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(K, K')-QUASICONFORMAL HARMONIC MAPPINGS OF THE UPPER HALF PLANE ONTO ITSELF

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Abstract. In this paper we show that (K, K')-quasiconformal mappings with unbounded image domains are not Hölder continuous, which is different from the case with bounded image domains given by Kalaj and Mateljević. For a (K, K')-quasiconformal harmonic mapping of the upper half plane onto itself, we prove that it is Lipschitz and hyperbolically Lipschitz continuous. Moreover, we get four equivalent conditions for a harmonic mapping of the upper half plane onto itself to be a (K, K')-quasiconformal mapping.

1. Introduction

A function F is called harmonic [4] in a region Ω if its Laplacian vanishes in Ω. By Lewy's theorem [15], a locally univalent harmonic function F has a non-vanishing Jacobian. The real axis, the upper half plane and the unit disk are denoted by $\mathbf{R}$, $\mathbf{H}$ and $\mathbf{D}$, respectively. If $w \in L^\infty(\mathbf{R})$, then its Poisson extension [6]

$$F(z) = p[w](z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} p(z,t)w(t)\,dt, \tag{1.1}$$

is harmonic on $\mathbf{H}$, where

$$p(z,t) = \frac{y}{(x-t)^2 + y^2}, \quad z = x + iy,$$

is called the Poisson kernel of $\mathbf{H}$.

A topological mapping f of Ω is said to be (K, K')-quasiconformal if it satisfies

1) f is ACL in Ω;
2) $L_f^2 \le K L_f l_f + K'$, $K \ge 1$, $K' \ge 0$ a.e. in Ω, where $L_f = |f_z| + |f_{\bar z}|$, $l_f = |f_z| - |f_{\bar z}|$.

If $K' = 0$, then f is a K-quasiconformal mapping. If a harmonic mapping is also (K, K')-quasiconformal, then we call it a (K, K')-quasiconformal harmonic mapping. For convenience, quasiconformal mapping and quasiconformal harmonic mapping are abbreviated by qc mapping and qch mapping, respectively.

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The class of K-qch mappings of $\mathbf{D}$ onto itself was first studied by Martio [16]. Pavlović [20] proved that a K-qch mapping of $\mathbf{D}$ onto itself is bi-Lipschitz. Its explicit bi-Lipschitz constants were given by Partyka and Sakan [19]. Zhu and Huang [23] used Heinz's inequality to improve the result. Kalaj and Pavlović [13] obtained some characterizations including the bi-Lipschitz continuity of K-qch mappings of $\mathbf{H}$ onto iteself. If f is K-qch and ψ is conformal, then $f \circ \psi$ is also K-qch. However, $\psi \circ f$ rarely preserves the harmonicity. Hence, the image domains of K-qch mappings can not be always confined to a canonical domain such as the unit disk or the upper half plane. Kalaj [9–13] did a lot of work on studying the Lipschitz continuity for different image domains from $\mathbf{D}$, for example he proved that every K-qch mapping between a Jordan domain with $C^{1,\alpha}$ ($\alpha < 1$) and a Jordan domain with $C^{1,1}$ compact boundary is bi-Lipschitz [9]. The fact that every K-qch mapping of $\mathbf{D}$ or $\mathbf{H}$ onto itself is hyperbolically bi-Lipschitz has been showed by Knežević and Mateljević [14]. Chen and Fang [3] generalized the above result to the case of convex image domain and gave the sharp bi-Lipschitz constants. The hyperbolically bi-Lipschitz continuity of some other classes of qc mappings were considered in [17], [18], [22].

Finn and Serrin [5] and Simon [21] obtained a Hölder estimate for (K, K')-qc mappings. Recently, Kalaj and Mateljević [12] studied the class of (K, K')-qc mappings with bounded image domains. They proved the following intrigue results: A (K, K')-qc mapping between two Jordan domains with C^2 boundaries is Hölder continuous. Moreover, if it is also harmonic, then it is Lipschitz continuous. In this paper, we study the class of (K, K')-qch mappings with unbounded image domains.

In Section 2, we first construct a (K, K')-qc mapping of an angular domain onto $\mathbf{H}$ which is neither Lipschitz nor Hölder continuous (see Example 2.1). A harmonic mapping with unbounded image domain is not necessarily Hölder or Lipschitz continuous (see Example 2.2 in [17]). However, we can construct an example of (K, K')-qch mappings of $\mathbf{H}$ onto itself which is Lipschitz but not bi-Lipschitz (see Example 2.2). In fact, after estimating the modulus of the gradient of (K, K')-qch mappings of $\mathbf{H}$ onto itself (see Lemma 2.1), we obtain the Lipschitz and hyperbolically Lipschitz continuity of (K, K')-qch mappings of $\mathbf{H}$ onto itself and its Lipschitz constants only depend on K and K' (see Theorems 2.1 and 2.2).

In Section 3, combining Theorem 2.1 with the knowledge of a harmonic function and its harmonic conjugate function we get several equivalent conditions for a harmonic mapping of $\mathbf{H}$ onto itself to be a (K, K')-qc mapping. That is

Theorem 1.1. *If f is a harmonic mapping of $\mathbf{H}$ onto itself and continuous on $\mathbf{H} \cup \mathbf{R}$, then the following assertions are equivalent.*

(1) *f is a (K, K')-qc mapping of $\mathbf{H}$ onto itself;*
(2) *There are two positive constants c and M such that $0 < u_x \le M$ and $|u_y| \le \sqrt{(K+1)cM + K'}$ for all $z \in H$;*
(3) *f is a Lipschitz mapping of $\mathbf{H}$ onto $\mathbf{H}$;*
(4) *φ is absolutely continuous on $\mathbf{R}$, $\varphi' \in L^\infty(\mathbf{R})$ and $H[\varphi'] \in L^\infty(\mathbf{R})$, where φ is the boundary value of f on the real axis $\mathbf{R}$ and φ' is the derivative of φ.*

In Section 4, we estimate the Jacobian of (K, K')-qch mappings of $\mathbf{H}$ onto itself (see Lemma 4.1). As an application of Lemma 4.1, we obtain the euclidean and hyperbolic area distortion of (K, K')-qch mappings of $\mathbf{H}$ onto itself (see Theorems 4.1 and 4.2).

2. Lipschitz and hyperbolically Lipschitz continuity

We first construct a (K,K')-qc mapping with unbounded image domain which is neither Lipschitz nor Hölder continuous. This is different from the class of (K,K')-qc mappings with bounded image domains given by Kalaj and Mateljević [12].

Example 2.1. Let $f(z)=x(1+y)+e^{-y}\sin x+i\frac{1}{2}(y^2-x^2),\ y\geq |x|$. Then f is a $(3,4)$-qc mapping of the angular domain $\Delta=\{z=x+iy\mid y\geq |x|,\ x\in \mathbf{R}\}$ onto $\mathbf{H}$. Moreover, it is neither Lipschitz nor Hölder continuous.

Proof. If $f(z)=u(z)+iv(z)$ is a topological and ACL mapping in Ω, then we have

$$f_z=\tfrac{1}{2}(f_x-if_y)=\tfrac{1}{2}[u_x+v_y+i(v_x-u_y)],$$

and

$$f_{\bar z}=\tfrac{1}{2}(f_x+if_y)=\tfrac{1}{2}[u_x-v_y+i(v_x+u_y)].$$

Then it follows that

$$|f_z|^2-|f_{\bar z}|^2=u_xv_y-u_yv_x>0,$$

and

$$(|f_z|+|f_{\bar z}|)^2\leq u_x^2+v_y^2+u_y^2+v_x^2+u_xv_y-u_yv_x. \tag{2.1}$$

Let $f(z)=x(1+y)+e^{-y}\sin x+i\frac{1}{2}(y^2-x^2)$. Using the above formulas, we obtain

$$J_f=|f_z|^2-|f_{\bar z}|^2=x^2+y^2+y+ye^{-y}\cos x-xe^{-y}\sin x, \tag{2.2}$$

and

$$u_x^2+v_y^2+u_y^2+v_x^2\leq 2(x^2+y^2+y+ye^{-y}\cos x-xe^{-y}\sin x)+4. \tag{2.3}$$

Moreover,

$$\frac{|f_{\bar z}|}{|f_z|}=\frac{|1+e^{-y}\cos x-ie^{-y}\sin x|}{|1+e^{-y}\cos x+2y+i(e^{-y}\sin x-2x)|}\to 1,\ \text{as } y\to 0,\ x\to 0. \tag{2.4}$$

Combining (2.1) with (2.2) and (2.3), we have

$$(|f_z|+|f_{\bar z}|)^2\leq 3J_f+4.$$

It follows that $f(z)=x(1+y)+e^{-y}\sin x+i\frac{1}{2}(y^2-x^2),\ y\geq |x|$, is a (3,4)-qc mapping. By (2.4), f is not a K-qc mapping for any $K\geq 1$.

Choosing two points $z_1=iy_1, z_2=iy_2\in H$, we have

$$\frac{|f(z_1)-f(z_2)|}{|z_1-z_2|^\alpha}=\frac{|y_1^2-y_2^2|}{2|y_1-y_2|^\alpha}\to +\infty,\ \text{as } y_2\to+\infty,$$

for $0<\alpha\leq 1$. It concludes that f is neither Lipschitz nor Hölder continuous. □

The following Lemma A plays a key role in this paper.

Lemma A. [1] *Let $f=u+iv$ be a harmonic 1–1 mapping of* $\mathbf{H}$ *onto itself and continuous on $\mathbf{H}\cup\mathbf{R}$ such that $f(\infty)=\infty$. Then $v(z)=c\operatorname{Im} z$, where c is a positive constant. Especially, v has bounded partial derivatives on* $\mathbf{H}$.

Example 2.2. Let $h(x)=x+\sin x,\ x\in\mathbf{R}$. Then there exists a (K,K')-qch mapping f of $\mathbf{H}$ onto itself with the boundary value h. Moreover, f is a (2,1)-qch mapping but fails to be a K-qch mapping for any $K\geq 1$.

Proof. Define f as

$$f(z)=\int_{-\infty}^{+\infty}\frac{yh(t)}{(x-t)^2+y^2}\,dt+iy,$$

where the integral is in the sense of Cauchy principle value. We have

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{yh(t)}{(x-t)^2+y^2}\,dt&=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{(u+x)y}{u^2+y^2}\,du+\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{y\sin(u+x)}{u^2+y^2}\,du\\&=x+e^{-y}\sin x.\end{aligned}$$

So $f(z)=x+e^{-y}\sin x+iy$ is a harmonic mapping of $\mathbf{H}$ onto itself and $f|_R=x+\sin x=h(x)$. After some concrete calculations, we get

$$\begin{aligned}f_z&=\tfrac{1}{2}(f_x-if_y)=\tfrac{1}{2}(2+e^{-y}\cos x+ie^{-y}\sin x),\\f_{\bar z}&=\tfrac{1}{2}(f_x+if_y)=\tfrac{1}{2}(e^{-y}\cos x-ie^{-y}\sin x),\\L_fl_f&=1+e^{-y}\cos x,\\L_f^2&=\tfrac{1}{4}(4+4e^{-y}\cos x+2e^{-2y})+\tfrac{1}{2}e^{-y}\sqrt{4+4e^{-y}\cos x+e^{-2y}}\\&\le\tfrac{1}{4}(4+4e^{-y}\cos x+2e^{-2y})+\tfrac{1}{4}(4+4e^{-y}\cos x+2e^{-2y})\\&=2(1+e^{-y}\cos x)+e^{-2y}.\end{aligned}$$

From the above relations we have

$$L_f^2\le 2L_fl_f+1;\quad \frac{|f_{\bar z}|}{|f_z|}\to 1\ \text{ when }\ \cos x\to -1,\ y\to 0.$$

It is easy to see that f is not bi-Lipschitz. By the definition of (K,K')-qch mapping we conclude that $f(z)=x+e^{-y}\sin x+iy$ is a $(2,1)$-qch mapping but it is not $(K,0)$-qch mapping for any $K\ge 1$. □

Lemma 2.1. *Let f be a harmonic mapping of $\mathbf{H}$ onto itself and continuous up to its boundary with $f(\infty)=\infty$. If f is a (K,K')-qc mapping, then f can be represented by $f=u+icy$ and the gradient of f is such that*

$$L_f=|f_z|+|f_{\bar z}|\le cK+\sqrt{K'},\tag{2.5}$$

where c is a positive constant.

Proof. By the definition of (K,K')-qc mapping, we have

$$L_f^2\le KL_fl_f+K'.$$

This implies that

$$L_f\le\frac{Kl_f+\sqrt{K^2l_f^2+4K'}}{2},$$

and consequently

$$L_f\le Kl_f+\sqrt{K'}.\tag{2.6}$$

According to Lemma A, we can assume that $f=u+ic\operatorname{Im}z=\frac{1}{2}(g(z)+cz+\overline{g(z)-cz})$, where g is a holomorphic function in $\mathbf{H}$. Hence

$$|f_z|=\tfrac{1}{2}|g'(z)+c|,\quad |f_{\bar z}|=\tfrac{1}{2}|g'(z)-c|.\tag{2.7}$$

Using (2.6) and (2.7) we have

$$(K+1)|f_{\bar z}| \le (K-1)|f_z| + \sqrt{K'} \le (K-1)(|f_{\bar z}| + c) + \sqrt{K'}.$$

So

$$2|f_{\bar z}| \le c(K-1) + \sqrt{K'},$$

and

$$|g'(z) - c| \le c(K-1) + \sqrt{K'}. \tag{2.8}$$

It is easy to get

$$|f_{\bar z}| = \tfrac{1}{2}|g'(z) - c| \le \tfrac{1}{2}(c(K-1) + \sqrt{K'}),$$
$$|f_z| = \tfrac{1}{2}|g'(z) + c| \le \tfrac{1}{2}(c(K+1) + \sqrt{K'}),$$

thus we obtain

$$|f_z| + |f_{\bar z}| \le cK + \sqrt{K'}. \qquad \square$$

Theorem 2.1. *Let $f(z) = u(z) + iv(z)$ be a harmonic mapping of $\mathbf{H}$ onto itself and continuous up to its boundary with $f(\infty) = \infty$. If f is also a (K, K')-qc mapping, then f is Lipschitz. Moreover,*

$$|f(z_1) - f(z_2)| \le (cK + \sqrt{K'})|z_1 - z_2|,$$

where c is a positive constant.

Proof. Let ℓ' be the line segment connecting z_1 and z_2. By Lemma 2.1 we have

$$|f(z_1) - f(z_2)| = \int_{f(\ell')} |df| \le \int_{\ell'} L_f |dz| \le (cK + \sqrt{K'})|z_1 - z_2|. \qquad \square$$

Theorem 2.2. *Let f be a harmonic mapping of $\mathbf{H}$ onto itself and continuous on $\boldsymbol{H} \cup \mathbf{R}$ with $f(\infty) = \infty$. If f is a (K, K')-qc mapping, then f is hyperbolically Lipschitz. Moreover,*

$$\mathcal{H}(f(z_1), f(z_2)) \le (cK + \sqrt{K'})\mathcal{H}(z_1, z_2),$$

where $\mathcal{H}(\cdot,\cdot)$ denotes the hyperbolic distance and c is a positive constant.

Proof. Let ℓ be the hyperbolic geodesic connecting two arbitrary points z_1 and z_2 in $\mathbf{H}$. Let ρ represent the hyperbolic metric density of $\mathbf{H}$. Since $\rho(f(z)) = \rho(z),\ z \in \mathbf{H}$, we have from Lemma 2.1 that

$$\begin{aligned}\mathcal{H}(f(z_1), f(z_2)) &= \int_{f(\ell)} \rho(f(z))|df| \le \int_{\ell} \rho(f(z)) L_f |dz| \\ &\le (cK + \sqrt{K'}) \int_{\ell} \rho(z)|dz| = (cK + \sqrt{K'})\mathcal{H}(z_1, z_2). \qquad \square\end{aligned}$$

Remark 2.1. In fact we can extend the above results of the Theorems 2.1 and 2.2 a little. That is, if $f = u + iv\colon \mathbf{H} \to \mathbf{H}$ is a (K, K')-quasiconformal mapping with the assumption that $f \in C^1$ and $v(z) = cy$ for some constant $c > 0$, then f is Lipschitz continuous with respect to the euclidean and hyperbolic metric.

Remark 2.2. The results of Theorem 2.1 and 2.2 generalize the results obtained by Knežević and Mateljević [14]. If $f(z) = Kx + iy,\ z \in \mathbf{H}$, then f is a K-qch mapping of $\mathbf{H}$ onto itself and satisfies the equality $L_f = K$. Hence, the above results are asymptotically sharp as K' tends to 0.

3. Equivalent conditions of (K, K')-qch mappings

In order to prove Theorem 1.1, we would like to introduce some useful knowledge about harmonic conjugate functions [6]. Actually, for every harmonic function F on $\mathbf{H}$ with a boundary value $w \in L^\infty(\mathbf{R})$, there exists a harmonic conjugate function denoted by $\widetilde{F}$ with $\widetilde{F}(i) = 0$ and the representation of $\widetilde{F}$ is given by

$$\widetilde{F}(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \left(q(z,t) + \frac{t}{t^2+1} \right) w(t)\, dt,$$

where

$$q(z,t) = \frac{x-t}{|z-t|^2}$$

is the conjugate Poisson kernel.

The Hilbert transformation of $w \in L^\infty(R)$ is given by the formula

$$H[w](x) = \lim_{\varepsilon \to 0} H_\varepsilon w(x) = \lim_{\varepsilon \to 0} \int_{|x-t|>\varepsilon} \frac{1}{\pi} \left(\frac{1}{x-t} + \frac{t}{t^2+1} \right) w(t)\, dt. \tag{3.1}$$

The connection between $H[w](z)$ and $\widetilde{F}(z)$ is given by the formulas

$$\lim_{y \to 0} H_y[w](z) = \lim_{y \to 0} \widetilde{F}(z), \tag{3.2}$$

and

$$\widetilde{F}(z) = \frac{1}{\pi} \int_{-\infty}^{+\infty} p(z,t) H[w](t)\, dt. \tag{3.3}$$

In fact, if F is a harmonic function in $\mathbf{H}$, then $F_y(i) - F_y(z)$ is a harmonic conjugate function of F_x, where F_x and F_y denote the partial derivatives of F.

The following three lemmas are valid for (K, K')-qch mappings of $\mathbf{H}$ onto itself and extend the results obtained by Kalaj and Pavlović [13] a little. For completeness, we also give their proofs point by point as follows.

For a harmonic mapping $f = u + iv$ on $\mathbf{H}$, let

$$f(i) = a + ic, \quad \phi(z) = \frac{\partial u}{\partial z} = \frac{1}{2}(u_x - iu_y).$$

We get from Lemma A that the following holds.

Lemma 3.1. *Let f be a harmonic mapping of $\mathbf{H}$ onto itself and continuous on $\mathbf{H} \cup \mathbf{R}$ with $f(\infty) = \infty$. If f is a (K, K')-qc mapping, then it has a representation of the from*

$$f(z) = 2\Re \int_i^z \phi(\zeta)\, d\zeta + a + icy, \tag{3.4}$$

where

1. *$a + ic$ is a point in $\mathbf{H}$,*
2. *ϕ is a holomorphic function on $\mathbf{H}$ and $\phi(H)$ lies in a bounded subset of the right half plane.*

Conversely, if (1) and (2) are satisfied, then the function f defined by (3.4) is a (K, K')-qch mapping defined on $\mathbf{H}$.

Proof. Let $f=u+iv$ be a (K,K')-qch mapping of $\mathbf{H}$ onto itself and continuous up to its boundary. By the definition of (K,K')-qc mapping we have

$$(|f_z|+|f_{\bar z}|)^2\le K(|f_z|^2-|f_{\bar z}|^2)+K',$$

where

$$f_z=\tfrac12(f_x-if_y)=\frac12[u_x+v_y+i(v_x-u_y)],$$
$$f_{\bar z}=\tfrac12(f_x+if_y)=\frac12[u_x-v_y+i(v_x+u_y)].$$

Then from Lemma A we have

$$\tfrac12(u_x^2+u_y^2+c^2)+\tfrac12[(u_x^2+u_y^2+c^2)^2-4c^2u_x^2]^{\frac12}\le Kcu_x+K',$$
$$J_f=|f_z|^2-|f_{\bar z}|^2=u_xv_y-u_yv_x=cu_x.$$

Combining the above relations we obtain

$$u_x^2+u_y^2+c^2\le(K+1)cu_x+K'.$$

Hence there exists a constant M such that

$$0<u_x\le\frac{(K+1)c+\sqrt{(K+1)^2c^2-4(c^2-K')}}{2}=M, \tag{3.5}$$

and

$$|u_y|\le\sqrt{(K+1)cM+K'}. \tag{3.6}$$

Let $\phi(z)=u_z=\frac12(u_x-iu_y)$. Then $\phi(z)$ is holomorphic on $\mathbf{H}$. By Green formula it follows that

$$2\Re\int_i^z\phi(\zeta)\,d\zeta=u(z)-u(i).$$

Let $a=u(i)$. Then

$$f(z)=2\Re\int_i^z\phi(\zeta)\,d\zeta+a+icy,$$

and ϕ satisfies that $\phi(\mathbf{H})$ lies in a bounded subset in the right half plane.

Conversely, if f is represented by (3.4) and satisfies the conditions (1) and (2) in Lemma 3.1, then there exist two positive constants M' and M'' such that

$$0<u_x\le M',\quad |u_y|\le M''.$$

So we conclude that

$$L_f^2=(|f_z|+|f_{\bar z}|)^2\le u_x^2+u_y^2+c^2\le M'^2+M''^2+c^2.$$

Hence, f is $(1,M'^2+M''^2+c^2)$-qch mapping of $\mathbf{H}$ onto itself. □

Lemma 3.2. *Let $f=u+iv$ be a harmonic mapping of $\mathbf{H}$ onto itself and continuous on $\mathbf{H}\cup\mathbf{R}$ with $f(\infty)=\infty$. If f is a (K,K')-qc mapping, then the restriction φ of f to the real axis $\mathbf{R}$ is Lipschitz and the relations*

$$u_x(z)=\frac1\pi\int_{-\infty}^{+\infty}p(z,t)\varphi'(t)\,dt, \tag{3.7}$$

and

$$u_y(z)-u_y(i)=-\frac1\pi\int_{-\infty}^{+\infty}(q(z,t)+\frac{t}{t^2+1})\varphi'(t)\,dt \tag{3.8}$$

hold.

Proof. Using Lemma A and Theorem 2.1, we have that f is Lipschitz on $\mathbf{H}$. So the restriction φ of f to $\mathbf{R}$ is Lipschitz and $|\varphi'| \in L^\infty(\mathbf{R})$. By (3.5) we have that the function u_x is bounded on $\mathbf{H}$ and according to Fatou's theorem the limit

$$\lim_{y\to 0} u_x(x,y) = w(x)$$

exists for almost all $x \in \mathbf{R}$. Hence, $w(t) \in L^\infty(\mathbf{R})$. Furthermore, we have

$$u_x(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} p(z,t)w(t)\,dt.$$

By (3.5) we easily know that $u_t(t,y)$ is bounded. Then u satisfies the relation

$$u(x,y) - u(0,y) = \int_0^x u_t(t,y)\,dt.$$

By the dominated convergence theorem, we naturally obtain

$$\lim_{y\to 0}\int_0^x u_t(t,y)\,dt = \int_0^x w(t)\,dt.$$

On the other hand,

$$\lim_{y\to 0}(u(x,y) - u(0,y)) = \varphi(x) - \varphi(0),$$

and therefore

$$\varphi(x) - \varphi(0) = \int_0^x w(t)\,dt.$$

Hence $\varphi'(x) = w(x)$ a.e. Thus (3.7) naturally holds. The validity of (3.8) now follows from (3.7) by the fact that the function $u_y(i) - u_y(z)$ is a harmonic conjugate function of $u_x(z)$. □

Lemma 3.3. *Let $f = u + iv$ be a harmonic mapping of $\mathbf{H}$ onto itself and continuous up to its boundary with $f(\infty) = \infty$. If f is a (K,K')-qc mapping and φ is the restriction of f on $\mathbf{R}$, then the function $H[\varphi'](z) \in L^\infty(\mathbf{R})$ and the equation*

$$u_y(z) - u_y(i) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} p(z,t)H[\varphi'](t)\,dt \tag{3.9}$$

holds.

Proof. By the inequality (3.6) it follows that the function $\widetilde{v}(z) = u_y(i) - u_y(z)$ is bounded on $\mathbf{H}$. Since $\widetilde{v}$ is the harmonic conjugate of $\widetilde{u} = u_x$, we have from (3.1), (3.2) and (3.3) that

$$H[\varphi'] = \lim_{y\to 0} H_y[\varphi'](z) = \lim_{y\to 0}(u_y(i) - u_y(z)).$$

Hence, $H[\varphi'](z) \in L^\infty(R)$ and

$$u_y(i) - u_y(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} p(z,t)H[\varphi'](t)\,dt. \qquad \square$$

Next, using the above three Lemmas we will give the proof of Theorem 1.1.

Proof. By Lemma 3.1, we get (1) ⇒ (2). The inequalities (3.5) and (3.6) imply that

$$L_f \le |u_x| + |u_y| + c \le M + c + \sqrt{(K+1)cM + K'}.$$

So L_f is bounded. Hence (2) $\Rightarrow$ (3).

To prove (3) $\Rightarrow$ (1), we note that $f(z)$ is a Lipschitz mapping of $\mathbf{H}$ onto $\mathbf{H}$. So there exists a constant $L>0$ such that

$$|f(z')-f(z)| \le L|z'-z|.$$

Let $\zeta = z + re^{i\theta}$. We have

$$\lim_{\zeta\to z}\frac{|f(\zeta)-f(z)|}{|\zeta-z|} = \lim_{r\to 0}\frac{|f(z+re^{i\theta})-f(z)|}{r} = |f_\zeta e^{i\theta} + f_{\bar\zeta}e^{-i\theta}|,$$

and

$$|f_\zeta| + |f_{\bar\zeta}| = \sup_\theta |f_\zeta e^{i\theta} + f_{\bar\zeta}e^{-i\theta}| \le L.$$

Thus we get

$$(|f_\zeta| + |f_{\bar\zeta}|)^2 \le |f_\zeta|^2 - |f_{\bar\zeta}|^2 + L^2.$$

Therefore f is $(1,L^2)$-qc mapping.

That (1) implies (4) follows from Lemma 3.2 and 3.3. To prove (4) implies (1), we assume that φ is the restriction of f to $\mathbf{R}$. By the conditions of (4), that is, $\varphi' \in L^\infty(\mathbf{R}),\ H[\varphi'] \in L^\infty(\mathbf{R})$, we can define a harmonic function $\widetilde{u}$ and its harmonic conjugate $\widetilde{v}$ on $\mathbf{H}$ as

$$\widetilde{u}(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} p(z,t)\varphi'(t)\,dt, \tag{3.10}$$

$$\widetilde{v}(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \left(q(z,t) + \frac{t}{t^2+1}\right)\varphi'(t)\,dt. \tag{3.11}$$

Let $\phi(z) = \frac{1}{2}(\widetilde{u}(z) + i\widetilde{v}(z))$. Then ϕ is a holomorphic function on $\mathbf{H}$. Define g by

$$g(z) = 2\Re\int_i^z \phi(\zeta)\,d\zeta + icy. \tag{3.12}$$

Since φ is absolutely continuous and $\varphi' \in L^\infty(\mathbf{R})$, it follows that $0 \le \varphi' < \lambda$, where λ is a positive constant. Thus, by (3.10) we get $0 \le \widetilde{u}(z) < \lambda,\ z \in \mathbf{H}$. By (3.11) there exists a positive constant such that $|\widetilde{v}(z)| \le M$. So the function ϕ maps $\mathbf{H}$ into a bounded subset of the right half plane. According to Lemma 3.1 we conclude that g is a (K,K')-qch mapping on $\mathbf{H}$. By (3.12)

$$(\Re g)_x = \widetilde{u}(z) = \frac{1}{\pi}\int_{-\infty}^{+\infty} p(z,t)\varphi'(t)\,dt$$

holds on $\mathbf{H}$. Since $\widetilde{u}$ is bounded on $\mathbf{H}$, we have that $\Re g$ satisfies the relation

$$\Re g(x,y) - \Re g(0,y) = \int_0^x (\Re g)_x(t,y)\,dt.$$

By the dominated convergence theorem, we naturally have

$$\lim_{y\to 0}\int_0^x (\Re g)_x(t,y)\,dt = \int_0^x \varphi'(t)\,dt.$$

By the fact that g is continuous on its boundary, then the restriction $\widetilde{\varphi}$ of g to $\mathbf{R}$ is Lipschitz and satisfies that $|\widetilde{\varphi}'| \in L^\infty(\mathbf{R})$ and

$$\lim_{y\to 0}(\Re g(x,y) - \Re g(0,y)) = \widetilde{\varphi}(x) - \widetilde{\varphi}(0).$$

Therefore

$$\widetilde{\varphi}(x) - \widetilde{\varphi}(0) = \int_0^x \varphi'(t)\,dt.$$

Hence $\varphi' = \widetilde{\varphi}'$ a.e. on $\mathbf{R}$. Since φ and $\widetilde{\varphi}$ are absolutely continuous, $\varphi = \widetilde{\varphi} + a'$ for some $a' \in \mathbf{R}$. Thus $f = g + a'$ and $\varphi = f|_{\mathbf{R}}$. So f is also a (K, K')-qc mapping on $\mathbf{H}$. □

4. Area distortion

In order to estimate the euclidean and hyperbolic area distortion of a (K, K')-qch mapping of $\mathbf{H}$ onto itself, we first estimate its Jacobian.

Lemma 4.1. *Assume that f is a harmonic mapping of $\mathbf{H}$ onto itself and continuous on $\mathbf{H} \cup \mathbf{R}$ with $f(\infty) = \infty$. If f is a (K, K')-qc mapping, then $f(z) = \frac{1}{2}(g(z) + cz + \overline{g(z) - cz})$, where g is a holomorphic function in $\mathbf{H}$ and c is a positive constant and the Jacobian of f is such that*

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 \le c^2 K + c\sqrt{K'}. \tag{4.1}$$

Proof. According to the definition of (K, K')-qc mappings and the inequality (2.7), we get

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 = \tfrac{1}{4}(|g'(z) + c|^2 - |g'(z) - c|^2) = \tfrac{c}{2}(g'(z) + \overline{g'(z)}) \le c|g'(z)|.$$

By (2.8) we obtain

$$||g'(z)| - c| \le |g'(z) - c| \le c(K-1) + \sqrt{K'},$$

so we conclude that

$$J_f \le c|g'(z)| \le c^2 K + c\sqrt{K'}.$$ □

Theorem 4.1. *Let $f = u + iv$ be a harmonic mapping of $\mathbf{H}$ onto itself and continuous up to its boundary with $f(\infty) = \infty$. If f is a (K, K')-qc mapping, then for any measurable subset $E \subset \mathbf{H}$,*

$$A_{\mathrm{euc}}(f(E)) \le (c^2 K + c\sqrt{K'}) A_{\mathrm{euc}}(E),$$

where $A_{\mathrm{euc}}(\cdot)$ denotes the euclidean area and c is a positive constant.

Proof. According to Lemma 4.1, we obtain

$$A_{\mathrm{euc}}(f(z)) = \iint_E J_f(z)|dz|^2 \le (c^2 K + c\sqrt{K'}) A_{\mathrm{euc}}(E).$$ □

Theorem 4.2. *Let f be a harmonic mapping of $\mathbf{H}$ onto itself and continuous up to its boundary with $f(\infty) = \infty$. If f is a (K, K')-qc mapping, then for any measurable subset $E \subset H$, we get*

$$A_{\mathrm{hyp}}(f(E)) \le (c^2 K + c\sqrt{K'}) A_{\mathrm{hyp}}(E),$$

where A_{hyp} denotes the hyperbolic area and c is a positive constant.

Proof. Let ρ be the hyperbolic metric density of $\mathbf{H}$. By the fact that $\rho(f(z)) = \rho(z)$, $z \in \mathbf{H}$, we obtain by Lemma 4.1 that

$$\begin{aligned} A_{\mathrm{hyp}}(f(E)) &= \iint_{f(E)} \rho^2(f(z))|df(z)|^2 = \iint_E \rho^2(z) J_f(z)|dz|^2 \\ &\le (c^2 K + c\sqrt{K'}) A_{\mathrm{hyp}}(E). \end{aligned}$$ □

Remark 4.1. When $K' = 0$, the above results are sharp. If $f(z) = Kx + iy,\ z \in H$, then f is a K-qch mapping of $\mathbf{H}$ onto itself and satisfies the equalities $J_f = K,\ L_f = K$. The results of Theorem 4.1 and 4.2 generalize the results of Chen [2].

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