Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 37, 2012, 229–243

LIPSCHITZ EQUIVALENCE OF A CLASS OF SELF-SIMILAR SETS WITH COMPLETE OVERLAPS

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Abstract. Fix $r \in (0, 1/3]$. We discuss a class of self-similar sets $\{K_n\}_{n\geq 1}$ with complete overlaps, where $K_n = (rK_n) \cup (rK_n + r^n(1-r)) \cup (rK_n + 1 - r)$. We prove that for any $n_1, n_2 \geq 1$, K_{n_1} and K_{n_2} are Lipschitz equivalent.

1. Introduction

Suppose $\{h_i\colon \mathbf{R}^m \to \mathbf{R}^m\}_{i=1}^k$ are contractive similitudes. We say that a compact set $\Lambda = h_1(\Lambda) \cup \cdots \cup h_k(\Lambda)$ is a self-similar set with *overlaps*, if there are $i \neq j$ such that $h_i(\Lambda) \cap h_j(\Lambda) \neq \varnothing$.

Self-similar sets with overlaps have very *complicated* structures. For example, the open set condition (OSC), which means the overlaps are little, was introduced by Moran [18] and studied by Hutchinson [10]. Schief [24], Bandt and Graf [1] showed the relation between the open set condition and the positive Hausdorff measure. Falconer [6] proved some "generic" results on Hausdorff dimension of self-similar sets without the assumption about the open set condition. One useful notion "transversality" to study self-similar sets (or measures) with overlaps can be found e.g. in Keane, Smorodinsky and Solomyak [11], Pollicott and Simon [19], Simon and Solomyak [25] and Solomyak [26]. Feng and Lau [9], Lau and Ngai [14] studied the weak separation condition. Please refer to Bandt and Hung [2], Sumi [27] for some recent work.

For self-similar set $E_\lambda = E_\lambda/3 \cup (E_\lambda/3 + \lambda/3) \cup (E_\lambda/3 + 2/3)$, a conjecture of Furstenberg says that $\dim_{\mathrm{H}} E_\lambda = 1$ for any λ irrational. Świątek and Veerman [28] proved that $\dim_{\mathrm{H}} E_\lambda > 0.767$ for every λ irrational. Kenyon [12], Rao and Wen [23] obtained that $\mathcal{H}^1(E_\lambda) > 0$ if and only if $\lambda = p/q \in \mathbf{Q}$ with $p \equiv q \not\equiv 0 \pmod 3$. The key idea of [23] is "graph-directed struture" introduced by Mauldin and Williams [17].

In particular, Rao and Wen [23] studied the self-similar sets with "complete overlaps".

doi:10.5186/aasfm.2012.3712
2010 Mathematics Subject Classification: Primary 28A80.
Key words: Fractals, self-similar set, Lipschitz equivalence, complete overlaps.
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This work is supported by National Natural Science Foundation of China (No. 11071224), Program for Excellent Talents in University of China, NSF of Ningbo (No. 2011A610176) and Post-Doctoral Science Project of Jiangsu Province (No. 1001080c).

Definition 1. We say that a self-similar set $\Lambda = h_1(\Lambda) \cup \cdots \cup h_k(\Lambda)$ has complete overlaps, if there are words $i_1 i_2 \cdots i_{n_1} \in \{1, \cdots, k\}^{n_1}$, $j_1 j_2 \cdots j_{n_2} \in \{1, \cdots, k\}^{n_2}$ with $i_1 \neq j_1$ such that

$$h_{i_1} \circ h_{i_2} \circ \cdots \circ h_{i_{(n_1)}}(x) \equiv h_{j_1} \circ h_{j_2} \circ \cdots \circ h_{j_{(n_2)}}(x). \tag{1.1}$$

Here (1.1) implies $h_{i_1} \circ h_{i_2} \circ \cdots \circ h_{i_{(n_1)}}(\Lambda) = h_{j_1} \circ h_{j_2} \circ \cdots \circ h_{j_{(n_2)}}(\Lambda)$.

Remark 1. The cases according to "complete ovelaps" and "OSC" are quite different. For example, suppose $h_1, \cdots, h_k$ have the same ratio α. Then (1.1) implies $n_1 = n_2 = n$ and $\dim_{\mathrm{H}} \Lambda \leq -\frac{\log(k^n-1)}{\log \alpha^n} < -\log k/\log \alpha$. However OSC implies $\dim_{\mathrm{H}} \Lambda = -\log k/\log \alpha$.

Remark 2. It is shown in [23] that for $\lambda \in \mathbf{Q}$, $\dim_{\mathrm{H}} E_\lambda < 1$ if and only if E_λ has complete overlaps, $E_{2/3^n}$ has complete overlaps and $\dim_{\mathrm{H}} E_{2/3^n} = \log_3 \frac{3+\sqrt{5}}{2}$ for all $n \geq 1$.

The other interesting topic on self-similar sets is their Lipschitz equivalence. Here two compact subsets X_1, X_2 of Euclidean spaces are said to be Lipschitz equivalent, if there is a bijection $f\colon X_1 \to X_2$ and a constant $C > 0$ such that for all $x, y \in X_1$,

$$C^{-1}|x-y| \leq |f(x) - f(y)| \leq C|x-y|.$$

If compact sets X_1 and X_2 are Lipschitz equivalent, then $\dim_{\mathrm{H}} X_1 = \dim_{\mathrm{H}} X_2$. However, it is worth pointing out that Cooper and Pignataro [3], Falconer and Marsh [8], David and Semmes[4] and Wen and Xi [30] showed that two self-similar sets *need not* be Lipschitz equivalent although they have the same Hausdorff dimension.

For self-similar sets *without* overlaps, Cooper and Pignataro [3], Falconer and Marsh [8], David and Semmes [4] and Xi [32] posed some algebraic conditions upon the ratios of similitudes for two given self-similar sets without overlaps to be Lipschitz equivalent. The quasi-Lipschitz equivalence, which is weaker than the Lipschitz equivalence, was studied for self-conformal sets and Ahlfors–David regular sets in Xi [31] and Wang and Xi [29], respectively. The Lipschitz embedding of fractals can be found in Llorente and Mattila [15], Mattila and Saaranen [16] and Deng and Wen et al. [5]. Please also refer to Rao, Ruan and Yang [21] and Rao, Ruan and Wang [22].

For self-similar sets *with* overlaps, an interesting result is on the {1,3,5}-{1,4,5} Problem which was posed by David and Semmes [4]. Let $H_1 = (H_1/5) \cup (H_1/5 + 2/5) \cup (H_1/5 + 4/5)$ be the {1,3,5} self-similar set, and $H_2 = (H_2/5) \cup (H_2/5 + 3/5) \cup (H_2/5 + 4/5)$ the {1,4,5} self-similar set. The problem asks about the Lipschitz equivalence of H_1 and H_2. Rao, Ruan and Xi [20] proved that H_1 and H_2 are Lipschitz equivalent. Furthermore, Xi and Ruan [33] proved that for given $r_1, r_2, r_3 \in (0, 1)$ with $r_1 + r_2 + r_3 < 1$, self-similar sets J_1 and J_2 are Lipschitz equivalent if and only if $\log r_1/\log r_2 \in \mathbf{Q}$, where

$$J_1 = (r_1 J_1) \cup (r_2 J_1 + 1 - r_2 - r_3) \cup (r_3 J_1 + 1 - r_3),$$
$$J_2 = (r_1 J_2) \cup (r_2 J_2 + \frac{1 + r_1 - r_2 - r_3}{2}) \cup (r_3 J_2 + 1 - r_3).$$

Xi and Xiong [34] generalized the result on the {1,3,5}-{1,4,5} Problem to the higher dimensional spaces. Given integers $n \geq 2$ and $m \geq 1$, for $A, B \subset \{0, 1, \cdots, (n-1)\}^m$, let $E_A = \cup_{a \in A} \frac{E_A + a}{n}$ and $E_B = \cup_{b \in B} \frac{E_B + b}{n}$ be self-similar sets in $\mathbf{R}^m$. Suppose that E_A and E_B are totally disconnected. Then it is proved in [34] that E_A and E_B are

Lipschitz equivalent if and only if A and B have the same cardinality. Please also refer to Wen and Xi [30], Xi and Xiong [35] and Zhu et al. [36].

We notice that for the self-similar sets considered in [20], [30], [33]–[36], the *open set condition* holds, which means two pieces of self-similar sets touch a little. In this paper, we will consider a quite different case according to *complete overlaps*.

Fix $r \in (0, 1/3]$. For any integer $n \geq 1$, let K_n be a self-similar set satisfying

$$K_n = (rK_n) \cup (rK_n + r^n(1-r)) \cup (rK_n + 1 - r).$$

The main result of the paper is stated as follows.

Theorem 1. *For any $n_1, n_2 \geq 1$, K_{n_1} and K_{n_2} are Lipschitz equivalent.*

Remark 3. K_n has *complete overlaps.* In fact, K_n is generated by

$$S_1(x) = rx, \quad S_2(x) = rx + r^n(1-r), \quad S_3(x) = rx + (1-r). \tag{1.2}$$

Let $S_{i_1 i_2 \cdots i_k} = S_{i_1} \circ S_{i_2} \circ \cdots \circ S_{i_k}$ and $[1]^t$ be the word composed of t digits 1. Then $S_{[1]^n 3}(x) \equiv S_{2[1]^n}(x) \equiv r^{n+1}x + r^n(1-r)$.

Remark 4. For $r = 1/3$, $K_n = E_{2/3^n}$ and it is shown in [23] that $\{K_n\}_n$ have the same Hausdorff dimension. In fact, the technique in [23] can deal with dimension for any self-similar set $\Lambda = \cup_{i=1}^k (\Lambda/n + b_i)$ where $n \in \mathbf{N}$ and $b_i \in \mathbf{Q}$ for all i. In this paper, fix any ratio $r(\leq 1/3)$ rational or *irrational*, for the special fractals $\{K_n\}_n$, we prove that $\{K_n\}_n$ belong to the same Lipschitz equivalent class, which implies $\{K_n\}_n$ have the same dimension.

Remark 5. For $r \in (0, 1/3]$, let $K = (r^{1/2}K) \cup (rK + 1 - r)$ be a self-similar set without overlaps. Proposition 5 in Section 6 says that K_n and K are Lipschitz equivalent for any n.

We organize the paper as follows. Section 2 is the preliminaries, including the counting function $L(m)$ (Lemma 1), the graph-directed sets (Lemma 2) and their corresponding criterion for Lipschitz equivalence (Lemma 3). In Section 3, for K_n we construct the graph-directed sets with ratio r and adjacency matrix M_n defined in (3.1). In Section 4, for K_1 we construct some graph-directed sets with ratio r and the same adjacency matrix M_n. Then it follows from Lemma 3 that K_1 and K_n are Lipschitz equivalent, hence Theorem 1 is proved. In Section 5, we obtain the Perron–Frobenius eigenvector of the matrix M_n in terms of the counting function $L(m)$ and the Fibonacci sequence. Section 6 shows that K_n is Lipschitz equivalent to a self-similar set K (in Remark 4) without overlaps.

2. Preliminaries

Fix $r \in (0, 1/3]$ and integer $n \geq 1$. Let E, F be the self-similar sets satisfying

$$E = rE \cup (rE + r(1-r)) \cup (rE + 1 - r), \tag{2.1}$$

$$F = rF \cup (rF + r^n(1-r)) \cup (rF + 1 - r). \tag{2.2}$$

Notice that $E = K_1$ and $F = K_n$.

Therefore for proving Theorem 1, we only need to prove the following proposition (in Section 4).

Proposition 1. *E and F are Lipschitz equivalent.*

Given any integer $m \in [0, 2^n - 1]$, let

$$m = x_0 x_1 \cdots x_{n-1} \quad (x_t = 0 \text{ or } 1),$$

be the dyadic representation of m, that is $m = 2^{n-1}x_0 + 2^{n-2}x_1 + \cdots + x_{n-1}$. Then we define the counting function

$$L(m) = L(x_0 x_1 \cdots x_{n-1}) = \sum\nolimits_{i=0}^{n-1} x_i, \tag{2.3}$$

i.e., the number of digits 1 in $x_0 x_1 \cdots x_{n-1}$. We have the following lemma.

Lemma 1. *Suppose* $m = x_0 x_1 \cdots x_{n-1} \in \mathbf{Z} \cap [0, 2^n - 1]$. *Then*

$$\begin{array}{ll} L(2m) = L(m) \text{ and } L(2m+1) = L(m) + 1 & \text{when } m < 2^{n-1}, \\ L(2(m - 2^{n-1})) = L(m) - 1 \text{ and } L(2(m - 2^{n-1}) + 1) = L(m) & \text{when } m \geq 2^{n-1}. \end{array}$$

Proof. When $m = x_0 \cdots x_{n-1} \in [0, 2^{n-1})$, we have $x_0 = 0$,

$$L(2m) = L(x_1 \cdots x_{n-1} 0) = L(0 x_1 \cdots x_{n-1}) = L(m),$$

and $L(2m+1) = L(x_1 \cdots x_{n-1} 1) = L(0 x_1 \cdots x_{n-1}) + 1 = L(m) + 1$.

When $m \in x_0 \cdots x_{n-1} = [2^{n-1}, 2^n - 1]$, we have $x_0 = 1$,

$$L(2(m - 2^{n-1})) = L(x_1 \cdots x_{n-1} 0) = L(1 x_1 \cdots x_{n-1}) - 1 = L(m) - 1,$$

and $L(2(m - 2^{n-1}) + 1) = L(x_1 \cdots x_{n-1} 1) = L(1 x_1 \cdots x_{n-1}) = L(m)$. □

Recall the notion of "graph-directed structure" [17] as follows.

Let $(V, \mathcal{E})$ be a directed graph, where $V = \{0, \cdots, (m-1)\}$ and $\mathcal{E}$ are vertex set and edge set respectively. Denote by $\mathcal{E}_{i,j}$ the set of edges from vertex i to vertex j. For this graph, we consider the *adjacency matrix* $\mathcal{A} = (a_{i,j})_{0 \leq i,j \leq m-1}$ defined by $a_{i,j} = \#\mathcal{E}_{i,j}$. Let $\rho(\mathcal{A})$ be the spectral radius of $\mathcal{A}$. The graph is said to be *transitive*, if for any vertexes i and j, there exists a directed path starting at i and ending at j.

To simplify the structure, we assume there exists $\tau \in (0,1)$ such that every edge $e \in \mathcal{E}$ is equipped with a contracting similitude $S_e \colon \mathbf{R}^n \to \mathbf{R}^n$ with ratio τ. By [17], there exists a unique family of non-empty compact sets $\Gamma_0, \Gamma_1, \cdots, \Gamma_{m-1}$ such that for any i,

$$\Gamma_i = \bigcup_{j=0}^{m-1} \bigcup_{e \in \mathcal{E}_{i,j}} S_e(\Gamma_j). \tag{2.4}$$

Here $\{\Gamma_0, \Gamma_1, \cdots, \Gamma_{m-1}\}$ are called graph-directed sets on $(V, \mathcal{E})$ with ratio τ. Furthermore, we say that $\{\Gamma_i\}_{i=0}^{m-1}$ are *dust-like* graph-directed sets on $(V, \mathcal{E})$, if the right hand of (2.4) is a disjoint union for each $0 \leq i \leq m-1$.

By [17], we have the following lemma.

Lemma 2. *Suppose that* $\{\Gamma_i\}_{i=0}^{m-1}$ *are dust-like graph-directed sets on a transitive graph with ratio* τ *and adjacency matrix* $\mathcal{A}$. *Then for every* i,

$$\dim_{\mathrm{H}} \Gamma_i = -\log \rho(\mathcal{A}) / \log \tau.$$

Theorem 2.1 of [20] yields the following lemma for Lipschitz equivalence.

Lemma 3. *Suppose* $\{\Gamma_i\}_{i=0}^{m-1}$ *and* $\{\Gamma_i'\}_{i=0}^{m-1}$ *are dust-like graph-directed sets with the same ratio and the same adjacency matrix. Then* Γ_i *and* Γ_i' *are Lipschitz equivalent for any* i.

For example, for $\{1,3,5\}$ set H_1 and $\{1,4,5\}$ set H_2 mentioned above, we obtained

$$\begin{array}{ll} \Gamma_0 = H_1, & \Gamma'_0 = H_2, \\ \Gamma_1 = H_1 \cup (H_1+2), & \Gamma'_1 = H_2 \cup (H_2+1), \\ \Gamma_2 = H_1 \cup (H_1+2) \cup (H_1+4), & \Gamma'_2 = H_2 \cup (H_2+1) \cup (H_2+2). \end{array}$$

It is pointed out in [20] that $\{\Gamma_i\}_{i=0}^2$ and $\{\Gamma'_i\}_{i=0}^2$ have the same ratio $1/5$ and adjacency matrix $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, which implies that $\Gamma_0(=H_1)$ and $\Gamma'_0(=H_2)$ are Lipschitz equivalent.

3. Graph-directed structure

Given $m = x_0x_1\cdots x_{n-1} \in \{0,1,2,\cdots,2^n-1\}$, set

$$A_m = \bigcup_{i_0i_1\cdots i_{n-1}\in\{0,1\}^n} (F + \sum_{t=0}^{n-1} r^t(1-r)(i_tx_t)),$$

where F is defined in (2.2). In particular, $A_0 = F$.

Let $M_n = (g_{i,j})_{0\le i,j\le 2^n-1}$ be a $2^n \times 2^n$ integer matrix, where

$$g_{i,j} = \begin{cases} 1, & \text{if } i < 2^{n-1},\ j = 2i, \\ 1, & \text{if } i < 2^{n-1},\ j = 2i+1, \\ 1, & \text{if } i \ge 2^{n-1},\ j = 2(i-2^{n-1}), \\ 2, & \text{if } i \ge 2^{n-1},\ j = 2(i-2^{n-1})+1, \\ 0, & \text{otherwise.} \end{cases}$$

That means

$$M_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2 \end{pmatrix}_{2^n\times 2^n}. \tag{3.1}$$

Proposition 2. *$\{A_m\}_{0\le m\le 2^n-1}$ are dust-like graph-directed sets with ratio r and adjacency matrix M_n, where $A_0 = F$.*

Proof. Let $m = x_0x_1\cdots x_{n-1}$ be the dyadic representation of m. We will discuss two cases.

Case 1. $0 \le m < 2^{n-1}$, that is $x_0 = 0$. We only need to check

$$rA_{2m+1} \cup (rA_{2m} + 1 - r) = A_m.$$

For this, we notice that

$$A_m = A_{0x_1\cdots x_{n-1}} = \bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} \left(F + \sum_{t=1}^{n-1} r^t(1-r)(i_t x_t)\right).$$

Applying equation (2.2), we have

$$A_m = T_1 \cup T_2 \cup T_3,$$

where

$$T_1 = \bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} \left(rF + \sum_{t=1}^{n-1} r^t(1-r)(i_t x_t)\right),$$
$$T_2 = \bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} \left(rF + r^{n-1}(1-r) + \sum_{t=1}^{n-1} r^t(1-r)(i_t x_t)\right),$$
$$T_3 = \bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} \left(rF + 1 - r + \sum_{t=1}^{n-1} r^t(1-r)(i_t x_t)\right).$$

Therefore, we have

$$\begin{aligned} T_1 \cup T_2 &= \left[\bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} r\left(F + \sum_{t=0}^{n-2} r^t(1-r)(i_{t+1}x_{t+1}) + r^{n-1}(1-r)(0\cdot 1)\right)\right] \\ &\quad \cup \left[\bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} r\left(F + \sum_{t=0}^{n-2} r^t(1-r)(i_{t+1}x_{t+1}) + r^{n-1}(1-r)(1\cdot 1)\right)\right] \\ &= \bigcup_{i_1\cdots i_{n-1}i_n\in\{0,1\}^{n}} r\left(F + \sum_{t=0}^{n-2} r^t(1-r)(i_{t+1}x_{t+1}) + r^{n-1}(1-r)(i_n\cdot 1)\right) \\ &= rA_{x_1\cdots x_m 1} = rA_{2m+1.} \end{aligned}$$

In the same way, we have

$$\begin{aligned} T_3 &= \bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} \left(rF + 1 - r + \sum_{t=1}^{n-1} r^t(1-r)(i_t x_t)\right) \\ &= r\left[\bigcup_{i_1\cdots i_{n-1}i_n\in\{0,1\}^{n}} \left(F + \sum_{t=0}^{n-2} r^t(1-r)(i_{t+1}x_{t+1}) + r^{n-1}(1-r)(i_n\cdot 0)\right)\right] + (1-r) \\ &= rA_{x_1\cdots x_{n-1}0} + (1-r) \\ &= rA_{2m} + (1-r). \end{aligned}$$

Since $rA_{2m+1} \subset [0,r]$, $rA_{2m} + (1-r) \subset [1-r,1]$ and $r \le 1/3$, we have the disjoint union

$$A_m = rA_{2m+1} \cup (rA_{2m} + 1 - r). \tag{3.2}$$

Case 2. $m \geq 2^{n-1}$, that is $x_0 = 1$,

$$A_m = A_{1x_1\cdots x_{n-1}} = \bigcup_{i_0=0}^{1} \bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} (F + \sum_{t=0}^{n-1} r^t(1-r)(i_t x_t)).$$

Let

$$R_{i_0} = \bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} (F + (1-r)(i_0) + \sum_{t=1}^{n-1} r^t(1-r)(i_t x_t)).$$

Then

$$A_m = R_0 \cup R_1.$$

Applying (2.2), we have

$$\begin{aligned} R_0 = &\left[\bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} (rF + \sum_{t=1}^{n-1} r^t(1-r)(i_t x_t))\right] \\ &\cup \left[\bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} (rF + r^n(1-r) + \sum_{t=1}^{n-1} r^t(1-r)(i_t x_t))\right] \\ &\cup \left[\bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} (rF + (1-r) + \sum_{t=1}^{n-1} r^t(1-r)(i_t x_t))\right] \\ = &\, R_{0,1} \cup R_{0,2} \cup R_{0,3}. \end{aligned}$$

In the same way as above, we obtain

$$R_{0,1} \cup R_{0,2} = rA_{x_1\cdots x_{n-1}1} = rA_{2m+1}.$$

Applying (2.2), we also have

$$\begin{aligned} R_1 = &\left[\bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} (rF + (1-r) + \sum_{t=1}^{n-1} r^t(1-r)(i_t x_t))\right] \\ &\cup \left[\bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} (rF + r^n(1-r) + (1-r) + \sum_{t=1}^{n-1} r^t(1-r)(i_t x_t))\right] \\ &\cup \left[\bigcup_{i_1\cdots i_{n-1}\in\{0,1\}^{n-1}} (rF + 2(1-r) + \sum_{t=1}^{n-1} r^t(1-r)(i_t x_t))\right] \\ = &\, R_{1,1} \cup R_{1,2} \cup R_{1,3}. \end{aligned}$$

Here $R_{1,1} = R_{0,3}$. In the same way,

$$R_{1,1} \cup R_{1,2} = rA_{2m+1} + (1-r) \text{ and } R_{1,3} = rA_{2m} + 2(1-r).$$

Therefore

$$\begin{aligned} A_m &= R_{0,1} \cup R_{0,2} \cup R_{0,3} \cup R_{1,1} \cup R_{1,2} \cup R_{1,3} \\ &= (R_{0,1} \cup R_{0,2}) \cup (R_{1,1} \cup R_{1,2}) \cup R_{1,3} \qquad (3.3) \\ &= rA_{2m+1} \cup (rA_{2m+1} + (1-r)) \cup (rA_{2m} + 2(1-r)). \end{aligned}$$

Notice that

$$\begin{aligned} rA_{2m+1} &\subset [0, r + \sum_{t=1}^{n} r^t(1-r)] \subset \left[0, r + \sum_{t=1}^{\infty} r^t(1-r)\right) \\ &= [0, 2r) \subset [0, 1-r) \qquad \text{(since } r \le 1/3\text{)}. \end{aligned}$$

Similarly,

$$\begin{aligned} (rA_{2m+1} + (1-r)) &\subset [1-r, 2(1-r)), \\ (rA_{2m} + 2(1-r)) &\subset [2(1-r), \infty). \end{aligned}$$

Then the above three intervals are disjoint, which implies the union (3.3) is disjoint.

This proposition follows from (3.2) and (3.3). □

Remark 6. For $r = 1/3$, Rao and Wen [23] pointed out the adjacency matrix w.r.t F is M_n. Our approach to prove Proposition 2, which can deal with the case of r *irrational*, is quite different from the technique in [23].

4. Proof of Proposition 1

Notice that $A_0 = F$ and $\{A_m\}_{0 \le m \le 2^n - 1}$ is dust-like with ratio r and adjacency matrix M_n. To prove the Lipschitz equivalence of E and F, we shall construct graph-directed sets $\{B_m\}_{0 \le m \le 2^n - 1}$ such that $B_0 = E$ and the corresponding adjacency matrix is M_n.

Let B_m be the set defined by

$$B_m = B_{x_0 x_1 \cdots x_{n-1}} = \begin{cases} r^{-k} E & \text{if } L(m) = 2k, \\ r^{-k}(E \cup (E + 1 - r)) & \text{if } L(m) = 2k+1, \end{cases} \tag{4.1}$$

where E is defined in (2.1). Notice that $B_0 = E$.

Lemma 4. *When $m \in [0, 2^{n-1})$ and $L(m) = 2k$, we get the disjoint decomposition*

$$B_m = rB_{2m+1} \cup (rB_{2m} + r^{-k}(1-r)). \tag{4.2}$$

Proof. When $m = x_0 \cdots x_{n-1} \in [0, 2^{n-1})$, we have $x_0 = 0$. Then $L(2m) = 2k$ and $L(2m+1) = 2k+1$ by Lemma 1.

Applying equation (2.1), we have

$$\begin{aligned} B_m &= B_{x_0 x_1 \cdots x_{n-1}} = r^{-k} E \\ &= r^{-k+1} E \cup (r^{-k+1} E + r^{-k+1}(1-r)) \cup (r^{-k+1} E + r^{-k}(1-r)) \\ &= \left[r^{-k+1}(E \cup (E + 1 - r))\right] \cup \left[r^{-k+1} E + r^{-k}(1-r)\right] \\ &= rB_{x_1 \cdots x_{n-1} 1} \cup (rB_{x_1 \cdots x_{n-1} 0} + r^{-k}(1-r)) \\ &= rB_{2m+1} \cup (rB_{2m} + r^{-k}(1-r)). \end{aligned}$$

Here $rB_{2m+1} \subset [0, r^{-k+1}(2-r)]$ and $(rB_{2m} + r^{-k}(1-r)) \subset [r^{-k}(1-r), r^{-k}]$, where

$$r^{-k+1}(2-r) < r^{-k}(1-r)$$

since $r \le 1/3$. Then the decomposition in (4.2) is disjoint, which is shown in Figure 1. In this figure, we get the decomposition of $r^{-k}E$ as two parts in shadows. □

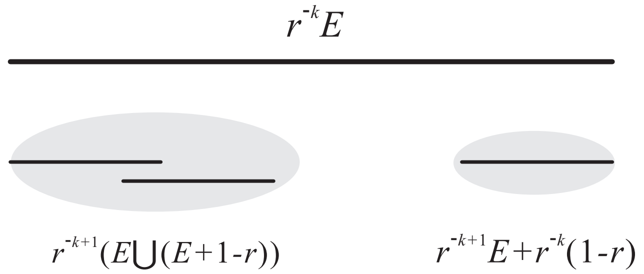

Figure 1. The decomposition of B_m with $m \in [0, 2^{n-1})$ and $L(m) = 2k$.

Lemma 5. *When $m \in [0, 2^{n-1})$ and $L(m) = 2k + 1$, we get the disjoint decomposition*

$$B_m = rB_{2m} \cup (rB_{2m+1} + r^{-k}(1 - r)). \tag{4.3}$$

Proof. When $m = x_0 \cdots x_{n-1} \in [0, 2^{n-1})$, it should holds that $x_0 = 0$. Then $L(2m) = 2k + 1$ and $L(2m + 1) = 2(k + 1)$ by Lemma 1.

Applying the equation (2.1), we have

$$B_m = B_{x_0 x_1 \cdots x_{n-1}} = r^{-k}E \cup \big[r^{-k}(E + 1 - r)\big] = \big[r^{-k+1}E \\ \cup (r^{-k+1}E + r^{-k+1}(1 - r)) \cup (r^{-k+1}E + r^{-k}(1 - r))\big] \cup \big[r^{-k}E + r^{-k}(1 - r)\big]$$

We conclude that

$$(r^{-k+1}E + r^{-k}(1 - r)) \subset r^{-k}E + r^{-k}(1 - r).$$

In fact, by (2.1), we also have

$$r^{-k}E + r^{-k}(1 - r) \\ = (r^{-k+1}E + r^{-k}(1 - r)) \cup (r^{-k+1}E + r^{-k}(1 - r^2)) \cup (r^{-k+1}E + 2r^{-k}(1 - r))$$

Therefore, we have

$$\begin{aligned} B_m &= \big[r^{-k+1}E \cup (r^{-k+1}E + r^{-k+1}(1 - r))\big] \cup \big[r^{-k}E + r^{-k}(1 - r)\big] \\ &= \big[r^{-k+1}(E \cup (E + 1 - r))\big] \cup \big[r \cdot r^{-(k+1)}E + r^{-k}(1 - r)\big] \\ &= rB_{x_1 \cdots x_{n-1} 0} \cup (rB_{x_1 \cdots x_{n-1} 1} + r^{-k}(1 - r)) \\ &= rB_{2m} \cup (rB_{2m+1} + r^{-k}(1 - r)). \end{aligned}$$

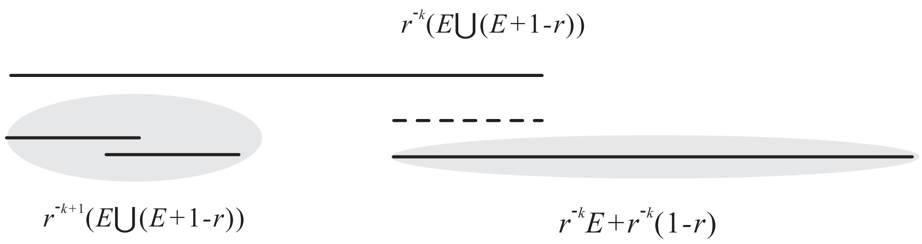

Figure 2. The decomposition of B_m with $m \in [0, 2^{n-1})$ and $L(m) = 2k + 1$.

Here $rB_{2m} \subset [0, r^{-k+1}(2-r)]$ and $(rB_{2m+1} + r^{-k}(1-r)) \subset [r^{-k}(1-r), r^{-k}(2-r)]$, where

$$r^{-k+1}(2 - r) < r^{-k}(1 - r)$$

since $r \leq 1/3$. That means the decomposition in (4.3) is disjoint as shown in Figure 2. In this figure, we get the decomposition of $r^{-k}(E \cup (E + 1 - r))$ as two parts in shadows. $\square$

Lemma 6. *When $m \in [2^{n-1}, 2^n - 1]$ and $L(m) = 2k$, we get the disjoint decomposition*

$$B_m = rB_{2(m-2^{n-1})} \cup \left[rB_{2(m-2^{n-1})+1} + r^{-k+1}(1-r)\right] \cup (rB_{2(m-2^{n-1})+1} + r^{-k}(1-r)). \tag{4.4}$$

Proof. When $m = x_0x_1 \cdots x_n \in [2^{n-1}, 2^n - 1]$, it should holds that $x_0 = 1$. Then $L(2(m-2^{n-1})) = 2(k-1)+1$ and $L(2(m-2^{n-1})+1) = 2k$ by Lemma 1. Therefore, by (2.1), we have

$$\begin{aligned} B_m &= B_{x_0x_1\cdots x_{n-1}} = r^{-k}E \\ &= r^{-k+1}E \cup \left[r^{-k+1}E + r^{-k+1}(1-r)\right] \cup \left[r^{-k+1}E + r^{-k}(1-r)\right] \\ &= Q_1 \cup Q_2 \cup Q_3. \end{aligned}$$

By (2.1), we also have

$$\begin{aligned} Q_1 &= r^{-k+1}E \\ &= \left[r^{-k+2}E \cup (r^{-k+2}E + r^{-k+2}(1-r)) \cup (r^{-k+2}E + r^{-k+1}(1-r))\right], \end{aligned}$$

where

$$(r^{-k+2}E + r^{-k+1}(1-r)) \subset Q_2,$$

since

$$Q_2 = (r^{-k+2}E + r^{-k+1}(1-r)) \cup (r^{-k+2}E + r^{-k+1}(1-r^2)) \cup (r^{-k+2}E + 2r^{-k+1}(1-r))$$

by using (2.1) again. Hence,

$$\begin{aligned} B_m &= \left[r^{-k+2}E \cup (r^{-k+2}E + r^{-k+2}(1-r))\right] \cup Q_2 \cup Q_3 \\ &= \left[r^{-k+2}(E \cup (E+1-r))\right] \cup (r^{-k+1}E + r^{-k+1}(1-r)) \cup (r^{-k+1}E + r^{-k}(1-r)) \\ &= rB_{x_1\cdots x_{n-1}0} \cup (rB_{x_1\cdots x_{n-1}1} + r^{-k+1}(1-r)) \cup (rB_{x_1\cdots x_{n-1}1} + r^{-k}(1-r)) \\ &= rB_{2(m-2^{n-1})} \cup (rB_{2(m-2^{n-1})+1} + r^{-k+1}(1-r)) \cup (rB_{2(m-2^{n-1})+1} + r^{-k}(1-r)). \end{aligned}$$

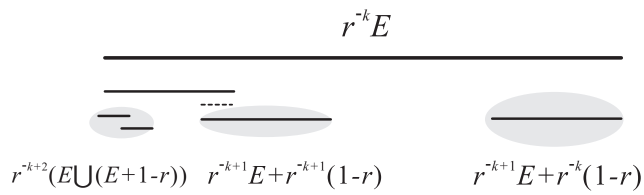

Figure 3. The decomposition of B_m with $m \geq 2^{n-1}$ and $L(m) = 2k$.

In the same way, the decomposition in (4.4) is disjoint, which is shown in Figure 3. In this figure, we get the decomposition of $r^{-k}E$ as three parts in shadows. □

Lemma 7. *When $m \in [2^{n-1}, 2^n - 1]$ and $L(m) = 2k + 1$, we get the disjoint decomposition*

$$B_m = rB_{2(m-2^{n-1})+1} \cup (rB_{2(m-2^{n-1})+1} + r^{-k}(1-r)) \cup (rB_{2(m-2^{n-1})} + 2r^{-k}(1-r)). \tag{4.5}$$

Proof. When $m = x_0 \cdots x_{n-1} \in [2^{n-1}, 2^n - 1]$, it should hold that $x_0 = 1$. Then $L(2(m-2^{n-1})) = 2k$ and $L(2(m-2^{n-1})+1) = 2k+1$ by Lemma 1. By (2.1), as in Figure 4, we have

$$B_m = B_{x_0x_1\cdots x_{n-1}} = r^{-k}E \cup [r^{-k}(E+1-r)] = P_1 \cup P_2 \cup P_3,$$

where

$$P_1 = r^{-k+1}E \cup (r^{-k+1}E + r^{-k+1}(1-r)) \cup (r^{-k+1}E + r^{-k}(1-r)),$$
$$P_2 = (r^{-k+1}E + r^{-k}(1-r)) \cup (r^{-k+1}E + r^{-k+1}(1-r) + r^{-k}(1-r)),$$
$$P_3 = r^{-k+1}E + 2r^{-k}(1-r).$$

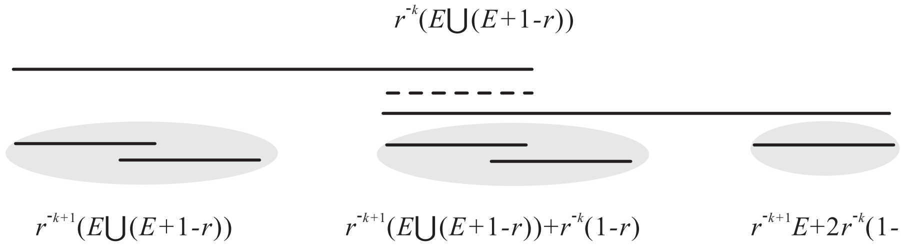

Figure 4. The decomposition of B_m with $m \geq 2^{n-1}$ and $L(m) = 2k+1$.

Notice that P_1 and P_2 have a common part $r^{-k+1}E + r^{-k}(1-r)$. Therefore, we have

$$B_m = \left[r^{-k+1}E \cup (r^{-k+1}E + r^{-k+1}(1-r))\right] \cup P_2 \cup P_3.$$

We also have

$$r^{-k+1}E \cup (r^{-k+1}E + r^{-k+1}(1-r)) = rB_{x_1 \cdots x_{n-1}1},$$

and

$$P_2 = rB_{x_1 \cdots x_{n-1}1} + r^{-k}(1-r),$$
$$P_3 = rB_{x_1 \cdots x_{n-1}0} + 2r^{-k}(1-r).$$

Hence

$$B_m = rB_{2(m-2^{n-1})+1} \cup (rB_{2(m-2^{n-1})+1} + r^{-k}(1-r)) \cup (rB_{2(m-2^{n-1})} + 2r^{-k}(1-r)).$$

In the same way, the decomposition in (4.5) is disjoint, which is shown in Figure 4. In this figure, we get the decomposition of $r^{-k}(E \cup (E+1-r))$ as three parts in shadows. □

By Lemmas 4–7, we have the following proposition.

Proposition 3. *$\{B_m\}_{0 \leq m \leq 2^n-1}$ are dust-like graph-directed sets with ratio r and adjacency matrix M_n, where $B_0 = E$.*

The proof of Proposition 1. It follows from Lemma 3, Proposition 2 and Proposition 3 that A_0 $(= F)$ and B_0 $(= E)$ are Lipschitz equivalent. This completes the proof of Proposition 1. □

5. Properties of adjacency matrix

In the section, for the adjacency matrix M_n defined in (3.1), we will discuss its Perron–Frobenius eigenvalue and Perron–Frobenius eigenvector.

Let $a = \frac{1+\sqrt{5}}{2}$, we have

$$a^2 = a + 1 \quad \text{and} \quad a^3 = 2a + 1. \tag{5.1}$$

Let $\{F_k\}_{k \geq 1} = \{1, 1, 2, 3, 5, \cdots\}$ be the Fibonacci sequence. Set $F_{-1} = 1$, $F_0 = 0$. We have

$$F_{k+1} = F_k + F_{k-1} \quad \text{for all } k \geq 0. \tag{5.2}$$

By induction, we have

$$a^k = aF_k + F_{k-1} \quad \text{for all } k \geq 0. \tag{5.3}$$

In fact, $a^0 = 1 = aF_0 + F_{-1}$, assume that $a^k = aF_k + F_{k-1}$, by (5.1) and (5.2) we can find

$$\begin{aligned} a^{k+1} &= a(aF_k + F_{k-1}) = a^2 F_k + aF_{k-1} = (a+1)F_k + aF_{k-1} \\ &= a(F_k + F_{k-1}) + F_k = aF_{k+1} + F_k. \end{aligned}$$

Proposition 4. *Given $n \geq 1$, the Perron–Frobenius eigenvalue of M_n is $\omega = \frac{3+\sqrt{5}}{2}$. Let $V_n = (v_0, v_1, \cdots, v_{2^n-1})^T > 0$ be the Perron–Frobenius eigenvector of M_n with $v_0 = 1$. Then*

$$V_n = (v_m)^T_{0 \leq m \leq 2^n-1} \quad \text{with } v_m = a^{L(m)} = aF_{L(m)} + F_{L(m)-1}. \tag{5.4}$$

Proof. For given $n \geq 1$, by Lemma 2 and Proposition 2, we have

$$\dim_{\mathrm{H}}(K_n) = -\frac{\log \omega_n}{\log r}, \tag{5.5}$$

where ω_n is the Perron–Frobenius (PF) eigenvalue of M_n. In particular, we easily check that $\dim_{\mathrm{H}}(K_1) = -\frac{\log \omega_1}{\log r}$, where $\omega_1 = \frac{3+\sqrt{5}}{2}$ is the PF eigenvalue of $M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Since K_n and K_1 are Lipschitz equivalent, we have

$$\dim_{\mathrm{H}}(K_n) = \dim_{\mathrm{H}}(K_1).$$

Then by (5.5),

$$\omega_n = \omega_1 = \omega = \frac{3+\sqrt{5}}{2} (= a^2).$$

Since the matrix M_n is primitive, we notice that the eigenspace associated to the PF eigenvalue is one dimensional. Therefore, to prove that V_n defined in (5.4) is the unique PF eigenvector of M_n with $v_0 = a^0 = 1$, we only need to check that

$$M_n V_n = \omega V_n,$$

i.e.,

$$\sum\nolimits_{j=0}^{2^n-1} g_{mj} v_j = \omega \cdot v_m \quad \text{for every } m. \tag{5.6}$$

We will check (5.6) for two cases.

Case 1. When $0 \leq m = x_0 x_1 \cdots x_{n-1} < 2^{n-1}$, that is $x_0 = 0$. Then by Lemma 1,

$$L(2m) = L(m) \quad \text{and} \quad L(2m+1) = L(m) + 1.$$

By the definition of M_n and (5.1), we have

$$\begin{aligned} \sum\nolimits_{j=0}^{2^n-1} g_{mj} v_j &= v_{2m} + v_{2m+1} = a^{L(2m)} + a^{L(2m+1)} \\ &= a^{L(m)} + a^{L(m)+1} = a^{L(m)}(1+a) = a^2 \cdot a^{L(m)} = \omega \cdot v_m. \end{aligned}$$

Case 2. When $2^{n-1} \leq m = x_0 x_1 \cdots x_{n-1} \leq 2^n - 1$, that is $x_0 = 1$. Then by Lemma 1,

$$L(2(m - 2^{n-1})) = L(m) - 1 \quad \text{and} \quad L(2(m - 2^{n-1}) + 1) = L(m).$$

By the definition of M_n and (5.1), we have

$$\begin{aligned}\sum_{j=0}^{2^n-1} g_{mj}v_j &= v_{2(m-2^{n-1})} + 2v_{2(m-2^{n-1})+1}\\ &= a^{L(2(m-2^{n-1}))} + 2a^{L(2(m-2^{n-1})+1)}\\ &= a^{L(m)-1} + 2a^{L(m)} = (1+2a)a^{L(m)-1}\\ &= a^3 \cdot a^{L(m)-1} = a^2 \cdot a^{L(m)} = \omega \cdot v_m. \qquad \square\end{aligned}$$

Remark 7. Rao and Wen [23] pointed out that ω can be proved to be the PF eigenvalue of M_n *by induction.* This conclusion is also a consequence of Proposition 1 (or Theorem 1) as shown in the proof of Proposition 4. Rao and Wen [23] also obtained the recurrent structure

$$V_1 = (1, a)^T, \cdots, V_n = (V_{n-1}, aV_{n-1})^T, \cdots.$$

6. Lipschitz equivalent to a self-similar set without overlaps

Fix $r \in (0, 1/3]$. Let

$$K = (rK + 1 - r) \cup (r^{1/2}K)$$

be a self-similar set without overlaps.

In this section, we will prove the following proposition.

Proposition 5. *K_n and K are Lipschitz equivalent for any n.*

Proof. By Theorem 1, we only need to show that K_1 and K are Lipschitz equivalent.

In fact, let

$$\Gamma_0 = K \quad \text{and} \quad \Gamma_1 = r^{-1/2}K.$$

Then we have the following disjoint decomposition

$$\Gamma_0 = K = (rK + 1 - r) \cup r^{1/2}K = (r\Gamma_0 + 1 - r) \cup r\Gamma_1. \tag{6.1}$$

By using (6.1), we also have the following disjoint decomposition

$$\begin{aligned}\Gamma_1 = r^{-1/2}K &= r^{-1/2}[(rK + 1 - r) \cup r^{1/2}K]\\ &= [r(r^{-1/2}K) + r^{-1/2}(1-r)] \cup K\\ &= [r\Gamma_1 + r^{-1/2}(1-r)] \cup (r\Gamma_0 + 1 - r) \cup r\Gamma_1.\end{aligned} \tag{6.2}$$

The above decompositions (6.1)–(6.2) show that $\{\Gamma_0, \Gamma_1\}$ are dust-like graph-directed sets with ratio r and adjacency matrix

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Applying Proposition 2 to the case of $n = 1$, we get graph-directed sets $\{B_0, B_1\}$ with ratio r and adjacency matrix M_1, where $B_0 = K_1$.

It follows from Lemma 3 that K and K_1 are Lipschitz equivalent. $\square$

Acknowledgements. The corresponding author acknowledges helpful discussion with Professor Z. Y. Wen.

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Received 20 May 2011