad\square$

Remark 4.7. We say that X is Ahlfors Q-regular if there exists a positive constant C such that

$$C^{-1} r^Q \leq \mu(B(x, r)) \leq C r^Q \quad \text{for every } B(x, r).$$

If X is Ahlfors Q-regular with $Q \geq p$, then μ satisfies (4.1). Moreover if X supports a $(1, p)$-Poincaré inequality and X is Ahlfors Q-regular with $Q \geq p$, then X is LLC (see [6, Proposition 4.5]). Therefore, if X is Ahlfors Q-regular with $Q \geq p$ and $\psi \in \mathcal{M}_0$ satisfies (4.2), then Ω satisfies the LHMD(ψ) property if and only if Ω satisfies the GHMD(ψ) property.

Remark 4.8. Let $\psi = \psi_{\alpha\beta}$. If $\alpha > 0$, then $\psi_{\alpha\beta}$ satisfies (4.2). Therefore if X is Ahlfors Q-regular with $Q \geq p$, then the LHMD($\psi_{\alpha\beta}$) property and the GHMD($\psi_{\alpha\beta}$) property are equivalent. On the other hand, $\psi_{0\beta}$ does not satisfy (4.2), and we do not know whether the equivalence holds or not.

5. Proof of Theorem 1.3 and Corollary 1.4

In this section we give the proof of Theorem 1.3 and Corollary 1.4.

Proof of Theorem 1.3. Let $a \in \partial\Omega$ and $u = \mathcal{P}_\Omega \tau_{a,\psi}$. We will show (ii) in Theorem 1.2 holds, i.e., $u(x) \le C\psi(d(x,a))$. For $\rho > 0$ we define a function $f(\rho)$ by

$$f(\rho) = \sup_{\Omega \cap S(a,\rho)} u(x).$$

It is sufficient to show that

$$(5.1) \qquad\qquad f(\rho) \le C\psi(\rho)$$

for small $\rho > 0$. Let $0 < \rho < r < \operatorname{diam}(\Omega)$. By definition of $\tau_{a,\psi}$ we see that $u \le \psi(r) + f(r)\chi_{\Omega \cap \partial B(a,r)}$ on $\partial(\Omega \cap B(a,r))$. The comparison principle yields

$$u(x) \le \psi(r) + f(r)\omega_p(x, \Omega \cap \partial B(a,r), \Omega \cap B(a,r))$$

for all $x \in \Omega \cap B(a,r)$. Hence, the LHMD(ψ_1) property implies

$$(5.2) \qquad f(\rho) \le \psi(r) + C_1 f(r)\frac{\psi_1(\rho)}{\psi_1(r)} = \psi(r) + C_1 f(r)\frac{\psi(\rho)}{\psi(r)}\frac{\psi_2(\rho)}{\psi_2(r)}.$$

Without loss of generality, we assume that $r_0 < \operatorname{diam}(\Omega)$. We can find a positive integer N such that $C_3^N \le 1/(2C_1)$. By (1.3) we have

$$(5.3) \qquad M = \sup_{0 < \rho < r \le r_0} \left\{ \frac{\psi(r)}{\psi(\rho)} : \frac{\psi_2(\rho)}{\psi_2(r)} = C_3^N \right\} < \infty.$$

We can find the number $0 < r_0' < r_0$ such that

$$\frac{\psi_2(r_0')}{\psi_2(r_0)} = C_3^N.$$

Let $0 < r < r_0'$. Then by (5.3) we can find a sequence $\{\rho_j\}_{j=1}^n$ such that $r = \rho_0 < \rho_1 < \ldots < \rho_{n-1} < r_0' \le \rho_n < r_0$,

$$\frac{\psi_2(\rho_j)}{\psi_2(\rho_{j+1})} = C_3^N \le \frac{1}{2C_1} \quad \text{for } j = 0, 1, \ldots, n-1,$$

and

$$\frac{\psi(\rho_{j+1})}{\psi(\rho_j)} \le M \quad \text{for } j = 0, 1, \ldots, n-1.$$

Hence, by (5.2) we obtain

$$f(\rho_j) \le \psi(\rho_{j+1}) + \frac{1}{2} f(\rho_{j+1})\frac{\psi(\rho_j)}{\psi(\rho_{j+1})} \quad \text{for } j = 0, 1, \ldots, n-1.$$

These inequalities imply that

$$f(r) = f(\rho_0) \leq \psi(\rho_1) + \psi(\rho_0) \sum_{j=1}^{n-1} \frac{1}{2^j} \frac{\psi(\rho_{j+1})}{\psi(\rho_j)} + \frac{1}{2^n} f(\rho_n) \frac{\psi(\rho_0)}{\psi(\rho_n)}$$

$$\leq M\psi(\rho_0) + M\psi(\rho_0) \sum_{j=1}^{n-1} \frac{1}{2^j} + f(\rho_n) \frac{\psi(\rho_0)}{\psi(\rho_n)}$$

$$\leq M\psi(\rho_0) + M\psi(\rho_0) + f(\rho_n) \frac{\psi(\rho_0)}{\psi(\rho_n)}$$

$$\leq (2M + \frac{\psi(\mathrm{diam}(\Omega))}{\psi(r_0')})\psi(r),$$

where $f \leq \psi(\mathrm{diam}(\Omega))$ and $r_0' \leq \rho_n$ are used in the last inequality. Thus (5.1) follows, and so (ii) in Theorem 1.2 holds. Hence $\|\mathcal{P}_\Omega\|_\psi < \infty$ by Theorem 1.2. $\qquad\square$

Proof of Corollary 1.4. Let us prove (1.3) with

$$C_3 = \sup_{0<r\leq r_0} \frac{\psi_2(C_4 r)}{\psi_2(r)} < 1.$$

Fix $0 < r \leq r_0$. Then

$$\frac{\psi_2(C_4 r)}{\psi_2(r)} \leq C_3.$$

By the monotonicity of ψ_2 we can find a number ρ such that $C_4 r \leq \rho < r$ and

$$\frac{\psi_2(\rho)}{\psi_2(r)} = C_3.$$

Proposition 2.12 yields that

$$\frac{\psi(r)}{\psi(\rho)} \leq \frac{\psi(r)}{\psi(C_4 r)} \leq \frac{1}{C_4}.$$

Hence we have

$$\sup_{0<\rho<r\leq r_0} \left\{ \frac{\psi(r)}{\psi(\rho)} : \frac{\psi_2(\rho)}{\psi_2(r)} = C_3 \right\} \leq \frac{1}{C_4} < \infty.$$

Next we prove that $\lim_{r\to 0} \psi_2(r) = 0$. By the monotonicity of ψ_2 the limit of $\psi_2(r)$ exists, as $r \to 0$. If $\lim_{r\to 0} \psi_2(r) \neq 0$, then we would have

$$\lim_{r\to 0} \frac{\psi_2(r)}{\psi_2(C_4 r)} = 1.$$

This would contradict (1.4). Hence $\lim_{r\to 0} \psi_2(r) = 0$. Since the assumptions of Theorem 1.3 are satisfied, it follows that $\|\mathcal{P}_\Omega\|_\psi < \infty$. $\qquad\square$

6. Proof of Corollaries 1.5, 1.6 and 1.7

In this section we prove Corollaries 1.5, 1.6, and 1.7.

Proof of Corollary 1.5. We divide the proof into the following two cases.

Case 1: (i) or (ii) holds. Let $\psi = \psi_{\alpha\beta}$, $\psi_1 = \psi_{\alpha'\beta'}$, and $\psi_2 = \psi_1/\psi$. Let r_0 be a small positive number. Then

$$\psi_2(r) = r^{\alpha'-\alpha}(-\log r)^{-\beta'+\beta} \quad \text{for } 0 < r \leq r_0.$$

Hence ψ_2 is increasing on $(0, r_0)$, and for some constant $C_4 \in (0, 1)$

$$\inf_{0 < r \le r_0} \frac{\psi_2(r)}{\psi_2(C_4 r)} > 1.$$

Since the assumptions of Corollary 1.4 are satisfied, we have $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$.

Case 2: (iii) holds. Let $\psi = \psi_{0\beta}$, $\psi_1 = \psi_{0\beta'}$, and $\psi_2 = \psi_1/\psi$. Let r_0 be a small positive number. Then

$$\psi_2(r) = (\log r)^{-\beta'+\beta} \quad \text{for } 0 < r \le r_0.$$

Hence $\lim_{r \to 0} \psi_2(r) = 0$ and ψ_2 is increasing on $(0, r_0)$. Fix a constant $0 < \eta < 1$ and $0 < r \le r_0$. Let $\lambda = \eta^{1/(\beta-\beta')}$ and $\rho = r^\lambda$. Then we have

$$\frac{\psi_2(\rho)}{\psi_2(r)} = \lambda^{-\beta'+\beta} = \eta,$$

and

$$\frac{\psi(r)}{\psi(\rho)} = \lambda^\beta.$$

Hence

$$\sup_{0 < \rho < r \le r_0} \left\{ \frac{\psi(r)}{\psi(\rho)} : \frac{\psi_2(\rho)}{\psi_2(r)} = \eta \right\} = \lambda^\beta < \infty.$$

Thus it follows from Theorem 1.3 that $\|\mathcal{P}_\Omega\|_{\psi_{0\beta}} < \infty$. $\qquad \square$

To prove Corollaries 1.6 and 1.7 we observe the following lemma (see [3, Lemma 6.1]).

Lemma 6.1. *A domain Ω satisfies the LHMD(φ_{α_2}) property for some $\alpha_2 > 0$ if and only if $X \setminus \Omega$ is uniformly p-fat.*

Proof of Corollary 1.6. First suppose that $X \setminus \Omega$ is uniformly p-fat. It follows from Lemma 6.1 that there is a constant $\alpha_2 > 0$ such that Ω satisfies the LHMD(φ_{α_2}) property. Let $\alpha_1 = \min\{\alpha_0, \alpha_2\}$. Then Ω satisfies the LHMD(φ_{α_1}) property. Corollary 1.5 yields that $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for $0 < \alpha < \alpha_1$ and $\beta \in \mathbf{R}$.

Conversely, suppose that $\|\mathcal{P}_\Omega\|_{\psi_{\alpha\beta}} < \infty$ for some $0 < \alpha < \alpha_0$ and $\beta \in \mathbf{R}$. Assume that X is Ahlfors Q-regular with $Q \ge p$. By Theorem 1.2 Ω satisfies the GHMD($\psi_{\alpha\beta}$) property. It follows from Remark 4.8 that Ω satisfies the LHMD($\psi_{\alpha\beta}$) property. Let $0 < \alpha' < \alpha$. By Corollary 1.5 we obtain that $\|\mathcal{P}_\Omega\|_{\varphi_{\alpha'}} < \infty$. Theorem 1.2 and Theorem 4.1 imply that Ω satisfies the LHMD($\varphi_{\alpha'}$) property. Lemma 6.1 yields that $X \setminus \Omega$ is uniformly p-fat. $\qquad \square$

Proof of Corollary 1.7. Suppose that $X \setminus \Omega$ is uniformly p-fat. It follows from Lemma 6.1 that there is a constant $\alpha_2 > 0$ such that Ω satisfies the LHMD(φ_{α_2}) property. Let $\alpha_1 = \min\{\alpha_0, \alpha_2\}$. Then Ω satisfies the LHMD(φ_{α_1}) property. Corollary 1.5 yields that $\|\mathcal{P}_\Omega\|_{\psi_{0\beta}} < \infty$ for every $\beta \in \mathbf{R}$. $\qquad \square$

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