RENORMALIZED SOLUTIONS FOR
A NON-UNIFORMLY PARABOLIC EQUATION

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Abstract. In this paper we prove the existence of nonnegative renormalized solutions for the initial-boundary value problem of a non-uniformly parabolic equation. Some well-known parabolic equations are the special cases of this equation.

1. Introduction

Suppose that Ω is a bounded domain of $\mathbf{R}^N$ ($N \geq 2$) with Lipschitz boundary $\partial\Omega$, and T is a positive number. Denote $\Omega_T = \Omega \times (0, T], \Sigma = \partial\Omega \times (0, T]$. In this paper we study the following non-uniformly parabolic initial-boundary value problem

$$(1.1) \qquad \begin{cases} u_t - \operatorname{div}\left(D_\xi \Phi(\nabla u)\right) = f & \text{in } \Omega_T, \\ u(x, t) = 0 & \text{on } \Sigma, \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases}$$

where $\Phi \colon \mathbf{R}^N \mapsto \mathbf{R}^+$ is a C^1 nonnegative, strictly convex function, $D_\xi \Phi \colon \mathbf{R}^N \to \mathbf{R}$ represents the gradient of $\Phi(\xi)$ with respect to ξ and ∇u represents the gradient with respect to the spatial variables x. Without loss of generality we may assume that $\Phi(0) = 0$.

Our main assumptions are that $\Phi(\xi)$ satisfies the super-linear condition (or 1-coercive condition, see [23, Chapter E])

$$(1.2) \qquad \lim_{|\xi| \to \infty} \frac{\Phi(\xi)}{|\xi|} = \infty,$$

and the symmetric condition: there exists a positive number $C > 0$ such that

$$(1.3) \qquad \Phi(-\xi) \leq C\Phi(\xi), \quad \xi \in \mathbf{R}^N.$$

In this paper we assume

$$(1.4) \qquad u_0 \in L^1(\Omega) \quad \text{and} \quad f \in L^1(\Omega_T)$$

with

$$(1.5) \qquad u_0 \geq 0 \quad \text{and} \quad f \geq 0.$$

There are numerous examples of $\Phi(\xi)$ satisfying structure assumptions (1.2) and (1.3). The well-known are listed as follows.

Example 1.1.

$$\Phi(\xi) = \frac{1}{p}|\xi|^p, \quad p > 1.$$

In this case, equation (1.1) is the parabolic counterpart of the p-Laplacian.

Example 1.2.

$$\Phi(\xi) = \frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2} + \cdots + \frac{1}{p_N}|\xi_N|^{p_N}, \quad p_i > 1, \ i = 1, 2, \ldots, N,$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$. In this case, equation (1.1) is the parabolic counterpart of the anisotropic p-Laplacian. (See [25, Chapter 2].)

Example 1.3.

$$\Phi(\xi) = |\xi| \log(1 + |\xi|).$$

(See [12, Chapter 4] and [20].)

Example 1.4.

$$\Phi(\xi) = |\xi| L_k(|\xi|),$$

where $L_i(s) = \log(1 + L_{i-1}(s)) \, (i = 1, 2, \ldots, k)$ and $L_0(s) = \log(1 + s)$ for $s \geq 0$. (See [22].)

Example 1.5.

$$\Phi(\xi) = e^{\frac{|\xi|^2}{2}} - 1.$$

(See [17], [24] and [28].)

Recently, Cai and Zhou [10] considered problem (1.1) without the right hand side f under structure assumptions (1.2), (1.3) and integrability condition $u_0 \in L^2(\Omega)$ and proved the existence and uniqueness of weak solutions. This paper is a continuation of [10]. We are interested in the study of the well-posedness of problem (1.1) with L^1 data. The existence techniques akin to the techniques of this paper were used in the elliptic case by Boccardo and Gallouët in [7] and [8]. The first paper [7] also contained parabolic results, further developed in [6]. However, the counterexamples by Serrin, [32], indicated that with nontrivial right hand side uniqueness might fail when $n \geq 3$ with the usual definition. Thus some extra conditions on the distributional solutions were needed in order to ensure both existence and uniqueness. The three different definitions for this purpose were independently introduced by Bénilan et al. [2] (entropy solutions), by Dall'Aglio [16] (SOLA, Solutions Obtained as Limit of Approximations) and by Lions and Murat [27] (renormalized solutions, see also [20]). In the parabolic case, one should consult [16] (SOLA), [3] (renormalized solutions) and [29] (entropy solutions).

One result on this topic can be found in [34] where via the introduction of the notion of entropy solutions. We proved that there exists an entropy solution for problem (1.1) under assumptions (1.2), (1.3) and (1.4). Besides, in [33] we studied the following nonlinear parabolic problem

(1.6)
$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f & \text{in } \Omega_T, \\ u = 0 & \text{on } \Sigma, \\ u(x, 0) = u_0(x) & \text{on } \Omega, \end{cases}$$

where the variable exponent $p\colon \overline{\Omega} \to (1, +\infty)$ is a continuous function, $f \in L^1(\Omega_T)$ and $u_0 \in L^1(\Omega)$. We proved the existence and uniqueness of both renormalized solutions and entropy solutions for problem (1.6) and discovered the equivalence of renormalized solutions and entropy solutions. This motivates us to study problem (1.1) in the framework of renormalized solutions. The notion of renormalized solutions was first introduced by DiPerna and Lions [20] for the study of Boltamann equation. It was then adapted to the study of some nonlinear elliptic and parabolic problems and evolution problems in fluid mechanics ([3, 4, 5, 9, 15, 26]). We hope that the renormalized solution is still existent and unique, and it is equivalent to the entropy solution of problem (1.1). However, so far we can only get the existence of renormalized solutions for problem (1.1) under (1.2), (1.3), (1.4) and additional (1.5). The uniqueness of renormalized solutions and the equivalence of renormalized solutions and entropy solutions remain open. The main difficulty lies on that there is no growth condition for function $\Phi(\xi)$. To our knowledge, growth conditions such as polynomial growth or exponential growth for function $\Phi(\xi)$ played an extremely important role, for example in [1, 11, 24], when parabolic problems or variational problems related to (1.1) were studied. If some condition such as Δ_2-condition, i.e., there exists a positive constant K such that for every $\xi > 0$ such that

$$\Phi(2\xi) \leq K\Phi(\xi),$$

is further assumed on function $\Phi(\xi)$, it is possible to prove the uniqueness result of the renormalized solutions, and the equivalence of renormalized solutions and entropy solutions for problem (1.1) without the nonnegativity assumption (1.5).

Let T_k denote the truncation function at height $k \geq 0$:

$$T_k(r) = \min\{k, \max\{r, -k\}\} = \begin{cases} k & \text{if } r \geq k, \\ r & \text{if } |r| < k, \\ -k & \text{if } r \leq -k, \end{cases}$$

and its primitive $\Theta_k\colon \mathbf{R} \to \mathbf{R}^+$ by

$$(1.7) \qquad \Theta_k(r) = \int_0^r T_k(s)\,ds = \begin{cases} \frac{r^2}{2} & \text{if } |r| \leq k, \\ k|r| - \frac{k^2}{2} & \text{if } |r| \geq k. \end{cases}$$

It is obvious that $\Theta_k(r) \geq 0$ and $\Theta_k(r) \leq k|r|$.

Next we define the very weak gradient of a measurable function u with $T_k(u) \in L^1(0, T; W_0^{1,1}(\Omega))$. As a matter of the fact, working as in Lemma 2.1 of [2] we can prove the following result:

Proposition 1.6. *For every measurable function u on Ω_T such that $T_k(u)$ belongs to $L^1(0, T; W_0^{1,1}(\Omega))$ for every $k > 0$, there exists a unique measurable function $v\colon \Omega_T \to \mathbf{R}^N$, such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \quad \text{almost everywhere in } \Omega_T \text{ and for every } k > 0,$$

where χ_E denotes the characteristic function of a measurable set E. Moreover, if u belongs to $L^1(0, T; W_0^{1,1}(\Omega))$, then v coincides with the weak gradient of u.

From Proposition 1.6, we denote $v = \nabla u$, which is called the very weak gradient of u. The notion of the very weak gradient allows us to give the following definition of renormalized solutions for problem (1.1). Denote $z = (x, t)$, $dz = dx\,dt$.

Definition 1.7. A nonnegative function $u \in C([0,T]; L^1(\Omega))$ such that $T_k(u) \in L^1(0,T; W_0^{1,1}(\Omega))$ is called a renormalized solution to problem (1.1) if the following conditions are satisfied:

(i) $\displaystyle \lim_{n \to \infty} \int_{\{(x,t) \in \Omega_T \,:\, n \leq |u(x,t)| \leq n+1\}} D_\xi \Phi(\nabla u) \cdot \nabla u \, dz = 0;$

(ii) For every nonnegative function $\varphi \in C^1(\overline{\Omega}_T)$ with $\varphi(\cdot, T) = 0$ and $S \in C^\infty(\mathbf{R}^+)$ satisfying that S' has a compact support and S'' is non-positive,

$$
\begin{aligned}
(1.8) \qquad &- \int_\Omega S(u_0)\varphi(x,0)\,dx - \int_0^T \int_\Omega S(u)\frac{\partial \varphi}{\partial t}\,dz \\
&+ \int_0^T \int_\Omega [S'(u)D_\xi \Phi(\nabla u) \cdot \nabla \varphi + S''(u)D_\xi \Phi(\nabla u) \cdot \nabla u \varphi]\,dz \\
&\geq \int_0^T \int_\Omega f S'(u)\varphi\,dz
\end{aligned}
$$

holds.

Now we state our main result.

Theorem 1.8. *Under structure assumptions* (1.2), (1.3), *integrability condition* (1.4) *and nonnegativity condition* (1.5), *there exists a renormalized solution for problem* (1.1).

The rest of this paper is organized as follows. In Section 2, we state some basic results that will be used later. We will prove the main result in Section 3. In the following sections C will represent a generic constant that may change from line to line even if in the same inequality.

2. Preliminaries

Let $\Phi(\xi)$ be a nonnegative convex function. We define the polar function of $\Phi(\xi)$ as

$$
(2.1) \qquad\qquad \Psi(\eta) = \sup_{\xi \in \mathbf{R}^N} \{\eta \cdot \xi - \Phi(\xi)\},
$$

which is also known as the Legendre transform of $\Phi(\xi)$. It is obvious that $\Psi(\eta)$ is a convex function. In the following we will list several lemmas.

Definition 2.1. [23, Definition 4.1.3] Let $C \subset \mathbf{R}^N$ be convex. The mapping $F \colon C \to \mathbf{R}^N$ is said to be monotone [resp. strictly monotone] on C when, for all x and x' in C,

$$
\langle F(x) - F(x'), x - x' \rangle \geq 0,
$$

$$
[\text{resp. } \langle F(x) - F(x'), x - x' \rangle > 0 \quad \text{whenever} \quad x \neq x'].
$$

Lemma 2.2. [23, Theorem 4.1.4] *Let f be a function differentiable on an open set $\Omega \subset \mathbf{R}^N$ and let C be a convex subset of Ω. Then, f is convex [resp. strictly convex] on C if and only if its gradient ∇f is monotone [resp. strictly monotone] on C.*

Lemma 2.3. *Suppose that $\Phi(\xi)$ is a convex C^1 function with $\Phi(0) = 0$. Then we have for all $\xi, \zeta \in \mathbf{R}^N$,*

$$(2.2) \qquad\qquad\qquad \Phi(\xi) \leq \xi \cdot D\Phi(\xi),$$

$$(2.3) \qquad\qquad (D\Phi(\xi) - D\Phi(\zeta)) \cdot (\xi - \zeta) \geq 0.$$

Lemma 2.4. [10, Lemma 2.2] *Suppose that $\Phi(\xi)$ is a nonnegative convex C^1 function and $\Psi(\eta)$ is its polar function. Then we have for $\xi, \eta, \zeta \in \mathbf{R}^N$,*

$$(2.4) \qquad\qquad\qquad \xi \cdot \eta \leq \Phi(\xi) + \Psi(\eta),$$

$$(2.5) \qquad\qquad \Psi(D\Phi(\zeta)) + \Phi(\zeta) = D\Phi(\zeta) \cdot \zeta.$$

Lemma 2.5. [21, Chapter 3] *Suppose that $\Phi(\xi)$ is a nonnegative convex function with $\Phi(0) = 0$, which satisfies (1.2). Then its polar function $\Psi(\eta)$ in (2.1) is a well-defined, nonnegative function in $\mathbf{R}^N$, which also satisfies (1.2).*

Lemma 2.6. [30, Chapter 4] *Let $D \subset \mathbf{R}^N$ be measurable with finite Lebesgue measure and $f_k \in L^1(D)$ and $g_k \in L^1(D)$ $(k = 1, 2, \dots)$, and*

$$|f_k(x)| \leq g_k(x) \ \text{ a.e. } x \in D, \ k = 1, 2, \dots.$$

If

$$\lim_{k \to \infty} f_k(x) = f(x), \quad \lim_{k \to \infty} g_k(x) = g(x) \text{ a.e. } x \in D,$$

and

$$\lim_{k \to \infty} \int_D g_k(x) \, dx = \int_D g(x) \, dx < +\infty,$$

then we have

$$\lim_{k \to \infty} \int_D f_k(x) \, dx = \int_D f(x) \, dx.$$

Lemma 2.7. [19, Proposition 9.1c] *or* [34, Lemma 2.7] *Let $D \subset \mathbf{R}^N$ be measurable with finite Lebesgue measure, and let $\{f_n\}$ be a sequence of functions in $L^p(D)(p \geq 1)$ such that*

$$f_n \rightharpoonup f \quad \text{weakly in } L^p(D),$$
$$f_n \to g \quad \text{a.e. in } D.$$

Then $f = g$ a.e. in D.

Lemma 2.8. *Let $D \subset \mathbf{R}^N$ be measurable with finite Lebesgue measure. Suppose that $\{a_n(x)\} \subset L^\infty(D)$ and $\{g_n(x)\} \subset L^1(D)$ are two sequences such that*

$$a_n \to a \quad \text{a.e. in } D \quad \text{and} \quad a_n \rightharpoonup a \quad \text{weakly-* in } L^\infty(D)$$

and

$$g_n \rightharpoonup g \quad \text{weakly in } L^1(D).$$

Then

$$a_n g_n \rightharpoonup ag \quad \text{weakly in } L^1(D).$$

Proof. By assumptions, we can know that there exist two positive numbers M_1, M_2 such that

$$\|a_n\|_{L^\infty(D)}, \|a\|_{L^\infty(D)} \leq M_1, \quad \|g_n\|_{L^1(D)}, \|g\|_{L^1(D)} \leq M_2.$$

According to Egorov theorem, for every $\delta > 0$, there exists a measurable set $E_\delta \subset D$ such that $|D\backslash E_\delta| \leq \delta$ and

$$a_n \to a \quad \text{uniformly in } E_\delta.$$

Then for every $\varepsilon > 0$, there exists $N_1 > 0$ such that when $n > N_1$,

$$\sup_{E_\delta} |a_n - a| < \frac{\varepsilon}{3M_2 M_3},$$

where M_3 is a positive number determined later.

For fixed $\psi \in L^\infty(D)$, we can write

$$\int_D (a_n g_n - ag)\psi \, dx = \int_D (a_n g_n - a g_n)\psi \, dx + \int_D (g_n - g)a\psi \, dx := I_1 + I_2.$$

Since $g_n \rightharpoonup g$ weakly in $L^1(D)$ and $a\psi \in L^\infty(D)$, it is easy to see that there exists $N_2 > 0$ such that

$$|I_2| < \frac{\varepsilon}{3}$$

when $n > N_2$.

Next we estimate I_1 as follows. Set

$$\|\psi\|_{L^\infty(D)} \leq M_3.$$

For all $n > N_0 = \max\{N_1, N_2\}$, we have

$$I_1 \leq \int_D |(a_n g_n - a g_n)\psi| \, dx = \int_{E_\delta} |(a_n - a)g_n \psi| \, dx + \int_{D\backslash E_\delta} |(a_n - a)g_n \psi| \, dx$$

$$\leq \sup_{E_\delta} |a_n - a| \cdot \|\psi\|_{L^\infty(D)} \|g_n\|_{L^1(D)} + (\|a_n\|_{L^\infty(D)} + \|a\|_{L^\infty(D)}) \|\psi\|_{L^\infty(D)} \int_{D\backslash E_\delta} |g_n| \, dx$$

$$\leq \frac{\varepsilon}{3} + 2M_1 M_3 \int_{D\backslash E_\delta} |g_n| \, dx.$$

We recall that Dunford–Pettis theorem ensures that a sequence in $L^1(D)$ is weakly convergent in $L^1(D)$ if and only if it is equi-integrable. Thus by Dunford–Pettis theorem, for $\varepsilon_0 = \varepsilon/(6M_1 M_3)$, there exists a positive number $\delta_0 = \delta_0(\varepsilon)$ such that

$$\int_A |g_n| \, dx < \varepsilon_0$$

for every subset $A \subset D$ with $|A| < \delta_0$. Choosing $\delta = \delta_0$ it follows that $|D\backslash E_\delta| < \delta_0$ and

$$2M_1 M_3 \int_{D\backslash E_\delta} |g_n| \, dx < \frac{\varepsilon}{3}.$$

Therefore, combining the above estimates, we obtain

$$|I_1| < \frac{2\varepsilon}{3}.$$

Hence, we conclude that

$$\left| \int_D (a_n g_n - ag)\psi \, dx \right| < \varepsilon \quad \text{for all } n > N_0,$$

which implies that

$$\lim_{n\to\infty} \int_D (a_n g_n - ag)\psi \, dx = 0 \quad \text{for all } \psi \in L^\infty(D).$$

This finishes the proof. □

Lemma 2.9. [14, Chapter 3] and [31] *Suppose that $\Phi(\xi)$ is a nonnegative convex function satisfying* (1.2). *Let $D \subset \mathbf{R}^N$ be a measurable with finite Lebesgue measure $|D|$ and let $\{f_k\} \subset L^1(D; \mathbf{R}^N)$ be a sequence satisfying that*

$$(2.6) \qquad \int_D \Phi(f_k) \, dx \leq C,$$

where C is a positive constant. Then there exist a subsequence $\{f_{k_j}\} \subset \{f_k\}$ and a function $f \in L^1(D; \mathbf{R}^N)$ such that

$$(2.7) \qquad f_{k_j} \rightharpoonup f \quad \text{weakly in} \ \ L^1(D; \mathbf{R}^N) \ \text{as} \ \ j \to \infty$$

and

$$(2.8) \qquad \int_D \Phi(f) \, dx \leq \liminf_{j \to \infty} \int_D \Phi(f_{k_j}) \, dx \leq C.$$

For the convenience of the readers, let us recall the definition of weak solutions for problem (1.1) and the main results in [10].

Definition 2.10. A function $u \colon \overline{\Omega} \times [0, T] \to \mathbf{R}$ is a weak solution of problem (1.1) if the following conditions are satisfied:

(i) $u \in C([0, T]; L^2(\Omega)) \cap L^1(0, T; W_0^{1,1}(\Omega))$ with

$$\int_{\Omega_T} D_\xi \Phi(\nabla u) \cdot \nabla u \, dz < \infty;$$

(ii) For any $\varphi \in C^1(\overline{\Omega}_T)$ with $\varphi(\cdot, T) = 0$ and $\varphi(\cdot, t)|_{\partial\Omega} = 0$, we have

$$(2.9) \qquad -\int_\Omega u_0(x)\varphi(x, 0) \, dx + \int_{\Omega_T} \left[-u\varphi_t + D_\xi \Phi(\nabla u) \cdot \nabla\varphi \right] dz = \int_{\Omega_T} f\varphi \, dz.$$

Lemma 2.11. [10, Theorem 1.2] *Let the structure assumptions* (1.2) *and* (1.3) *be satisfied. If $u_0 \in L^2(\Omega)$ and $f = 0$, then there exists a unique weak solution for the initial-boundary value problem* (1.1).

Corollary 2.12. *If we assume $f \in L^2(\Omega_T)$ the existence and uniqueness of weak solutions of problem* (1.1) *can be obtained working as in the proof of Lemma 2.11.*

Remark 2.13. Let u be a weak solution in Definition 2.10. By using the approximation technique (see [13, Chapter 3] or [18, Chapter 2]) we have for every $\varphi \in C^1(\overline{\Omega}_T)$ with $\varphi(\cdot, t)|_{\partial\Omega} = 0$, each $t \in [0, T]$,

$$(2.10) \qquad \int_\Omega u\varphi \, dx \Big|_0^t + \int_0^t \int_\Omega \left[-u\varphi_t + D_\xi \Phi(\nabla u) \cdot \nabla\varphi \right] dx \, d\tau = \int_0^t \int_\Omega f\varphi \, dx \, d\tau.$$

Remark 2.14. Let u be a weak solution in Definition 2.10 with $f = 0$. We can formally choose u as a test function in (2.10) to obtain an energy type estimate. That is, for a.e. $t \in [0, T]$,

$$\frac{1}{2}\|u(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega D_\xi \Phi(\nabla u) \cdot \nabla u \, dx d\tau = \frac{1}{2}\|u_0\|_{L^2(\Omega)}^2.$$

3. The proof of main result

Now we are ready to prove our main result: the existence of renormalized solutions of problem (1.1). Our proof follows rather standard (but nontrivial) procedure. First f and u_0 are estimated by smooth functions for which the existence of weak solutions is known. Then, by using the definition of a weak solution, we show that the sequence of solutions forms a Cauchy sequence in a suitable norm (passing to a subsequence if necessary). However, this is not enough to pass to a limit under the integral sign but more information is needed on the gradients. The next step is to show by a compactness argument that the gradients converge almost everywhere. This can be done without the growth bounds for function Φ. The final step is to validate that the limit is a renormalized solution. Some of the reasoning is based on the ideas developed in [2], [7], [8] and [29].

Proof of Theorem 1.8. We first introduce the approximate problems. Let $\{f_n\} \subset C_0^\infty(\Omega_T)$ and $\{u_{0n}\} \subset C_0^\infty(\Omega)$ be two nonnegative sequences of functions strongly convergent respectively to f in $L^1(\Omega_T)$ and to u_0 in $L^1(\Omega)$ with $f, u_0 \geq 0$, such that

$$(3.1) \qquad \|f_n\|_{L^1(\Omega_T)} \leq \|f\|_{L^1(\Omega_T)}, \ \|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}.$$

Let us consider the approximate problems

$$(3.2) \qquad \begin{cases} (u_n)_t - \operatorname{div}\left(D_\xi\Phi(\nabla u_n)\right) = f_n & \text{in } \Omega_T, \\ u_n = 0 & \text{on } \Sigma, \\ u_n(x,0) = u_{0n} & \text{on } \Omega. \end{cases}$$

By virtue of Lemma 2.11 (see also Corollary 2.12) we can find nonnegative $u_n \in C([0,T]; L^2(\Omega)) \cap L^1(0,T; W_0^{1,1}(\Omega))$, that is a weak solution of problem (3.2) in the sense of Definition 2.10. It follows from $f_n \in C_0^\infty(\Omega_T) \subset L^2(\Omega_T)$, $D_\xi\Phi(\nabla u_n) \in L^1(\Omega_T)$(see [10] for details) and the equation (3.2) that $(u_n)_t \in L^1(0,T; W^{-1,1}(\Omega)) + L^2(\Omega_T)$ and

$$(3.3) \qquad \int_{\Omega_T} D_\xi\Phi(\nabla u_n) \cdot \nabla u_n \, dz < \infty.$$

Our aim is to prove that a subsequence of these approximate solutions $\{u_n\}$ converges to a nonnegative measurable function u, which is an renormalized solution of problem (1.1). We will divide the proof into several steps. Although some of the arguments are not new, we present a self-contained proof for the sake of clarity and readability.

Using an approximation argument as in Remark 2.13 and Remark 2.14, we can choose $T_k(u_n)\chi_{(0,t)}$ as a test function in (3.2) to have

$$(3.4) \qquad \begin{aligned} &\int_\Omega \Theta_k(u_n)(t)\, dx - \int_\Omega \Theta_k(u_{0n})\, dx + \int_0^t\!\!\int_\Omega D_\xi\Phi(\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\, dx\, ds \\ &= \int_0^t\!\!\int_\Omega f_n T_k(u_n)\, dx\, ds. \end{aligned}$$

It follows from the definition (1.7) of $\Theta_k(r)$ and (3.1) that

$$(3.5) \qquad \begin{aligned} &\int_0^t\!\!\int_\Omega D_\xi\Phi(\nabla T_k(u_n)) \cdot \nabla T_k(u_n)\, dx\, ds + \int_\Omega \Theta_k(u_n)(t)\, dx \\ &\leq k\big(\|f_n\|_{L^1(\Omega_T)} + \|u_{0n}\|_{L^1(\Omega)}\big) \leq k\big(\|f\|_{L^1(\Omega_T)} + \|u_0\|_{L^1(\Omega)}\big). \end{aligned}$$

Recalling (2.2), we have

$$(3.6) \qquad \int_{\Omega_T} \Phi(\nabla T_k(u_n)) \, dz \le \int_{\Omega_T} D_\xi \Phi(T_k(\nabla u_n)) \cdot \nabla T_k(u_n) \, dz \le Ck,$$

which implies from (1.2) that

$$(3.7) \qquad \int_{\Omega_T} |\nabla T_k(u_n)| \, dz \le C(k+1),$$

that is $T_k(u_n)$ is bounded in $L^1(0, T; W_0^{1,1}(\Omega))$.

If we choose $k = 1$ in the inequality (3.5), then for a.e. $t \in [0, T]$,

$$\int_\Omega \Theta_1(u_n(t)) \, dx \le \|f\|_{L^1(\Omega_T)} + \|u_0\|_{L^1(\Omega)}.$$

Moreover,

$$\int_\Omega |u_n(t)| \, dx \le \mathrm{meas}(\Omega) + \|f\|_{L^1(\Omega_T)} + \|u_0\|_{L^1(\Omega)}.$$

Thus we obtain

$$(3.8) \qquad \|u_n\|_{L^\infty(0,T;L^1(\Omega))} \le C.$$

Step 1. We shall prove that $\{u_n\}$ converges in $C([0, T]; L^1(\Omega))$ and we shall find a subsequence which is almost everywhere convergent in Ω_T.

Let m and n be two integers, then from (3.2) we can write the weak form as

$$(3.9) \qquad \begin{aligned} &\int_0^T \langle (u_n - u_m)_t, \phi \rangle \, dt + \int_{\Omega_T} [D_\xi \Phi(\nabla u_n) - D_\xi \Phi(\nabla u_m)] \cdot \nabla \phi \, dz \\ &= \int_{\Omega_T} (f_n - f_m) \phi \, dz, \end{aligned}$$

for all $\phi \in C_0^1(\overline{\Omega}_T)$. Recalling (2.4), (2.5), (1.3) and (3.3), we observe that

$$\begin{aligned} |D_\xi \Phi(\nabla u_n) \cdot \nabla u_m| &\le \Phi(\nabla u_m) + \Phi(-\nabla u_m) + \Psi(D_\xi \Phi(\nabla u_n)) \\ &\le (C+1)\Phi(\nabla u_m) + D_\xi \Phi(\nabla u_n) \cdot \nabla u_n \\ &\le (C+1)D_\xi \Phi(\nabla u_m) \cdot \nabla u_m + D_\xi \Phi(\nabla u_n) \cdot \nabla u_n \in L^1(\Omega_T). \end{aligned}$$

Denote

$$(3.10) \qquad \alpha_{n,m} = \int_{\Omega_T} |f_n - f_m| \, dz + \int_\Omega |u_{0n} - u_{0m}| \, dx.$$

We know that

$$\lim_{n,m \to \infty} \alpha_{n,m} = 0.$$

Using an approximation argument as above, we conclude that $w = T_1(u_n - u_m)\chi_{(0,t)}$ with $t \le T$ can be a test function in (3.9). From (2.3), discarding the positive term we get

$$\begin{aligned} \int_\Omega \Theta_1(u_n - u_m)(t) \, dx &\le \int_\Omega \Theta_1(u_{0n} - u_{0m}) \, dx + \|f_n - f_m\|_{L^1(\Omega_T)} \\ &\le \|u_{0n} - u_{0m}\|_{L^1(\Omega)} + \|f_n - f_m\|_{L^1(\Omega_T)} = \alpha_{n,m}. \end{aligned}$$

Therefore, we conclude from the definition (1.7) of $\Theta_k(r)$ that

$$\int_{\{|u_n - u_m| < 1\}} \frac{|u_n - u_m|^2(t)}{2}\, dx + \int_{\{|u_n - u_m| \geq 1\}} \frac{|u_n - u_m|(t)}{2}\, dx$$

$$\leq \int_\Omega [\Theta_1(u_n - u_m)](t)\, dx \leq \alpha_{n,m}.$$

It follows from Hölder's inequality that

$$\int_\Omega |u_n - u_m|(t)\, dx = \int_{\{|u_n - u_m| < 1\}} |u_n - u_m|(t)\, dx + \int_{\{|u_n - u_m| \geq 1\}} |u_n - u_m|(t)\, dx$$

$$\leq \left(\int_{\{|u_n - u_m| < 1\}} |u_n - u_m|^2(t)\, dx \right)^{\frac{1}{2}} \text{meas}(\Omega)^{\frac{1}{2}} + 2\alpha_{n,m}$$

$$\leq (2\text{meas}(\Omega))^{\frac{1}{2}} \alpha_{n,m}^{\frac{1}{2}} + 2\alpha_{n,m}.$$

Thus we get

$$\|u_n - u_m\|_{C([0,T];L^1(\Omega))} \to 0 \quad \text{as } n, m \to \infty,$$

i.e., $\{u_n\}$ is a Cauchy sequence in $C([0,T];L^1(\Omega))$. Then u_n converges to u in $C([0,T];L^1(\Omega))$. We find an a.e. convergent subsequence (still denoted by $\{u_n\}$) in Ω_T such that

$$(3.11) \qquad\qquad\qquad u_n \to u \quad \text{a.e. in } \Omega_T.$$

Recalling (3.6) and Lemma 2.9, we may draw a subsequence (we also denote it by the original sequence for simplicity) such that

$$(3.12) \qquad\qquad\qquad \nabla T_k(u_n) \rightharpoonup \eta_k \quad \text{weakly in } L^1(\Omega_T)$$

and

$$\int_{\Omega_T} \Phi(\eta_k)\, dz \leq Ck.$$

In view of (3.11), we conclude that $\eta_k = \nabla T_k(u)$ a.e. in Ω_T.

Step 2. We shall prove that the sequence $\{\nabla u_n\}$ converges almost everywhere in Ω_T to ∇u (up to a subsequence).

We first claim that $\{\nabla u_n\}$ is a Cauchy sequence in measure. Let $\delta > 0$, and denote

$$E_1 := \{(x,t) \in \Omega_T \colon |\nabla u_n| > h\} \cup \{(x,t) \in \Omega_T \colon |\nabla u_m| > h\},$$
$$E_2 := \{(x,t) \in \Omega_T \colon |u_n - u_m| > 1\}$$

and

$$E_3 := \{(x,t) \in \Omega_T \colon |\nabla u_n| \leq h, |\nabla u_m| \leq h, |u_n - u_m| \leq 1, |\nabla u_n - \nabla u_m| > \delta\},$$

where h will be chosen later. It is obvious that

$$\{(x,t) \in \Omega_T \colon |\nabla u_n - \nabla u_m| > \delta\} \subset E_1 \cup E_2 \cup E_3.$$

For $k \geq 0$, we can write

$$\{(x,t) \in \Omega_T \colon |\nabla u_n| \geq h\} \subset \{(x,t) \in \Omega_T \colon |u_n| \geq k\} \cup \{(x,t) \in \Omega_T \colon |\nabla T_k(u_n)| \geq h\}.$$

Thus, applying (3.8), (1.2) and (3.7), there exist constants $C > 0$ such that

$$\text{meas}\{(x,t) \in \Omega_T \colon |\nabla u_n| \geq h\} \leq \frac{C}{k} + \frac{Ck}{h},$$

when h is large appropriately. By choosing $k = Ch^{\frac{1}{2}}$, we deduce that

$$\text{meas}\{(x,t) \in \Omega_T \colon |\nabla u_n| \geq h\} \leq Ch^{-\frac{1}{2}}.$$

Let $\varepsilon > 0$. We may let $h = h(\varepsilon)$ large enough such that

$$(3.13) \qquad \text{meas}(E_1) \leq \varepsilon/3 \quad \text{for all } n, m \geq 0.$$

On the other hand, by Step 1, we know that $\{u_n\}$ is a Cauchy sequence in $L^1(\Omega_T)$. Then there exists $N_1(\varepsilon) \in \mathbf{N}$ such that

$$(3.14) \qquad \text{meas}(E_2) \leq \varepsilon/3 \quad \text{for all } n, m \geq N_1(\varepsilon).$$

Moreover, since Φ is C^1 and strictly convex, then from Lemma 2.2 and Definition 2.1, there exists a real valued function $m(h, \delta) > 0$ such that

$$(3.15) \qquad (D\Phi(\xi) - D\Phi(\zeta)) \cdot (\xi - \zeta) \geq m(h, \delta) > 0$$

for all $\xi, \zeta \in \mathbf{R}^N$ with $|\xi|, |\zeta| \leq h$, $|\xi - \zeta| \geq \delta$. By taking $T_1(u_n - u_m)$ as a test function in (3.9), we obtain

$$\begin{aligned}
m(h, \delta) \, \text{meas}(E_3) &\leq \int_{E_3} [D_\xi \Phi(\nabla u_n) - D_\xi \Phi(\nabla u_m)] \cdot (\nabla u_n - \nabla u_m) \, dz \\
&\leq \int_{\Omega_T} [D_\xi \Phi(\nabla u_n) - D_\xi \Phi(\nabla u_m)] \cdot \nabla T_1(u_n - u_m) \, dz \\
&\leq \int_{\Omega_T} |f_n - f_m| \, dz + \int_{\Omega} |u_{0n} - u_{0m}| \, dx = \alpha_{n,m},
\end{aligned}$$

which implies that

$$\text{meas}(E_3) \leq \frac{\alpha_{n,m}}{m(h, \delta)} \leq \varepsilon/3,$$

for all $n, m \geq N_2(\varepsilon, \delta)$. It follows from (3.13) and (3.14) that

$$\text{meas}\{(x,t) \in \Omega_T \colon |\nabla u_n - \nabla u_m| > \delta\} \leq \varepsilon \quad \text{for all } n, m \geq \max\{N_1, N_2\},$$

that is $\{\nabla u_n\}$ is a Cauchy sequence in measure. Then we may choose a subsequence (denote it by the original sequence) such that

$$\nabla u_n \to v \quad \text{a.e. in } \Omega_T.$$

Thus, from Proposition 1.6 and $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $L^1(\Omega_T)$, we deduce from Lemma 2.7 that v coincides with the very weak gradient of u. Therefore, we have

$$(3.16) \qquad \nabla u_n \to \nabla u \quad \text{a.e. in } \Omega_T.$$

Step 3. We shall prove that u is a renormalized solution.

For given $a, k > 0$, define the function $T_{k,a}(s) = T_a(s - T_k(s))$ as

$$T_{k,a}(s) = \begin{cases} s - k \operatorname{sign}(s) & \text{if } k \leq |s| < k + a, \\ a \operatorname{sign}(s) & \text{if } |s| \geq k + a, \\ 0 & \text{if } |s| \leq k. \end{cases}$$

Using $T_{k,a}(u_n)$ with $u_n \geq 0$ as a test function in (3.2), we find

$$\int_{\{|u_n|>k\}} \Theta_a(u_n - k)(T) \, dx - \int_{\{|u_{0n}|>k\}} \Theta_a(u_{0n} - k) \, dx$$

$$+ \int_{\{k \leq |u_n| \leq k+a\}} D_\xi \Phi(\nabla u_n) \cdot \nabla u_n \, dz \leq \int_\Omega f_n T_{k,a}(u_n) \, dz,$$

which yields that

$$\int_{\{k \leq |u_n| \leq k+a\}} D_\xi \Phi(\nabla u_n) \cdot \nabla u_n \, dz \leq a \left(\int_{\{|u_n|>k\}} |f_n| \, dz + \int_{\{|u_{0n}|>k\}} |u_{0n}| \, dx \right).$$

Recalling the fact that u belongs to $C([0,T]; L^1(\Omega))$, we have

$$(3.17) \qquad \lim_{k \to \infty} \text{meas}\{(x,t) \in \Omega_T : |u| > k\} = 0.$$

Since $D_\xi \Phi(\nabla u_n) \cdot \nabla u_n \geq 0$, by Fatou's lemma, (3.11) and (3.16), we get

$$\int_{\{k \leq |u| \leq k+a\}} D_\xi \Phi(\nabla u) \cdot \nabla u \, dz = \int_{\Omega_T} D_\xi \Phi(\nabla u) \cdot \nabla u \chi_{\{k \leq |u| \leq k+a\}} \, dz$$

$$= \int_{\Omega_T} \liminf_{n \to \infty} D_\xi \Phi(\nabla u_n) \cdot \nabla u_n \chi_{\{k \leq |u_n| \leq k+a\}} \, dz$$

$$\leq \liminf_{n \to \infty} \int_{\{k \leq |u_n| \leq k+a\}} D_\xi \Phi(\nabla u_n) \cdot \nabla u_n \, dz.$$

Furthermore, since

$$\int_{\{|u_n|>k\}} |f_n| \, dz = \int_{\{|u_n|>k\}} |f_n - f| \, dz + \int_{\{|u_n|>k\}} |f| \, dz$$

$$\leq \int_{\Omega_T} |f_n - f| \, dz + \int_{\{|u_n|>k\}} |f| \, dz,$$

from (3.17) we have

$$\lim_{k \to \infty} \lim_{n \to \infty} \int_{\{|u_n|>k\}} |f_n| \, dz = 0.$$

Similarly, we know

$$\lim_{k \to \infty} \lim_{n \to \infty} \int_{\{|u_{0n}|>k\}} |u_{0n}| \, dx = 0.$$

Therefore, passing to the limit first in n then in k, we conclude that

$$\lim_{k \to \infty} \int_{\{(x,t) \in \Omega_T : k \leq |u(x,t)| \leq k+a\}} D_\xi \Phi(\nabla u) \cdot \nabla u \, dz = 0.$$

Choosing $a = 1$, we obtain the renormalized condition, i.e.,

$$\lim_{k \to \infty} \int_{\{(x,t) \in \Omega_T : k \leq |u(x,t)| \leq k+1\}} D_\xi \Phi(\nabla u) \cdot \nabla u \, dz = 0.$$

Let $S \in C^\infty(\mathbf{R}^+)$ be such that $\text{supp} S' \subset [0,M]$ for some $M > 0$. For every nonnegative function $\varphi \in C^\infty(\overline{\Omega}_T)$ with $\varphi(x,T) = 0$, $S'(u_n)\varphi$ is a test function in

(3.2). It yields

(3.18)
$$\int_{\Omega_T} \frac{\partial S(u_n)}{\partial t} \varphi \, dz + \int_{\Omega_T} [S'(u_n) D_\xi \Phi(\nabla u_n) \cdot \nabla \varphi$$
$$+ S''(u_n) D_\xi \Phi(\nabla u_n) \cdot \nabla u_n \varphi] \, dz = \int_{\Omega_T} f_n S'(u_n) \varphi \, dz.$$

First we consider the first term on the left-hand side of (3.18). Since S is bounded and continuous, (3.11) implies that $S(u_n)$ converges to $S(u)$ a.e. in Ω_T and weakly-* in $L^\infty(\Omega_T)$. Then $\frac{\partial S(u_n)}{\partial t}$ converges to $\frac{\partial S(u)}{\partial t}$ in $D'(\Omega_T)$ as $n \to \infty$, that is

$$\int_{\Omega_T} \frac{\partial S(u_n)}{\partial t} \varphi \, dz \to \int_{\Omega_T} \frac{\partial S(u)}{\partial t} \varphi \, dz.$$

For the other terms on the left-hand side of (3.18), because of supp$S' \subset [0, M]$ we know

$$S'(u_n) D_\xi \Phi(\nabla u_n) = S'(u_n) D_\xi \Phi(\nabla T_M(u_n)).$$

Using (3.11) and $S \in C^\infty(\mathbf{R}^+)$, we have

(3.19)
$$S'(u_n) \to S'(u) \quad \text{a.e. in } \Omega_T.$$

In view of (3.5) and (2.5), we know that

$$\int_{\Omega_T} \Psi(D_\xi \Phi(\nabla T_M(u_n))) \, dz \leq C.$$

Applying Lemma 2.5, Lemma 2.9 and (3.12), we conclude that (up to a subsequence)

(3.20)
$$D_\xi \Phi(\nabla T_M(u_n)) \rightharpoonup D_\xi \Phi(\nabla T_M(u)) \quad \text{weakly in } L^1(\Omega_T).$$

Then (3.19), (3.20), the boundedness of S' and Lemma 2.8 yield that

$$S'(u_n) D_\xi \Phi(\nabla T_m(u_n)) \rightharpoonup S'(u) D_\xi \Phi(\nabla T_M(u)) \quad \text{weakly in } L^1(\Omega_T).$$

Noting that

$$S'(u) D_\xi \Phi(\nabla u) = S'(u) D_\xi \Phi(\nabla T_M(u)),$$

we deduce that

$$\int_{\Omega_T} S'(u_n) D_\xi \Phi(\nabla u_n) \cdot \nabla \varphi \, dz \to \int_{\Omega_T} S'(u) D_\xi \Phi(\nabla u) \cdot \nabla \varphi \, dz,$$

as $n \to \infty$.

Moreover, since $S''(u_n) \leq 0$, $\varphi \geq 0$ and

$$D_\xi \Phi(\nabla u_n) \cdot \nabla u_n \geq 0,$$

then

$$-S''(u_n) D_\xi \Phi(\nabla u_n) \cdot \nabla u_n \varphi \geq 0.$$

Thus from Fatou's lemma, (3.11) and (3.16), we obtain that

$$-\int_{\Omega_T} S''(u) D_\xi \Phi(\nabla u) \cdot \nabla u \varphi \, dz \leq -\liminf_{n \to \infty} \int_{\Omega_T} S''(u_n) D_\xi \Phi(\nabla u_n) \cdot \nabla u_n \varphi \, dz.$$

Using the strong convergence of f_n, (3.11) and the Lebesgue dominated convergence theorem, we can pass to the limits as $n \to \infty$ for the right-hand side of (3.18) to conclude that

$$- \int_\Omega S(u_0)\varphi(x,0)\,dx - \int_0^T \int_\Omega S(u)\frac{\partial \varphi}{\partial t}\,dz$$
$$+ \int_0^T \int_\Omega [S'(u)D_\xi \Phi(\nabla u) \cdot \nabla \varphi + S''(u)D_\xi \Phi(\nabla u) \cdot \nabla u \varphi]\,dz$$
$$\geq \int_0^T \int_\Omega f S'(u)\varphi\,dz$$

for all $k > 0$ and $\varphi \in C^1(\overline{\Omega}_T)$ with $\varphi \geq 0$ and $\varphi|_\Sigma = 0$. This finishes the proof. $\qquad\square$

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