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THE ESSENTIAL NORM OF A COMPOSITION OPERATOR ON THE MINIMAL MÖBIUS INVARIANT SPACE

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Abstract. We derive a formula for the essential norm of a composition operator on the minimal Möbius invariant space of analytic functions. This extends a recent result due to Wulan and Xiong, and completes the picture of the situation in the Besov space setting. Our methods carry over to the case of the Bergman space A_1, so we are able to complement a result of Vukotić concerning the essential norm of an operator on that space. Moreover, we show that the essential norm of a non-compact composition operator is at least 1. We also obtain lower bounds depending on the behavior of the symbol near the boundary, and calculate the order of magnitude of the essential norm of composition operators induced by finite Blaschke products.

Let $\mathbf{D} \subset \mathbf{C}$ be the open unit disk. For $\alpha \in \overline{\mathbf{D}}$, we put

$$\varphi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}, \quad z \in \mathbf{D}.$$

The minimal space $\mathscr{M}$ (or analytic Besov B_1 space) is defined to be the set of all analytic functions f on $\mathbf{D}$ for which there exist a sequence of points $\alpha_n \in \overline{\mathbf{D}}$, and a sequence of complex numbers $\lambda = (\lambda(n))_{n=1}^{\infty} \in \ell^1$ such that

$$f(z) = \sum_{n=1}^{\infty} \lambda(n)\, \varphi_{\alpha_n}(z).$$

So, a function in $\mathscr{M}$ has an "atomic" decomposition as a sum of Möbius transformations. We norm $\mathscr{M}$ by

$$\|f\|_{\mathscr{M}} = \inf \left\{ \|\lambda\|_{\ell^1} : f = \sum_{n=1}^{\infty} \lambda(n)\, \varphi_{\alpha_n}, \text{ for some } \lambda \in \ell^1,\, \alpha_n \in \overline{\mathbf{D}} \right\}.$$

The minimal space was introduced and extensively studied in [1], where it was shown that if one defines appropriately the notion of a "Möbius invariant space", then $\mathscr{M}$ is the smallest one. In fact, its norm is stronger than the norm of any other such space. $\mathscr{M}$ may be thought of as the first entry in the hierarchy of the analytic Besov spaces B_p. Note that functions in $\mathscr{M}$ extend continuously to the boundary, so $\mathscr{M}$ is a "boundary regular" space. Moreover, as shown in [1], $\mathscr{M}$ coincides with the set of all analytic functions on $\mathbf{D}$ with integrable second derivative. More specifically, there exists a constant $C > 0$ such that for every $f \in \mathscr{M}$

$$C^{-1} \|f''\|_1 \leq \|f - f(0) - f'(0)\, z\|_{\mathscr{M}} \leq C \|f''\|_1,$$

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where

$$\|f''\|_1 = \iint\limits_{\mathbf{D}} |f''(z)|\, dA(z),$$

and dA is normalized area measure. Every $f \in \mathscr{M}$ can be recovered from its second derivative by means of the formula

$$f(z) = f(0) + f'(0)\, z - \iint\limits_{\mathbf{D}} \frac{\varphi_\alpha(z)}{\overline{\alpha}} f''(\alpha)\, dA(\alpha).$$

In what follows, for positive x, y, the symbols $x \simeq y$ and $x \lesssim y$ mean $C^{-1}x \le y \le C\,x$ and $x \le C\,y$ respectively, where $C > 0$ is an absolute numerical constant, not necessarily the same each time it occurs.

Now let ψ be an analytic map of $\mathbf{D}$ into itself. It is clear that the composition operator

$$C_\psi \colon \mathscr{M} \to \mathscr{M}, \quad C_\psi f = f \circ \psi,$$

is bounded if and only if

$$\sup_{|\alpha|<1} \|(\varphi_\alpha \circ \psi)''\|_1 < \infty. \tag{1}$$

Of course, this is equivalent to

$$\sup_{|\alpha|<1} \iint\limits_{\mathbf{D}} \left| \frac{1-|\alpha|^2}{(1-\overline{\alpha}\psi)^2}\psi'' + 2\overline{\alpha}\frac{1-|\alpha|^2}{(1-\overline{\alpha}\psi)^3}(\psi')^2 \right| dA < +\infty.$$

In [1] the authors prove that this is equivalent to

$$\sup_{|\alpha|<1} \iint\limits_{\mathbf{D}} \frac{1-|\alpha|^2}{|1-\overline{\alpha}\psi|^2}|\psi''|\, dA < +\infty$$

together with

$$\sup_{|\alpha|<1} \iint\limits_{\mathbf{D}} \frac{1-|\alpha|^2}{|1-\overline{\alpha}\psi|^3}|\psi'|^2\, dA < +\infty.$$

One direction of this equivalence is, of course, trivial and the other direction is done in [1] by proving, via interpolation with the Dirichlet space, that, if C_ψ is bounded on $\mathscr{M}$, then the second of the last two relations holds and hence also the first. Here is an alternative proof. If C_ψ is bounded on $\mathscr{M}$, then, since

$$\iint\limits_{\mathbf{D}} |f'|^2\, dA \lesssim \|f\|^2_{\mathscr{M}},$$

we get that

$$\sup_{|\alpha|<1} \iint\limits_{\mathbf{D}} \frac{(1-|\alpha|^2)^2}{|1-\overline{\alpha}\psi|^4}|\psi'|^2\, dA < +\infty.$$

This is equivalent to

$$\sup_{|\alpha|<1} \iint\limits_{\mathbf{D}} \frac{(1-|\alpha|^2)^2}{|1-\overline{\alpha}z|^4} n_\psi(z)\, dA(z) < +\infty,$$

where $n_\psi(z)$ is the cardinality of the inverse image of z under ψ. Now, it is standard to show that this is equivalent to

$$\sup_I \frac{1}{|I|^2} \iint\limits_{S(I)} n_\psi(z)\, dA(z) < +\infty,$$

where I ranges over all arcs on the unit circle $\mathbf{T}$ and $S(I)$ is the usual Carleson square over I. Finally, it is also standard to show that this is equivalent to

$$\sup_{|\alpha|<1} \iint\limits_{\mathbf{D}} \frac{1-|\alpha|^2}{|1-\overline{\alpha}z|^3} n_\psi(z)\, dA(z) < +\infty,$$

which is equivalent to the original

$$\sup_{|\alpha|<1} \iint\limits_{\mathbf{D}} \frac{1-|\alpha|^2}{|1-\overline{\alpha}\psi|^3} |\psi'|^2\, dA < +\infty.$$

A sufficient condition for the boundedness of C_ψ, involving the integral means of ψ'', has been obtained by Blasco in [2], extending a series of partial results in [1]. As far as compactness is concerned, the minimal space satisfies the conditions of Theorem 2.1 in [4], therefore C_ψ is compact if and only if $\|\psi\|_\infty < 1$. Moreover, Wulan and Xiong proved in [7] that C_ψ is compact if and only if the "little Oh" version of (1) holds, namely

$$\limsup_{|\alpha|\to 1} \|(\varphi_\alpha \circ \psi)''\|_1 := \lim_{s\to 1} \sup_{|\alpha|>s} \|(\varphi_\alpha \circ \psi)''\|_1 = 0.$$

Here we are interested in estimating the essential norm of C_ψ which is defined to be the distance of C_ψ to the subspace of compact operators, that is

$$\|C_\psi\|_e = \inf\{\|C_\psi - K\| \colon K \text{ compact}\}.$$

We will prove the following asymptotic estimate which naturally extends the result in [7].

Theorem 1. *Let $C_\psi \colon \mathscr{M} \to \mathscr{M}$ be a bounded composition operator. Then*

$$\|C_\psi\|_e \simeq \limsup_{|\alpha|\to 1} \|(\varphi_\alpha \circ \psi)''\|_1.$$

Proof. For $R \in (0,1)$ we will use the notation

$$\{|\psi| < R\} = \{z \in \mathbf{D} \colon |\psi(z)| < R\}, \quad \{|\psi| \geq R\} = \{z \in \mathbf{D} \colon |\psi(z)| \geq R\}.$$

To prove the lower bound

$$\|C_\psi\|_e \gtrsim \limsup_{|\alpha|\to 1} \|(\varphi_\alpha \circ \psi)''\|_1, \tag{2}$$

choose a sequence $\alpha_n \in \mathbf{D}$ with $|\alpha_n| \to 1$, such that

$$\limsup_{|\alpha|\to 1} \|(\varphi_\alpha \circ \psi)''\|_1 = \lim_n \|(\varphi_{\alpha_n} \circ \psi)''\|_1,$$

and let $K \colon \mathscr{M} \to \mathscr{M}$ be a compact operator. Without loss of generality, we may assume that $\|K\varphi_{\alpha_n} - f\|_{\mathscr{M}} \to 0$, for some $f \in \mathscr{M}$. Then

$$\|C_\psi - K\| \geq \|C_\psi \varphi_{\alpha_n} - K\varphi_{\alpha_n}\|_{\mathscr{M}} \gtrsim \|(\varphi_{\alpha_n} \circ \psi)'' - f''\|_1 - \|K\varphi_{\alpha_n} - f\|_{\mathscr{M}}. \tag{3}$$

We may further assume that f'' is not identically zero, otherwise (2) follows from (3) upon taking the limit as $n \to \infty$. Now let $\varepsilon > 0$, and fix $R \in (0,1)$ such that

$$\iint\limits_{\{|\psi|\geq R\}} |f''|\, dA < \varepsilon.$$

On the other hand

$$\begin{aligned}\iint\limits_{\{|\psi|<R\}} |(\varphi_{\alpha_n}\circ\psi)''|\, dA &= \iint\limits_{\{|\psi|<R\}} \left|\frac{1-|\alpha_n|^2}{(1-\overline{a}_n\psi)^2}\psi'' + 2\overline{a}_n\frac{1-|\alpha_n|^2}{(1-\overline{a}_n\psi)^3}(\psi')^2\right| dA \\ &\lesssim \frac{1-|\alpha_n|^2}{(1-R)^3}\left(\|\psi''\|_1 + \|\psi'\|_2^2\right).\end{aligned}$$

Therefore, for n large enough

$$\iint\limits_{\{|\psi|<R\}} |(\varphi_{\alpha_n}\circ\psi)''|\, dA < \frac{1}{2}\iint\limits_{\{|\psi|<R\}} |f''|\, dA.$$

Consequently

$$\begin{aligned}\|(\varphi_{\alpha_n}\circ\psi)'' - f''\|_1 &= \iint\limits_{\{|\psi|<R\}} |(\varphi_{\alpha_n}\circ\psi)'' - f''|\, dA + \iint\limits_{\{|\psi|\geq R\}} |(\varphi_{\alpha_n}\circ\psi)'' - f''|\, dA \\ &\geq \iint\limits_{\{|\psi|<R\}} |f''|\, dA - \iint\limits_{\{|\psi|<R\}} |(\varphi_{\alpha_n}\circ\psi)''|\, dA \\ &\quad + \iint\limits_{\{|\psi|\geq R\}} |(\varphi_{\alpha_n}\circ\psi)''|\, dA - \varepsilon > \|(\varphi_{\alpha_n}\circ\psi)''\|_1 - \varepsilon.\end{aligned}$$

It follows that

$$\|C_\psi - K\| \gtrsim \limsup_{|\alpha|\to 1} \|(\varphi_\alpha\circ\psi)''\|_1 - \varepsilon.$$

The idea of the preceding argument is that for α close to the boundary, $(\varphi_\alpha\circ\psi)''$ and f'', have, loosely speaking of course, "disjoint supports".

To prove the upper bound

$$\|C_\psi\|_e \lesssim \limsup_{|\alpha|\to 1} \|(\varphi_\alpha\circ\psi)''\|_1,$$

let $\varepsilon, s, R, R' \in (0,1)$ with $R < R'$, and for $f(z) = \sum_{k=0}^\infty a_k z^k \in \mathscr{M}$ let

$$T_n f(z) = \sum_{k=0}^n \left(1 - \frac{k}{n+1}\right) a_k z^k, \quad Pf(z) = a_0 + a_1 z.$$

Then $T_n f \to f$ in $\mathscr{M}$ and $\|T_n f\|_{\mathscr{M}} \leq \|f\|_{\mathscr{M}}$ by [1]. Since T_n and P are compact operators, we have that

$$\|C_\psi\|_e \leq \|C_\psi - C_\psi T_n - P(C_\psi - C_\psi T_n)\|.$$

Now choose $f_n \in \mathscr{M}$ with $\|f_n\|_{\mathscr{M}} \leq 1$ such that

$$\|C_\psi - C_\psi T_n - P(C_\psi - C_\psi T_n)\| \leq \|(C_\psi - C_\psi T_n)f_n - P(C_\psi - C_\psi T_n)f_n\|_{\mathscr{M}} + \varepsilon.$$

It is not hard to see that weak-* convergence in $\mathscr{M}$ implies uniform convergence on compact sets. So by Alaoglou's theorem, we may assume that there exists $f \in \mathscr{M}$ such that $f_n \to f$ uniformly on compact sets. Now

$$\begin{aligned}(4)\quad \|C_\psi\|_e &\le \|(C_\psi - C_\psi T_n)f_n - P(C_\psi - C_\psi T_n)f_n\|_{\mathscr{M}} + \varepsilon \\ &\lesssim \|(C_\psi f_n - C_\psi T_n f_n)''\|_1 + \varepsilon \\ &= \iint\limits_{\{|\psi|\ge R\}} |(f_n\circ\psi - T_n f_n\circ\psi)''|\,dA + \iint\limits_{\{|\psi|<R\}} |(f_n\circ\psi - T_n f_n\circ\psi)''|\,dA + \varepsilon \\ &= I_1 + I_2 + \varepsilon.\end{aligned}$$

To estimate I_1, we use the reproducing formula

$$(f_n\circ\psi - T_n f_n\circ\psi)''(z) = -\iint\limits_{\mathbf{D}} \frac{1}{\overline{\alpha}}\,(\varphi_\alpha\circ\psi)''(z)(f_n - T_n f_n)''(\alpha)\,dA(\alpha).$$

Then

$$I_1 \le \left(\sup_{|\alpha|\le s}\iint\limits_{\{|\psi|\ge R\}} |(\varphi_\alpha\circ\psi)''|\,dA + \sup_{|\alpha|>s}\|(\varphi_\alpha\circ\psi)''\|_1\right)\iint\limits_{\mathbf{D}}\frac{1}{|\alpha|}\,|(f_n - T_n f_n)''(\alpha)|\,dA(\alpha).$$

For $|\alpha|\le s$ we have

$$\begin{aligned}\iint\limits_{\{|\psi|\ge R\}} |(\varphi_\alpha\circ\psi)''|\,dA &= \iint\limits_{\{|\psi|\ge R\}} \left|\frac{1-|\alpha|^2}{(1-\overline{\alpha}\psi)^2}\psi'' + 2\overline{a}\frac{1-|\alpha|^2}{(1-\overline{\alpha}\psi)^3}(\psi')^2\right| dA \\ &\lesssim \frac{1}{(1-s)^3}\iint\limits_{\{|\psi|\ge R\}} \left(|\psi''| + |\psi'|^2\right)\,dA.\end{aligned}$$

Moreover

$$\iint\limits_{\mathbf{D}}\frac{1}{|\alpha|}\,|(f_n - T_n f_n)''(\alpha)|\,dA(\alpha) \lesssim \|(f_n - T_n f_n)''\|_1 \lesssim \|f_n - T_n f_n\|_{\mathscr{M}} \le 2.$$

Therefore

$$I_1 \lesssim \frac{1}{(1-s)^3}\iint\limits_{\{|\psi|\ge R\}} \left(|\psi''| + |\psi'|^2\right)\,dA + \sup_{|\alpha|>s}\|(\varphi_\alpha\circ\psi)''\|_1.$$

To estimate I_2, note that

$$\begin{aligned}I_2 &\le \iint\limits_{\{|\psi|<R\}} \left(|f_n'\circ\psi - (T_n f_n)'\circ\psi|\cdot|\psi''| + |f_n''\circ\psi - (T_n f_n)''\circ\psi|\cdot|\psi'|^2\right)\,dA \\ &\le C_{R,R'}\left(\sup_{|z|\le R'}|f_n(z) - f(z)| + \|T_n f - f\|_{\mathscr{M}}\right)\left(\|\psi''\|_1 + \|\psi'\|_2^2\right).\end{aligned}$$

So, taking the limits as $n\to\infty$, $R\to 1$, $s\to 1$, $\varepsilon\to 0$ (in the indicated order), (4) gives

$$\|C_\psi\|_e \lesssim \limsup_{|\alpha|\to 1}\|(\varphi_\alpha\circ\psi)''\|_1. \qquad \square$$

Remark 1. The "standard" way to prove lower bounds for the essential norm is to choose an appropriate normalized sequence f_n which converges weakly to zero. Then

$$\|C_\psi\|_e \geq \limsup_n \|C_\psi f_n\|.$$

In our case such a sequence does not exist because the minimal space has the Schur property, that is, weak convergence is equivalent to convergence in the norm. Indeed, $\mathscr{M}$ is isomorphic to the Bergman space A^1 (see, for instance, [9]). But A^1 is isomorphic to ℓ^1, [6], and the latter space is known to have the Schur property. It might nevertheless be instructive to give a more direct proof of the fact that $\mathscr{M}$ and ℓ^1 are isomorphic. Using standard decomposition arguments one shows that there exist a sequence of points $\alpha_n \in \mathbf{D}$, and a sequence of functionals $\Lambda_n \in \mathscr{M}^*$ such that the operator

$$T\colon \ell^1 \to \mathscr{M}, \quad T\lambda = \sum_n \lambda(n)\, \varphi_{\alpha_n},$$

is onto, the operator

$$S\colon \mathscr{M} \to \ell^1, \quad Sf = (\Lambda_n(f))_{n=1}^\infty,$$

is an isomorphic embedding, and $TS = I$. This implies that

$$\ell^1 = \ker(T) \oplus S(\mathscr{M}).$$

Thus, $S(\mathscr{M})$ is complemented, therefore, by a theorem of Pelczynski [3], it is isomorphic to ℓ^1. Actually, we don't even need to know that $\mathscr{M}$ and ℓ^1 are isomorphic in order to establish the Schur property. Here is a completely independent proof. Suppose toward a contradiction that f_n is a sequence such that $f_n \xrightarrow{w} 0$ and $\|f_n''\|_1 = 1$. Weak convergence implies uniform convergence on compact sets, therefore we can find a subsequence f_{k_n} and an increasing sequence $R_n \in (0,1)$ such that

$$\iint\limits_{|z|<R_{n-1}} |f_{k_n}''(z)|\, dA(z) < \frac{1}{10}, \qquad \iint\limits_{R_{n-1}<|z|<R_n} |f_{k_n}''(z)|\, dA(z) > \frac{9}{10}.$$

Now choose real functions θ_n so that

$$|f_{k_n}''(z)| = f_{k_n}''(z)\, e^{i\theta_n(z)}, \quad R_{n-1} < |z| < R_n,$$

and put

$$h = \sum_n e^{i\theta_n} \chi_{\{z\colon R_{n-1}<|z|<R_n\}}.$$

Then $h \in L^\infty(\mathbf{D})$, hence

$$\Lambda_h(f) = \iint\limits_{\mathbf{D}} f'' h\, dA,$$

is an element of $\mathscr{M}^*$. However,

$$\left|\iint\limits_{\mathbf{D}} f_{k_n}'' h\, dA\right| \geq \left|\iint\limits_{R_{n-1}<|z|<R_n} f_{k_n}''(z)\, e^{i\theta_n(z)}\, dA(z)\right| - \iint\limits_{|z|<R_{n-1}} |f_{k_n}''(z)|\, dA(z)$$
$$- \iint\limits_{|z|>R_n} |f_{k_n}''(z)|\, dA(z) > \frac{9}{10} - \frac{1}{10} - \frac{1}{10},$$

contradicting $\Lambda_h(f_{k_n}) \to 0$.

Remark 2. One may use the argument in the proof of the lower bound in Theorem 1 to show that if ψ induces a bounded composition operator on the Bergman space A^1, then

$$\|C_\psi\|_e \geq \limsup_{|\alpha|\to 1} \|u_\alpha \circ \psi\|_1,$$

where

$$u_\alpha(z) = \frac{1-|\alpha|^2}{(1-\overline{\alpha}z)^3}, \quad \alpha, z \in \mathbf{D}.$$

Combining this with the growth estimate

$$|f(z)| \leq \frac{\|f\|_1}{(1-|z|^2)^2}$$

for f in A^1, we obtain

$$\|C_\psi\|_e \geq \limsup_{|\alpha|\to 1} \left(\frac{1-|\alpha|^2}{1-|\psi(\alpha)|^2}\right)^2.$$

This complements a result due to Vukotić [5], who proved that in the case of the reflexive Bergman spaces A^p, $p > 1$, we have

$$\|C_\psi\|_e \geq \limsup_{|\alpha|\to 1} \left(\frac{1-|\alpha|^2}{1-|\psi(\alpha)|^2}\right)^{2/p}.$$

Moreover, using the techniques in the proof of the upper bound in Theorem 1 and the decomposition

$$f = \sum_{n=1}^{\infty} \lambda(n) u_{\alpha_n}, \quad \lambda \in \ell^1,\ \alpha_n \in \mathbf{D},$$

for functions in A^1, we can prove the corresponding upper bound for composition operators on A^1, namely

$$\|C_\psi\|_e \lesssim \limsup_{|\alpha|\to 1} \|u_\alpha \circ \psi\|_1.$$

We omit the details.

Returning to the minimal space, we have already mentioned that the result in [4] implies that if $\|\psi\|_\infty = 1$ then C_ψ is not compact. We will now show that something much stronger holds. Namely, the essential norm of C_ψ is at least 1. We will need the following.

Lemma 1. *Let $f \in \mathscr{M}$ be such that $f(0) = f'(0) = 0$ and $f(1) = 1$. Then for every $0 < r \leq 1/2$ we have that*

$$\iint_{\Omega_r} |f''|\, dA \gtrsim r,$$

where

$$\Omega_r = \{z\colon |z| \leq r\} \cup \bigcup_{|z|\leq r} [z, 1].$$

Here, $[z, 1]$ is the line segment joining the points z and 1.

Proof. Integration by parts yields

$$|f(1-\varepsilon)| - \varepsilon\,|f'(1-\varepsilon)| \le \int_0^{1-\varepsilon} |1-u|\,|f''(u)|\,|du|,$$

for small $\varepsilon > 0$. Letting $\varepsilon \to 0$ and using the fact that f is in the little Bloch space, we get

$$1 \le \int_0^z |1-u|\,|f''(u)|\,|du| + \int_z^1 |1-u|\,|f''(u)|\,|du| = I_1(z) + I_2(z),$$

for all $|z| < 1$. All the integrals above are supposed to be along the corresponding line segments. Therefore

$$r^2 \lesssim \iint\limits_{|z|\le r} I_1(z)\,dA(z) + \iint\limits_{|z|\le r} I_2(z)\,dA(z).$$

However,

$$\begin{aligned}
\iint\limits_{|z|\le r} I_1(z)\,dA(z) &\le 2\int_0^1 \iint\limits_{|z|\le r} |z|\,|f''(tz)|\,dA(z)\,dt = 2\int_0^1 \frac{1}{t^3} \iint\limits_{|z|\le tr} |z|\,|f''(z)|\,dA(z)\,dt \\
&= \iint\limits_{|z|\le r} \left(\frac{r^2}{|z|^2} - 1\right) |z|\,|f''(z)|\,dA(z) \\
&\lesssim r^2 \iint\limits_{|z|\le r/2} \frac{|f''(z)|}{|z|}\,dA(z) + r \iint\limits_{r/2\le|z|\le r} |f''(z)|\,dA(z) \\
&\lesssim r \iint\limits_{|z|\le r} |f''(z)|\,dA(z) \le r \iint\limits_{\Omega_r} |f''(z)|\,dA(z).
\end{aligned}$$

Moreover,

$$\begin{aligned}
\iint\limits_{|z|\le r} I_2(z)\,dA(z) &\le \int_0^1 t \iint\limits_{|z|\le r} |1-z|^2 |f''(1-t+tz)|\,dA(z)\,dt \\
&= \int_0^1 \frac{1}{t^3} \iint\limits_{|z-(1-t)|\le tr} |1-z|^2 |f''(z)|\,dA(z)\,dt \\
&\lesssim \int_0^1 \frac{1}{t} \iint\limits_{|z-(1-t)|\le tr} |f''(z)|\,dA(z)\,dt \\
&\le \iint\limits_{\Omega_r} |f''(z)| \int_{|z-1|/(1+r)}^{|z-1|/(1-r)} \frac{dt}{t}\,dA(z) \lesssim r \iint\limits_{\Omega_r} |f''(z)|\,dA(z),
\end{aligned}$$

and we are done. □

Corollary 1. *Let C_ψ be bounded. If $\|\psi\|_\infty = 1$ then $\|C_\psi\|_e \ge 1$.*

Proof. By automorphism invariance, we may assume that $\psi(0) = 0$ and $\psi(1) = 1$. Let $\beta_n = 1 - 1/n$ and consider the functions

$$f_n(z) = \frac{g_n((1-\beta_n)z + \beta_n) - (1-\beta_n)g_n'(\beta_n)z}{1 - (1-\beta_n)g_n'(\beta_n)}, \quad |z| < 1,$$

where

$$g_n = \frac{1 - \overline{\psi(\beta_n)}}{\psi(\beta_n) - 1}\, \varphi_{\psi(\beta_n)} \circ \psi.$$

Then $f_n(0) = f_n'(0) = 0$ and $f_n(1) = 1$. So, using Lemma 1 with, say, $r = 1/2$, we get

$$\begin{aligned} 1 &\lesssim \iint_{\mathbf{D}} |f_n''(z)|\, dA(z) = \frac{(1-\beta_n)^2}{|1 - (1-\beta_n)g_n'(\beta_n)|} \iint_{\mathbf{D}} |g_n''((1-\beta_n)z + \beta_n)|\, dA(z) \\ &\le \frac{1}{|1 - (1-\beta_n)g_n'(\beta_n)|} \iint_{\mathbf{D}} |(\varphi_{\psi(\beta_n)} \circ \psi)''(z)|\, dA(z). \end{aligned}$$

By the Schwarz–Pick lemma

$$|g_n'(\beta_n)| = \frac{|\psi'(\beta_n)|}{1 - |\psi(\beta_n)|^2} \le \frac{1}{1 - \beta_n^2}.$$

Therefore

$$1 - \frac{1-\beta_n}{1-\beta_n^2} \lesssim \iint_{\mathbf{D}} |(\varphi_{\psi(\beta_n)} \circ \psi)''(z)|\, dA(z).$$

Letting $n \to +\infty$ we obtain

$$\|C_\psi\|_e \gtrsim 1.$$

Note that $\|C_{\psi_n}\|_e \le \|C_\psi\|_e^n$, where ψ_n is the n-fold self-composition $\psi_n = \psi \circ \cdots \circ \psi$. So, we actually have

$$\|C_\psi\|_e \ge 1. \qquad \square$$

This is reminiscent of the situation in H^∞, where the essential norm of a non-compact composition operator is exactly 1, see [8]. In our case, however, $\|C_\psi\|_e$ may take on arbitrarily large values.

Theorem 2. *If B is a Blaschke product of degree n, then $\|C_B\|_e \simeq \|C_B\| \simeq n$.*

Proof. To prove the essential norm estimate, we may assume that $B(0) = 0$ because

$$\limsup_{|\alpha|\to 1} \|(\varphi_\alpha \circ B)''\|_1 = \limsup_{|\alpha|\to 1} \|(\varphi_\alpha \circ \varphi_\beta \circ B)''\|_1$$

for any fixed $\beta \in \mathbf{D}$, and moreover $\varphi_\beta \circ B$ is a Blaschke product of degree n. Then

$$\|(\varphi_\alpha \circ B)''\|_1 = \iint_{\mathbf{D}} \left| 2\overline{\alpha}\,(1-|\alpha|^2)\, \frac{B'(z)^2}{(1 - \overline{\alpha}B(z))^3} + (1-|\alpha|^2)\, \frac{B''(z)}{(1 - \overline{\alpha}B(z))^2} \right| dA(z).$$

Using the change of variable formula for n-valent functions and subordination, we estimate

$$(1-|\alpha|^2) \iint_{\mathbf{D}} \frac{|B'(z)|^2}{|1 - \overline{\alpha}B(z)|^3}\, dA(z) = n\,(1-|\alpha|^2) \iint_{\mathbf{D}} \frac{dA(z)}{|1 - \overline{\alpha}z|^3} \simeq n,$$

$$(1-|\alpha|^2)\iint\limits_{\mathbf{D}}\frac{|B''(z)|}{|1-\overline{\alpha}B(z)|^2}\,dA(z)\le(1-|\alpha|^2)\,\|B''\|_\infty\iint\limits_{\mathbf{D}}\frac{dA(z)}{|1-\overline{\alpha}z|^2}$$
$$\simeq\|B''\|_\infty(1-|\alpha|^2)\,\log\frac{e}{1-|\alpha|^2}.$$

Therefore

$$\limsup_{|\alpha|\to1}\|(\varphi_\alpha\circ B)''\|_1\simeq n.$$

To prove the norm estimate, we write

$$B=\prod_{j=1}^{n}\psi_j,$$

where ψ_j is a Möbius function φ_{α_j} times a unimodular constant, and introduce the notation

$$B_j=\prod_{k\ne j}\psi_k.$$

Then

$$B''=\frac{(B')^2}{B}+\sum_j B_j\,\psi_j''-\sum_j B_j\,\frac{(\psi_j')^2}{\psi_j}.$$

Hence

$$|B''|\le\frac{|B'|^2}{|B|}+\sum_j|\varphi_{\alpha_j}''|+\sum_j\frac{|\varphi_{\alpha_j}'|^2}{|\varphi_{\alpha_j}|}.$$

Consequently

$$\|B\|_{\mathscr{M}}\simeq\iint\limits_{\mathbf{D}}|B''|\,dA+|B(0)|+|B'(0)|\lesssim n.$$

Therefore

$$\sup_B\|C_B\|=\sup_B\sup_{|\alpha|<1}\|\varphi_\alpha\circ B\|_{\mathscr{M}}=\sup_B\|B\|_{\mathscr{M}}\lesssim n.$$

Since $\|C_B\|_e\le\|C_B\|$, we see that $\|C_B\|$ is in fact comparable to n. □

Note that the argument in the proof of Theorem 2 shows that if $\|\psi''\|_\infty<\infty$, then

$$\|C_\psi\|_e\lesssim\limsup_{|z|\to1}n_\psi(z).$$

This leads to the conjecture that, at least for "nice" symbols, we have

$$\|C_\psi\|_e\simeq\limsup_{|z|\to1}n_\psi(z).$$

We do not know how to prove (or disprove) this. We will, nevertheless, show that the lower bound in Corollary 1 can be improved under certain assumptions on the valency of ψ near the boundary. First, we need some preparation. Let $\psi\in\mathscr{M}$ and

$$\Omega(\beta;w;s)=\{z\colon|z-\beta|\le s\}\cup\bigcup_{|z-\beta|\le s}[z,w],$$

where $\{z\colon |z-\beta|\le s\}\subseteq\mathbf{D}$, $w\in\overline{\mathbf{D}}$ and $s\le\frac{1}{2}|w-\beta|$. $[z,w]$ is the line segment joining z and w. We also let $\alpha=\psi(\beta)$ and we apply Lemma 1, after a trivial normalization, to get

$$\iint\limits_{\Omega(\beta;w;s)}|(\phi_\alpha\circ\psi)''(z)|\,dA(z)\ge c_0 s\left|\frac{\alpha-\psi(w)}{(w-\beta)(1-\overline{\alpha}\psi(w))}-\frac{\psi'(\beta)}{|\alpha|^2-1}\right|,$$

where c_0 is an absolute constant. In particular, if $|w-\beta|\ge 1-|\beta|$, then

$$\iint\limits_{\Omega(\beta;w;\frac{1-|\beta|}{2})}|(\phi_\alpha\circ\psi)''(z)|\,dA(z)\ge\frac{c_0}{2}(1-|\beta|)\left|\frac{\alpha-\psi(w)}{(w-\beta)(1-\overline{\alpha}\psi(w))}-\frac{\psi'(\beta)}{|\alpha|^2-1}\right|.$$

Lemma 2. *Let $\zeta\in\mathbf{T}$ and suppose that $\psi(\zeta)\in\mathbf{T}$. Then, for every sequence β_n converging to ζ, if we put $\alpha_n=\psi(\beta_n)$ we have that either*

$$\limsup\frac{|\zeta-\beta_n|}{1-|\beta_n|}\iint\limits_{\Omega(\beta_n;\zeta;\frac{1-|\beta_n|}{2})}\left|(\phi_{\alpha_n}\circ\psi)''(z)\right|dA(z)\ge\frac{c_0}{4}$$

or

$$\limsup\frac{|\zeta-\beta_n|}{1-|\beta_n|}\iint\limits_{\Omega(\beta_n;w;\frac{1-|\beta_n|}{2})}\left|(\phi_{\alpha_n}\circ\psi)''(z)\right|dA(z)\ge\frac{c_0}{4}$$

for all $w\in\mathbf{D}$.

Proof. To get a contradiction we assume that

$$\limsup\frac{|\zeta-\beta_n|}{1-|\beta_n|}\iint\limits_{\Omega(\beta_n;\zeta;\frac{1-|\beta_n|}{2})}|(\phi_{\alpha_n}\circ\psi)''(z)|\,dA(z)<\frac{c_0}{4}$$

and

$$\limsup\frac{|\zeta-\beta_n|}{1-|\beta_n|}\iint\limits_{\Omega(\beta_n;w;\frac{1-|\beta_n|}{2})}|(\phi_{\alpha_n}\circ\psi)''(z)|\,dA(z)<\frac{c_0}{4}$$

for at least one $w\in\mathbf{D}$. These imply

$$\frac{1}{2}>\limsup|\zeta-\beta_n|\left|\frac{\alpha_n-\psi(\zeta)}{(\zeta-\beta_n)(1-\overline{\alpha_n}\psi(\zeta))}-\frac{\psi'(\beta_n)}{|\alpha_n|^2-1}\right|$$

and

$$\frac{1}{2}>\limsup|\zeta-\beta_n|\left|\frac{\alpha_n-\psi(w)}{(w-\beta_n)(1-\overline{\alpha_n}\psi(w))}-\frac{\psi'(\beta_n)}{|\alpha_n|^2-1}\right|.$$

Adding, we get

$$1>\limsup\left|\frac{\alpha_n-\psi(\zeta)}{1-\overline{\alpha_n}\psi(\zeta)}-\frac{(\zeta-\beta_n)(\alpha_n-\psi(w))}{(w-\beta_n)(1-\overline{\alpha_n}\psi(w))}\right|$$

and, hence,

$$1>1.$$

□

Now, let $0<t\le 1$ and $n\in\mathbf{N}$. A point $\xi\in\mathbf{T}$ is called an $(n;t)$ value of ψ if there are $\zeta_1,\dots,\zeta_n\in\mathbf{T}$ and a sequence of points α_m converging to ξ, such that for each α_m there exist $\beta_{m,1},\dots,\beta_{m,n}$ so that

$$\psi(\beta_{m,1})=\cdots=\psi(\beta_{m,n})=\alpha_m$$

and

$$\frac{1-|\beta_{m,j}|}{|\zeta_j-\beta_{m,j}|}\geq t\quad (1\leq j\leq n).$$

In such a case, it is obvious that $\beta_{m,j}\to\zeta_j\,(1\leq j\leq n)$ and that

$$\psi(\zeta_1)=\cdots=\psi(\zeta_n)=\xi.$$

So, geometrically, ξ can be approximated by points whose preimages lie within a fixed number of Stoltz domains.

Theorem 3. *If there is at least one $(n;t)$ value of ψ, then $\|C_\psi\|_e\geq\frac{c_0}{4}nt$.*

Proof. It is obvious by Lemma 2, that for each $j=1,\dots,n$ we can choose a fixed w_j so that either $w_j\in\mathbf{D}$ or $w_j=\zeta_j$ and so that, for some subsequence α_{m_k} (and the corresponding subsequences $\beta_{m_k,j}\,(1\leq j\leq n)$), all regions $\Omega(\beta_{m_k,j};w_j;\frac{1-|\beta_{m_k,j}|}{2})$, $(1\leq j\leq n)$ are mutually disjoint and so that

$$\limsup_k\iint\limits_{\Omega(\beta_{m_k,j};w_j;\frac{1-|\beta_{m_k,j}|}{2})}\left|(\phi_{\alpha_{m_k}}\circ\psi)''(z)\right|dA(z)\geq\frac{c_0}{4}\,t.$$

Adding in j, we see that

$$\|C_\psi\|_e\geq\limsup_k\iint\limits_{\mathbf{D}}\left|(\phi_{\alpha_{m_k}}\circ\psi)''(z)\right|dA(z)\geq\frac{c_0}{4}\,nt.\qquad\square$$

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