LIPSCHITZ EQUIVALENCE OF A CLASS OF SELF-SIMILAR SETS WITH COMPLETE OVERLAPS

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Abstract. Fix $r \in (0, 1/3)$. We discuss a class of self-similar sets $\{K_n\}_{n \geq 1}$ with complete overlaps, where $K_n = (rK_n) \cup (rK_n + r^n(1-r)) \cup (rK_n + 1 - r)$. We prove that for any $n_1, n_2 \geq 1$, $K_{n_1}$ and $K_{n_2}$ are Lipschitz equivalent.

1. Introduction

Suppose $\{h_i: \mathbb{R}^m \to \mathbb{R}^m\}_{i=1}^k$ are contractive similitudes. We say that a compact set $\Lambda = h_1(\Lambda) \cup \cdots \cup h_k(\Lambda)$ is a self-similar set with overlaps, if there are $i \neq j$ such that $h_i(\Lambda) \cap h_j(\Lambda) \neq \emptyset$.

Self-similar sets with overlaps have very complicated structures. For example, the open set condition (OSC), which means the overlaps are little, was introduced by Moran [18] and studied by Hutchinson [10]. Schief [24], Bandt and Graf [1] showed the relation between the open set condition and the positive Hausdorff measure. Falconer [6] proved some “generic” results on Hausdorff dimension of self-similar sets without the assumption about the open set condition. One useful notion “transversality” to study self-similar sets (or measures) with overlaps can be found e.g. in Keane, Smorodinsky and Solomyak [11], Pollicott and Simon [19], Simon and Solomyak [25] and Solomyak [26]. Feng and Lau [9], Lau and Ngai [14] studied the weak separation condition. Please refer to Bandt and Hung [2], Sumi [27] for some recent work.

For self-similar set $E_\lambda = E_\lambda/3 \cup (E_\lambda/3 + \lambda/3) \cup (E_\lambda/3 + 2/3)$, a conjecture of Furstenberg says that $\dim_H E_\lambda = 1$ for any $\lambda$ irrational. Świetek and Veerman [28] proved that $\dim_H E_\lambda > 0.767$ for every $\lambda$ irrational. Kenyon [12], Rao and Wen [23] obtained that $H^1(E_\lambda) > 0$ if and only if $\lambda = p/q \in \mathbb{Q}$ with $p \equiv q \neq 0 \pmod{3}$. The key idea of [23] is “graph-directed struture” introduced by Mauldin and Williams [17].

In particular, Rao and Wen [23] studied the self-similar sets with “complete overlaps”.

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Definition 1. We say that a self-similar set $\Lambda = h_1(\Lambda) \cup \cdots \cup h_k(\Lambda)$ has complete overlaps, if there are words $i_1i_2\cdots i_n \in \{1, \cdots , k\}^n$, $j_1j_2\cdots j_n \in \{1, \cdots , k\}^n$ with $i_1 \neq j_1$ such that

\[ h_{i_1} \circ h_{i_2} \circ \cdots \circ h_{i_n}(x) \equiv h_{j_1} \circ h_{j_2} \circ \cdots \circ h_{j_n}(x). \tag{1.1} \]

Here (1.1) implies $h_{i_1} \circ h_{i_2} \circ \cdots \circ h_{i_n}(\Lambda) = h_{j_1} \circ h_{j_2} \circ \cdots \circ h_{j_n}(\Lambda)$.

Remark 1. The cases according to “complete overlaps” and “OSC” are quite different. For example, suppose $h_1, \cdots , h_k$ have the same ratio $\alpha$. Then (1.1) implies $n_1 = n_2 = n$ and $\dim_H \Lambda \leq -\frac{\log(k^n-1)}{\log \alpha^n} < -\log k/\log \alpha$. However OSC implies $\dim_H \Lambda = -\log k/\log \alpha$.

Remark 2. It is shown in [23] that for $\lambda \in Q$, $\dim_H E_\lambda < 1$ if and only if $E_\lambda$ has complete overlaps, $E_{2/3^\nu}$ has complete overlaps and $\dim_H E_{2/3^n} = \log_3 \frac{3+\sqrt{5}}{2}$ for all $n \geq 1$.

The other interesting topic on self-similar sets is their Lipschitz equivalence. Here two compact subsets $X_1, X_2$ of Euclidean spaces are said to be Lipschitz equivalent, if there is a bijection $f : X_1 \to X_2$ and a constant $C > 0$ such that for all $x, y \in X_1$,

\[ C^{-1}|x-y| \leq |f(x) - f(y)| \leq C|x-y|. \]

If compact sets $X_1$ and $X_2$ are Lipschitz equivalent, then $\dim_H X_1 = \dim_H X_2$. However, it is worth pointing out that Cooper and Pignataro [3], Falconer and Marsh [8], David and Semmes [4] and Wen and Xi [30] showed that two self-similar sets need not be Lipschitz equivalent although they have the same Hausdorff dimension.

For self-similar sets without overlaps, Cooper and Pignataro [3], Falconer and Marsh [8], David and Semmes [4] and Xi [32] posed some algebraic conditions upon the ratios of similitudes for two given self-similar sets without overlaps to be Lipschitz equivalent. The quasi-Lipschitz equivalence, which is weaker than the Lipschitz equivalence, was studied for self-conformal sets and Ahlfors–David regular sets in Xi [31] and Wang and Xi [29], respectively. The Lipschitz embedding of fractals can be found in Llorente and Mattila [15], Mattila and Saaranen [16] and Deng and Wen et al. [5]. Please also refer to Rao, Ruan and Yang [21] and Rao, Ruan and Wang [22].

For self-similar sets with overlaps, an interesting result is on the \{1,3,5\}-\{1,4,5\} Problem which was posed by David and Semmes [4]. Let $H_1 = (H_1/5) \cup (H_1/5 + 2/5) \cup (H_1/5 + 4/5)$ be the \{1,3,5\} self-similar set, and $H_2 = (H_2/5) \cup (H_2/5 + 3/5) \cup (H_2/5 + 4/5)$ the \{1,4,5\} self-similar set. The problem asks about the Lipschitz equivalence of $H_1$ and $H_2$. Rao, Ruan and Xi [20] proved that $H_1$ and $H_2$ are Lipschitz equivalent. Furthermore, Xi and Ruan [33] proved that for given $r_1, r_2, r_3 \in (0,1)$ with $r_1 + r_2 + r_3 < 1$, self-similar sets $J_1$ and $J_2$ are Lipschitz equivalent if and only if $\log r_1/\log r_2 \in Q$, where

\[ J_1 = (r_1J_1) \cup (r_2J_1 + 1 - r_2 - r_3) \cup (r_3J_1 + 1 - r_3), \]
\[ J_2 = (r_1J_2) \cup (r_2J_2 + \frac{1 + r_1 - r_2 - r_3}{2}) \cup (r_3J_2 + 1 - r_3). \]

Xi and Xiong [34] generalized the result on the \{1,3,5\}-\{1,4,5\} Problem to the higher dimensional spaces. Given integers $n \geq 2$ and $m \geq 1$, for $A, B \subset \{0, 1, \cdots , (n-1)\}^m$, let $E_A = \cup_{a \in A} E_{\frac{A}{a}}$ and $E_B = \cup_{b \in B} E_{\frac{B}{b}}$ be self-similar sets in $R^m$. Suppose that $E_A$ and $E_B$ are totally disconnected. Then it is proved in [34] that $E_A$ and $E_B$ are
Lipschitz equivalent if and only if $A$ and $B$ have the same cardinality. Please also refer to Wen and Xi [30], Xi and Xiong [35] and Zhu et al. [36].

We notice that for the self-similar sets considered in [20], [30], [33]–[36], the open set condition holds, which means two pieces of self-similar sets touch a little. In this paper, we will consider a quite different case according to complete overlaps.

Fix $r \in (0, 1/3]$. For any integer $n \geq 1$, let $K_n$ be a self-similar set satisfying

$$K_n = (rK_n) \cup (rK_n + r^n(1 - r)) \cup (rK_n + 1 - r).$$

The main result of the paper is stated as follows.

**Theorem 1.** For any $n_1, n_2 \geq 1$, $K_{n_1}$ and $K_{n_2}$ are Lipschitz equivalent.

**Remark 3.** $K_n$ has complete overlaps. In fact, $K_n$ is generated by

$$(1.2) \quad S_1(x) = rx, \quad S_2(x) = rx + r^n(1 - r), \quad S_3(x) = rx + (1 - r).$$

Let $S_{t_1t_2\cdots t_n} = S_{t_1} \circ S_{t_2} \circ \cdots \circ S_{t_n}$ and $[1]^t$ be the word composed of $t$ digits 1. Then $S_{[1]^t}(x) \equiv S_{2[1]^n}(x) \equiv r^{n+1}x + r^n(1 - r)$.

**Remark 4.** For $r = 1/3$, $K_n = E_{2/3^n}$ and it is shown in [23] that $\{K_n\}_n$ have the same Hausdorff dimension. In fact, the technique in [23] can deal with dimension for any self-similar set $\Lambda = \bigcup_{i=1}^k \Lambda/n + b_i$ where $n \in \mathbb{N}$ and $b_i \in \mathbb{Q}$ for all $i$. In this paper, fix any ratio $r(\leq 1/3)$ rational or irrational, for the special fractals $\{K_n\}_n$, we prove that $\{K_n\}_n$ belong to the same Lipschitz equivalent class, which implies $\{K_n\}_n$ have the same dimension.

**Remark 5.** For $r \in (0, 1/3]$, let $K = (r^{1/2}K) \cup (rK + 1 - r)$ be a self-similar set without overlaps. Proposition 5 in Section 6 says that $K_n$ and $K$ are Lipschitz equivalent for any $n$.

We organize the paper as follows. Section 2 is the preliminaries, including the counting function $L(m)$ (Lemma 1), the graph-directed sets (Lemma 2) and their corresponding criterion for Lipschitz equivalence (Lemma 3). In Section 3, for $K_n$ we construct the graph-directed sets with ratio $r$ and adjacency matrix $M_n$ defined in (3.1). In Section 4, for $K_1$ we construct some graph-directed sets with ratio $r$ and the same adjacency matrix $M_n$. Then it follows from Lemma 3 that $K_1$ and $K_n$ are Lipschitz equivalent, hence Theorem 1 is proved. In Section 5, we obtain the Perron–Frobenius eigenvector of the matrix $M_n$ in terms of the counting function $L(m)$ and the Fibonacci sequence. Section 6 shows that $K_n$ is Lipschitz equivalent to a self-similar set $K$ (in Remark 4) without overlaps.

## 2. Preliminaries

Fix $r \in (0, 1/3]$ and integer $n \geq 1$. Let $E, F$ be the self-similar sets satisfying

$$(2.1) \quad E = rE \cup (rE + r(1 - r)) \cup (rE + 1 - r),$$

$$(2.2) \quad F = rF \cup (rF + r^n(1 - r)) \cup (rF + 1 - r).$$

Notice that $E = K_1$ and $F = K_n$.

Therefore for proving Theorem 1, we only need to prove the following proposition (in Section 4).

**Proposition 1.** $E$ and $F$ are Lipschitz equivalent.
Given any integer \( m \in [0, 2^n - 1] \), let

\[
m = x_0x_1 \cdots x_{n-1} \quad (x_i = 0 \text{ or } 1),
\]

be the dyadic representation of \( m \), that is \( m = 2^{n-1}x_0 + 2^{n-2}x_1 + \cdots + x_{n-1} \). Then we define the counting function

\[
L(m) = L(x_0x_1 \cdots x_{n-1}) = \sum_{i=0}^{n-1} x_i,
\]
i.e., the number of digits 1 in \( x_0x_1 \cdots x_{n-1} \). We have the following lemma.

**Lemma 1.** Suppose \( m = x_0x_1 \cdots x_{n-1} \in \mathbb{Z} \cap [0, 2^n - 1] \). Then\[
L(2m) = L(m) \quad \text{and} \quad L(2m + 1) = L(m) + 1 \quad \text{when } n < 2^n - 1,
\]
and

\[
L(2(m - 2^{n-1})) = L(m) - 1 \quad \text{and} \quad L(2(m - 2^{n-1}) + 1) = L(m) \quad \text{when } n \geq 2^n - 1.
\]

**Proof.** When \( m = x_0 \cdots x_{n-1} \in [0, 2^n - 1] \), we have \( x_0 = 0 \),

\[
L(2m) = L(x_1 \cdots x_{n-1}0) = L(0x_1 \cdots x_{n-1}) = L(m),
\]
and \( L(2m + 1) = L(x_1 \cdots x_{n-1}1) = L(0x_1 \cdots x_{n-1}) + 1 = L(m) + 1 \).

When \( m = x_0 \cdots x_{n-1} = [2^{n-1}, 2^n - 1] \), we have \( x_0 = 1 \),

\[
L(2(m - 2^{n-1})) = L(x_1 \cdots x_{n-1}0) = L(1x_1 \cdots x_{n-1}) - 1 = L(m) - 1,
\]
and \( L(2(m - 2^{n-1}) + 1) = L(x_1 \cdots x_{n-1}1) = L(1x_1 \cdots x_{n-1}) = L(m) \). \( \square \)

Recall the notion of “graph-directed structure” [17] as follows.

Let \( (V, \mathcal{E}) \) be a directed graph, where \( V = \{0, \ldots, (m - 1)\} \) and \( \mathcal{E} \) are vertex set and edge set respectively. Denote by \( \mathcal{E}_{i,j} \) the set of edges from vertex \( i \) to vertex \( j \). For this graph, we consider the adjacency matrix \( \mathcal{A} = (a_{i,j})_{0 \leq i,j \leq m-1} \) defined by \( a_{i,j} = \#\mathcal{E}_{i,j} \). Let \( \rho(\mathcal{A}) \) be the spectral radius of \( \mathcal{A} \). The graph is said to be transitive, if for any vertexes \( i \) and \( j \), there exists a directed path starting at \( i \) and ending at \( j \).

To simplify the structure, we assume there exists \( \tau \in (0,1) \) such that every edge \( e \in \mathcal{E} \) is equipped with a contracting similitude \( S_e: \mathbb{R}^n \to \mathbb{R}^n \) with ratio \( \tau \). By [17], there exists a unique family of non-empty compact sets \( \Gamma_0, \Gamma_1, \ldots, \Gamma_{m-1} \) such that for any \( i \),

\[
\Gamma_i = \bigcup_{j=0}^{m-1} \bigcup_{e \in \mathcal{E}_{i,j}} S_e(\Gamma_j).
\]

Here \( \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{m-1}\} \) are called graph-directed sets on \( (V, \mathcal{E}) \) with ratio \( \tau \). Furthermore, we say that \( \{\Gamma_i\}_{i=0}^{m-1} \) are dust-like graph-directed sets on \( (V, \mathcal{E}) \), if the right hand of (2.4) is a disjoint union for each \( 0 \leq i \leq m - 1 \).

By [17], we have the following lemma.

**Lemma 2.** Suppose that \( \{\Gamma_i\}_{i=0}^{m-1} \) are dust-like graph-directed sets on a transitive graph with ratio \( \tau \) and adjacency matrix \( \mathcal{A} \). Then for every \( i \),

\[
\dim \Gamma_i = -\frac{\log \rho(\mathcal{A})}{\log \tau}.
\]

Theorem 2.1 of [20] yields the following lemma for Lipschitz equivalence.

**Lemma 3.** Suppose \( \{\Gamma_i\}_{i=0}^{m-1} \) and \( \{\Gamma'_i\}_{i=0}^{m-1} \) are dust-like graph-directed sets with the same ratio and the same adjacency matrix. Then \( \Gamma_i \) and \( \Gamma'_i \) are Lipschitz equivalent for any \( i \).
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For example, for \( \{1, 3, 5\} \) set \( H_1 \) and \( \{1, 4, 5\} \) set \( H_2 \) mentioned above, we obtained

\[
\begin{align*}
\Gamma_0 &= H_1, \\
\Gamma_1 &= H_1 \cup (H_1 + 2), \\
\Gamma_2 &= H_1 \cup (H_1 + 2) \cup (H_1 + 4), \\
\Gamma'_0 &= H_2, \\
\Gamma'_1 &= H_2 \cup (H_2 + 1), \\
\Gamma'_2 &= H_2 \cup (H_2 + 1) \cup (H_2 + 2).
\end{align*}
\]

It is pointed out in [20] that \( \{\Gamma_i\}_{i=0}^2 \) and \( \{\Gamma'_i\}_{i=0}^2 \) have the same ratio \( 1/5 \) and adjacency matrix

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
1 & 1 & 2
\end{pmatrix},
\]

which implies that \( \Gamma_0(= H_1) \) and \( \Gamma'_0(= H_2) \) are Lipschitz equivalent.

3. Graph-directed structure

Given \( m = x_0x_1 \cdots x_{n-1} \in \{0, 1, 2, \cdots, 2^n - 1\} \), set

\[
A_m = \bigcup_{i_0i_1 \cdots i_{n-1} \in \{0, 1\}^n} (F + \sum_{t=0}^{n-1} r^t(1 - r)(i_t x_t)),
\]

where \( F \) is defined in (2.2). In particular, \( A_0 = F \).

Let \( M_n = (g_{i,j})_{0 \leq i, j \leq 2^n - 1} \) be a \( 2^n \times 2^n \) integer matrix, where

\[
g_{i,j} = \begin{cases} 
1, & \text{if } i < 2^{n-1}, \; j = 2i, \\
1, & \text{if } i < 2^{n-1}, \; j = 2i + 1, \\
1, & \text{if } i \geq 2^{n-1}, \; j = 2(i - 2^{n-1}), \\
2, & \text{if } i \geq 2^{n-1}, \; j = 2(i - 2^{n-1}) + 1, \\
0, & \text{otherwise}. 
\end{cases}
\]

That means

\[
(3.1) \quad M_n = \begin{pmatrix}
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 \\
1 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2 \\
\end{pmatrix}_{2^n \times 2^n}.
\]

**Proposition 2.** \( \{A_m\}_{0 \leq m \leq 2^n - 1} \) are dust-like graph-directed sets with ratio \( r \) and adjacency matrix \( M_n \), where \( A_0 = F \).

*Proof.* Let \( m = x_0x_1 \cdots x_{n-1} \) be the dyadic representation of \( m \). We will discuss two cases.

**Case 1.** \( 0 \leq m < 2^{n-1} \), that is \( x_0 = 0 \). We only need to check

\[
rA_{2m+1} \cup (rA_{2m} + 1 - r) = A_m.
\]
For this, we notice that

\[ A_m = A_{0x_1 \cdots x_{n-1}} = \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} \left( F + \sum_{t=1}^{n-1} r^t (1 - r)(i_t x_t) \right). \]

Applying equation (2.2), we have

\[ A_m = T_1 \cup T_2 \cup T_3, \]

where

\[
T_1 = \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} \left( rF + \sum_{t=0}^{n-2} r^t (1-r)(i_{t+1} x_{t+1}) + r^{n-1}(1-r)(0 \cdot 1) \right),
\]

\[
T_2 = \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} \left( rF + r^{n-1}(1-r) + \sum_{t=1}^{n-1} r^t (1-r)(i_t x_t) \right),
\]

\[
T_3 = \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} \left( rF + 1 - r + \sum_{t=1}^{n-1} r^t (1-r)(i_t x_t) \right).
\]

Therefore, we have

\[
T_1 \cup T_2 = \left\{ \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} r \left( F + \sum_{t=0}^{n-2} r^t (1-r)(i_{t+1} x_{t+1}) + r^{n-1}(1-r)(0 \cdot 1) \right) \right\}
\]

\[
\cup \left\{ \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} r \left( F + \sum_{t=0}^{n-2} r^t (1-r)(i_{t+1} x_{t+1}) + r^{n-1}(1-r)(1 \cdot 1) \right) \right\}
\]

\[
= \bigcup_{i_1 \cdots i_{n-1} i_n \in \{0,1\}^n} r \left( F + \sum_{t=0}^{n-2} r^t (1 - r)(i_{t+1} x_{t+1}) + r^{n-1}(1-r)(i_n \cdot 1) \right)
= r A_{x_1 \cdots x_{n-1}} = r A_{2m+1}.
\]

In the same way, we have

\[
T_3 = \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} \left( rF + 1 - r + \sum_{t=1}^{n-1} r^t (1-r)(i_t x_t) \right)
\]

\[
= r \left[ \bigcup_{i_1 \cdots i_{n-1} i_n \in \{0,1\}^n} \left( F + \sum_{t=0}^{n-2} r^t (1-r)(i_{t+1} x_{t+1}) + r^{n-1}(1-r)(i_n \cdot 0) \right) \right] + (1-r)
\]

\[
= r A_{x_1 \cdots x_{n-1} 0} + (1-r)
= r A_{2m} + (1-r).
\]

Since \( r A_{2m+1} \subset [0, r] \), \( r A_{2m} + (1-r) \subset [1-r, 1] \) and \( r \leq 1/3 \), we have the disjoint union

\[ A_m = r A_{2m+1} \cup (r A_{2m} + 1 - r). \]
Case 2. $m \geq 2^{n-1}$, that is $x_0 = 1$,

\[ A_m = A_{1x_1 \cdots x_{n-1}} = \bigcup_{i_0 = 0}^{1} \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} (F + \sum_{t=0}^{n-1} r^t(1 - r)(i_t x_t)). \]

Let

\[ R_{i_0} = \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} (F + (1 - r)(i_0) + \sum_{t=1}^{n-1} r^t(1 - r)(i_t x_t)). \]

Then

\[ A_m = R_0 \cup R_1. \]

Applying (2.2), we have

\[
R_0 = \left[ \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} \left( rF + \sum_{t=1}^{n-1} r^t(1 - r)(i_t x_t) \right) \right] \\
\bigcup \left[ \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} \left( rF + r^n(1 - r) + \sum_{t=1}^{n-1} r^t(1 - r)(i_t x_t) \right) \right] \\
\bigcup \left[ \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} \left( rF + (1 - r) + \sum_{t=1}^{n-1} r^t(1 - r)(i_t x_t) \right) \right] \\
= R_{0,1} \cup R_{0,2} \cup R_{0,3}.
\]

In the same way as above, we obtain

\[ R_{0,1} \cup R_{0,2} = rA_{x_1 \cdots x_{n-1}1} = rA_{2m+1}. \]

Applying (2.2), we also have

\[
R_1 = \left[ \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} \left( rF + (1 - r) + \sum_{t=1}^{n-1} r^t(1 - r)(i_t x_t) \right) \right] \\
\bigcup \left[ \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} \left( rF + r^n(1 - r) + (1 - r) + \sum_{t=1}^{n-1} r^t(1 - r)(i_t x_t) \right) \right] \\
\bigcup \left[ \bigcup_{i_1 \cdots i_{n-1} \in \{0,1\}^{n-1}} \left( rF + 2(1 - r) + \sum_{t=1}^{n-1} r^t(1 - r)(i_t x_t) \right) \right] \\
= R_{1,1} \cup R_{1,2} \cup R_{1,3}.
\]

Here $R_{1,1} = R_{0,3}$. In the same way,

\[ R_{1,1} \cup R_{1,2} = rA_{2m+1} + (1 - r) \text{ and } R_{1,3} = rA_{2m} + 2(1 - r). \]

Therefore

\[
A_m = R_{0,1} \cup R_{0,2} \cup R_{0,3} \cup R_{1,1} \cup R_{1,2} \cup R_{1,3} \\
= (R_{0,1} \cup R_{0,2}) \cup (R_{1,1} \cup R_{1,2}) \cup R_{1,3} \\
= rA_{2m+1} \cup (rA_{2m+1} + (1 - r)) \cup (rA_{2m} + 2(1 - r)).
\]

(3.3)
Notice that
\[ rA_{2m+1} \subset [0, r + \sum_{t=1}^{n} r^t(1 - r)] \subset \left[0, r + \sum_{t=1}^{\infty} r^t(1 - r)\right] = [0, 2r) \subset [0, 1 - r) \quad \text{(since } r \leq 1/3).\]

Similarly,
\[ (rA_{2m+1} + (1 - r)) \subset [1 - r, 2(1 - r)), \]
\[ (rA_{2m} + 2(1 - r)) \subset [2(1 - r), \infty).\]

Then the above three intervals are disjoint, which implies the union (3.3) is disjoint.

This proposition follows from (3.2) and (3.3).

**Remark 6.** For \( r = 1/3 \), Rao and Wen [23] pointed out the adjacency matrix w.r.t \( F \) is \( M_n \). Our approach to prove Proposition 2, which can deal with the case of \( r \) irrational, is quite different from the technique in [23].

### 4. Proof of Proposition 1

Notice that \( A_0 = F \) and \( \{A_m\}_{0 \leq m \leq 2^n - 1} \) is dust-like with ratio \( r \) and adjacency matrix \( M_n \). To prove the Lipschitz equivalence of \( E \) and \( F \), we shall construct graph-directed sets \( \{B_m\}_{0 \leq m \leq 2^n - 1} \) such that \( B_0 = E \) and the corresponding adjacency matrix is \( M_n \).

Let \( B_m \) be the set defined by

\[ B_m = B_{x_0x_1\cdots x_{n-1}} = \begin{cases} r^{-k}E & \text{if } L(m) = 2k, \\ r^{-k}(E \cup (E + 1 - r)) & \text{if } L(m) = 2k + 1, \end{cases} \]

where \( E \) is defined in (2.1). Notice that \( B_0 = E \).

**Lemma 4.** When \( m \in [0, 2^n-1] \) and \( L(m) = 2k \), we get the disjoint decomposition

\[ B_m = rB_{2m+1} \cup (rB_{2m} + r^{-k}(1 - r)). \]

**Proof.** When \( m = x_0 \cdots x_{n-1} \in [0, 2^n-1] \), we have \( x_0 = 0 \). Then \( L(2m) = 2k \) and \( L(2m+1) = 2k+1 \) by Lemma 1.

Applying equation (2.1), we have

\[ B_m = B_{x_0x_1\cdots x_{n-1}} = r^{-k}E \]
\[ = r^{-k+1}E \cup (r^{-k+1}E + r^{-k+1}(1 - r)) \cup (r^{-k+1}E + r^{-k}(1 - r)) \]
\[ = [r^{-k+1}(E \cup (E + 1 - r))] \cup [r^{-k+1}E + r^{-k}(1 - r)] \]
\[ = rB_{x_1\cdots x_{n-1}} \cup (rB_{x_1\cdots x_{n-1}+1} + r^{-k}(1 - r)) \]
\[ = rB_{2m+1} \cup (rB_{2m} + r^{-k}(1 - r)). \]

Here \( rB_{2m+1} \subset [0, r^{-k+1}(2 - r)] \) and \( (rB_{2m} + r^{-k}(1 - r)) \subset [r^{-k}(1 - r), r^{-k}], \) where

\[ r^{-k+1}(2 - r) < r^{-k}(1 - r) \]

since \( r \leq 1/3 \). Then the decomposition in (4.2) is disjoint, which is shown in Figure 1. In this figure, we get the decomposition of \( r^{-k}E \) as two parts in shadows. \[ \square \]
Lemma 5. When \( m \in [0, 2^{n-1}) \) and \( L(m) = 2k + 1 \), we get the disjoint decomposition
\[
B_m = rB_{2m} \cup (rB_{2m+1} + r^{-k}(1 - r)).
\]

Proof. When \( m = x_0 \cdots x_{n-1} \in [0, 2^{n-1}) \), it should holds that \( x_0 = 0 \). Then
\( L(2m) = 2k + 1 \) and \( L(2m + 1) = 2(k + 1) \) by Lemma 1.

Applying the equation (2.1), we have
\[
B_m = B_{x_0x_1 \cdots x_{n-1}} = r^{-k}E \cup [r^{-k}(E + 1 - r)] = [r^{-k+1}E
\]
\[
\quad \cup (r^{-k+1}E + r^{-k+1}(1 - r)) \cup (r^{-k+1}E + r^{-k}(1 - r)) \] \[ \cup [r^{-k}E + r^{-k}(1 - r)]
\]
We conclude that
\[
(r^{-k+1}E + r^{-k}(1 - r)) \subset r^{-k}E + r^{-k}(1 - r).
\]
In fact, by (2.1), we also have
\[
r^{-k}E + r^{-k}(1 - r)
\]
\[
= (r^{-k+1}E + r^{-k}(1 - r)) \cup (r^{-k+1}E + r^{-k}(1 - r^2)) \cup (r^{-k+1}E + 2r^{-k}(1 - r))
\]
Therefore, we have
\[
B_m = [r^{-k+1}E \cup (r^{-k+1}E + r^{-k+1}(1 - r))] \cup [r^{-k}E + r^{-k}(1 - r)]
\]
\[
= [r^{-k+1}(E \cup (E + 1 - r))] \cup [r \cdot r^{-k+1}E + r^{-k}(1 - r)]
\]
\[
= rB_{x_1 \cdots x_{n-1} 0} \cup (rB_{x_1 \cdots x_{n-1} 1} + r^{-k}(1 - r))
\]
\[
= rB_{2m} \cup (rB_{2m+1} + r^{-k}(1 - r)).
\]

Therefore, we have
\[
B_m = r^k(E \cup E_{n+1})
\]

Figure 2. The decomposition of \( B_m \) with \( m \in [0, 2^{n-1}) \) and \( L(m) = 2k + 1 \).

Here \( rB_{2m} \subset [0, r^{-k+1}(2 - r)] \) and \( (rB_{2m+1} + r^{-k}(1 - r)) \subset [r^{-k}(1 - r), r^{-k}(2 - r)] \),
where
\[
r^{-k+1}(2 - r) < r^{-k}(1 - r)
\]
since \( r \leq 1/3 \). That means the decomposition in (4.3) is disjoint as shown in Figure 2.

In this figure, we get the decomposition of \( r^{-k}(E \cup (E + 1 - r)) \) as two parts in shadows. \( \square \)
Lemma 6. When \( m \in [2^{n-1}, 2^n - 1] \) and \( L(m) = 2k \), we get the disjoint decomposition

\[
B_m = rB_{2(m-2^{n-1})} \cup \left[ rB_{2(m-2^{n-1})+1} + r^{-k+1}(1-r) \right] \\
\cup (rB_{2(m-2^{n-1})+1} + r^{-k}(1-r)).
\]

Proof. When \( m = x_0x_1 \cdots x_n \in [2^{n-1}, 2^n - 1] \), it should holds that \( x_0 = 1 \). Then \( L(2(m-2^{n-1})) = 2(k-1) + 1 \) and \( L(2(m-2^{n-1})+1) = 2k \) by Lemma 1. Therefore, by (2.1), we have

\[
B_m = B_{x_0x_1 \cdots x_{n-1}} = r^{-k}E
\]

\[
= r^{-k+1}E \cup (r^{-k+1}E + r^{-k+1}(1-r)) \cup \left[ r^{-k+1}E + r^{-k}(1-r) \right]
\]

\[
= Q_1 \cup Q_2 \cup Q_3.
\]

By (2.1), we also have

\[
Q_1 = r^{-k+1}E
\]

\[
= \left[ r^{-k+2}E \cup (r^{-k+2}E + r^{-k+2}(1-r)) \cup (r^{-k+2}E + r^{-k+1}(1-r)) \right],
\]

where

\[
(r^{-k+2}E + r^{-k+1}(1-r)) \subset Q_2,
\]

since

\[
Q_2 = (r^{-k+2}E + r^{-k+1}(1-r)) \cup (r^{-k+2}E + r^{-k+1}(1-r^2)) \cup (r^{-k+2}E + 2r^{-k+1}(1-r))
\]

by using (2.1) again. Hence,

\[
B_m = \left[ r^{-k+2}E \cup (r^{-k+2}E + r^{-k+2}(1-r)) \right] \cup Q_2 \cup Q_3
\]

\[
= \left[ r^{-k+2}(E \cup (E+1-r)) \right] \cup (r^{-k+1}E + r^{-k+1}(1-r)) \cup (r^{-k+1}E + r^{-k}(1-r))
\]

\[
= rB_{x_1 \cdots x_{n-1}0} \cup (rB_{x_1 \cdots x_{n-1}1} + r^{-k+1}(1-r)) \cup (rB_{x_1 \cdots x_{n-1}1} + r^{-k}(1-r))
\]

\[
= rB_{2(m-2^{n-1})} \cup (rB_{2(m-2^{n-1})+1} + r^{-k+1}(1-r)) \cup (rB_{2(m-2^{n-1})+1} + r^{-k}(1-r)).
\]

\[
\begin{array}{cc}
\quad & \quad \\
r^{k}\cdot(E \cup (E+1-r)) & r^{k+1}E + r^{k+1}(1-r) \\
r^{k+1}E + r^{k+1}(1-r) & r^{k+1}E + r^{k}(1-r)
\end{array}
\]

Figure 3. The decomposition of \( B_m \) with \( m \geq 2^{n-1} \) and \( L(m) = 2k \).

In the same way, the decomposition in (4.4) is disjoint, which is shown in Figure 3. In this figure, we get the decomposition of \( r^{-k}E \) as three parts in shadows. \( \square \)

Lemma 7. When \( m \in [2^{n-1}, 2^n - 1] \) and \( L(m) = 2k + 1 \), we get the disjoint decomposition

\[
B_m = rB_{2(m-2^{n-1})+1} \cup (rB_{2(m-2^{n-1})+1} + r^{-k}(1-r)) \cup (rB_{2(m-2^{n-1})+1} + 2r^{-k}(1-r)).
\]

Proof. When \( m = x_0 \cdots x_{n-1} \in [2^{n-1}, 2^n - 1] \), it should holds that \( x_0 = 1 \). Then \( L(2(m-2^{n-1})) = 2k \) and \( L(2(m-2^{n-1})+1) = 2k + 1 \) by Lemma 1. By (2.1), as in Figure 4, we have

\[
B_m = B_{x_0x_1 \cdots x_{n-1}} = r^{-k}E \cup [r^{-k}(E + 1 - r)] = P_1 \cup P_2 \cup P_3,
\]
We have
\[ P_1 = r^{-k+1}E \cup (r^{-k+1}E + r^{-k+1}(1 - r)) \cup (r^{-k+1}E + r^{-k}(1 - r)), \]
\[ P_2 = (r^{-k+1}E + r^{-k}(1 - r)) \cup (r^{-k+1}E + r^{-k+1}(1 - r) + r^{-k}(1 - r)), \]
\[ P_3 = r^{-k+1}E + 2r^{-k}(1 - r). \]

Hence
\[ r^k(E \cup (E+1-r)) \]

Figure 4. The decomposition of \( B_m \) with \( m \geq 2^{n-1} \) and \( L(m) = 2k + 1 \).

Notice that \( P_1 \) and \( P_2 \) have a common part \( r^{-k+1}E + r^{-k}(1 - r) \). Therefore, we have
\[ B_m = \left[ r^{-k+1}E \cup (r^{-k+1}E + r^{-k+1}(1 - r)) \right] \cup P_2 \cup P_3. \]
We also have
\[ r^{-k+1}E \cup (r^{-k+1}E + r^{-k+1}(1 - r)) = rB_{x_1\cdots x_{n-1}1}, \]
and
\[ P_2 = rB_{x_1\cdots x_{n-1}1} + r^{-k}(1 - r), \]
\[ P_3 = rB_{x_1\cdots x_{n-1}0} + 2r^{-k}(1 - r). \]

Hence
\[ B_m = rB_{2(m-2^{n-1})+1} \cup (rB_{2(m-2^{n-1})+1} + r^{-k}(1 - r)) \cup (rB_{2(m-2^{n-1})} + 2r^{-k}(1 - r)). \]

In the same way, the decomposition in (4.5) is disjoint, which is shown in Figure 4. In this figure, we get the decomposition of \( r^{-k}(E \cup (E+1-r)) \) as three parts in shadows.

By Lemmas 4–7, we have the following proposition.

**Proposition 3.** \( \{B_m\}_{0 \leq m \leq 2^{n-1}} \) are dust-like graph-directed sets with ratio \( r \) and adjacency matrix \( M_n \), where \( B_0 = E \).

The proof of Proposition 1. It follows from Lemma 3, Proposition 2 and Proposition 3 that \( A_0 (= F) \) and \( B_0 (= E) \) are Lipschitz equivalent. This completes the proof of Proposition 1. \( \square \)

5. Properties of adjacency matrix

In the section, for the adjacency matrix \( M_n \) defined in (3.1), we will discuss its Perron–Frobenius eigenvalue and Perron–Frobenius eigenvector.

Let \( a = \frac{1+\sqrt{5}}{2} \), we have
\[ a^2 = a + 1 \quad \text{and} \quad a^3 = 2a + 1. \]

Let \( \{F_k\}_{k \geq 1} = \{1, 1, 2, 3, 5, \cdots \} \) be the Fibonacci sequence. Set \( F_{-1} = 1, \ F_0 = 0 \). We have
\[ F_{k+1} = F_k + F_{k-1} \quad \text{for all} \ k \geq 0. \]
By induction, we have

\[ a^k = aF_k + F_{k-1} \quad \text{for all } k \geq 0. \]

In fact, \( a^0 = 1 = aF_0 + F_{-1} \), assume that \( a^k = aF_k + F_{k-1} \), by (5.1) and (5.2) we can find

\[ a^{k+1} = a(aF_k + F_{k-1}) = a^2F_k + aF_{k-1} = (a + 1)F_k + aF_{k-1} \]
\[ = a(F_k + F_{k-1}) + F_k = aF_{k+1} + F_k. \]

**Proposition 4.** Given \( n \geq 1 \), the Perron–Frobenius eigenvalue of \( M_n \) is \( \omega = \frac{3+\sqrt{5}}{2} \). Let \( V_n = (v_0, v_1, \cdots, v_{2^n-1})^T > 0 \) be the Perron–Frobenius eigenvector of \( M_n \) with \( v_0 = 1 \). Then

\[ V_n = (v_m)_{0 \leq m \leq 2^n-1} \quad \text{with} \quad v_m = a^{L(m)} = aF_{L(m)} + F_{L(m)-1}. \]

**Proof.** For given \( n \geq 1 \), by Lemma 2 and Proposition 2, we have

\[ \dim_H(K_n) = -\frac{\log \omega_n}{\log r}, \]
where \( \omega_n \) is the Perron–Frobenius (PF) eigenvalue of \( M_n \). In particular, we easily check that \( \dim_H(K_1) = -\frac{\log \omega_1}{\log r} \), where \( \omega_1 = \frac{3+\sqrt{5}}{2} \) is the PF eigenvalue of \( M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \).

Since \( K_n \) and \( K_1 \) are Lipschitz equivalent, we have

\[ \dim_H(K_n) = \dim_H(K_1). \]

Then by (5.5),

\[ \omega_n = \omega_1 = \omega = \frac{3 + \sqrt{5}}{2} = a^2. \]

Since the matrix \( M_n \) is primitive, we notice that the eigenspace associated to the PF eigenvalue is one dimensional. Therefore, to prove that \( V_n \) defined in (5.4) is the unique PF eigenvector of \( M_n \) with \( v_0 = a^0 = 1 \), we only need to check that

\[ M_nV_n = \omega V_n, \]
i.e.,

\[ \sum_{j=0}^{2^n-1} g_{mj}v_j = \omega \cdot v_m \quad \text{for every } m. \]

We will check (5.6) for two cases.

**Case 1.** When \( 0 \leq m = x_0x_1 \cdots x_{n-1} < 2^n-1 \), that is \( x_0 = 0 \). Then by Lemma 1,

\[ L(2m) = L(m) \quad \text{and} \quad L(2m+1) = L(m)+1. \]

By the definition of \( M_n \) and (5.1), we have

\[ \sum_{j=0}^{2^n-1} g_{mj}v_j = v_{2m} + v_{2m+1} = a^{L(2m)} + a^{L(2m+1)} \]
\[ = a^{L(m)} + a^{L(m)+1} = a^{L(m)}(1 + a) = a^2 \cdot a^{L(m)} = \omega \cdot v_m. \]

**Case 2.** When \( 2^n-1 \leq m = x_0x_1 \cdots x_{n-1} \leq 2^n - 1 \), that is \( x_0 = 1 \). Then by Lemma 1,

\[ L(2(m - 2^{n-1})) = L(m) - 1 \quad \text{and} \quad L(2(m - 2^{n-1}) + 1) = L(m). \]
By the definition of $M_n$ and (5.1), we have
\[ \sum_{j=0}^{2^n-1} g_{m_j} v_j = v_{2(m-2^n-1)} + 2v_{2(m-2^n-1)+1} \]
\[ = a^{L(2(m-2^n-1))} + 2a^{L(2(m-2^n-1)+1)} \]
\[ = a^{L(m)-1} + 2a^{L(m)} = (1 + 2a)a^{L(m)-1} \]
\[ = a^3 \cdot a^{L(m)-1} = a^2 \cdot a^{L(m)} = \omega \cdot v_m. \]

**Remark 7.** Rao and Wen [23] pointed out that $\omega$ can be proved to be the PF eigenvalue of $M_n$ by induction. This conclusion is also a consequence of Proposition 1 (or Theorem 1) as shown in the proof of Proposition 4. Rao and Wen [23] also obtained the recurrent structure
\[ V_1 = (1, a)^T, \ldots, V_n = (V_{n-1}, aV_{n-1})^T, \ldots. \]

6. Lipschitz equivalent to a self-similar set without overlaps

Fix $r \in (0, 1/3]$. Let
\[ K = (rK + 1 - r) \cup (r^{1/2}K) \]
be a self-similar set without overlaps.

In this section, we will prove the following proposition.

**Proposition 5.** $K_n$ and $K$ are Lipschitz equivalent for any $n$.

**Proof.** By Theorem 1, we only need to show that $K_1$ and $K$ are Lipschitz equivalent.

In fact, let
\[ \Gamma_0 = K \quad \text{and} \quad \Gamma_1 = r^{-1/2}K. \]
Then we have the following disjoint decomposition
\[ (6.1) \quad \Gamma_0 = K = (rK + 1 - r) \cup r^{1/2}K = (r\Gamma_0 + 1 - r) \cup r\Gamma_1. \]

By using (6.1), we also have the following disjoint decomposition
\[ (6.2) \quad \Gamma_1 = r^{-1/2}K = r^{-1/2}[rK + 1 - r) \cup r^{1/2}K] \]
\[ = [r(r^{-1/2}K) + r^{-1/2}(1 - r)] \cup K \]
\[ = [r\Gamma_1 + r^{-1/2}(1 - r)] \cup (r\Gamma_0 + 1 - r) \cup r\Gamma_1. \]

The above decompositions (6.1)–(6.2) show that \{\Gamma_0, \Gamma_1\} are dust-like graph-directed sets with ratio $r$ and adjacency matrix
\[ M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \]

Applying Proposition 2 to the case of $n = 1$, we get graph-directed sets \{B_0, B_1\} with ratio $r$ and adjacency matrix $M_1$, where $B_0 = K_1$.

It follows from Lemma 3 that $K$ and $K_1$ are Lipschitz equivalent. \[ \square \]

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References


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