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HILBERT MATRIX OPERATOR ON SPACES OF ANALYTIC FUNCTIONS

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Abstract. We consider the action of the Hilbert matrix operator, H, on the Hardy space H^1, weighted Hardy spaces H^p_α ($\alpha \geq 0$), Bergman spaces with logarithmic weights, etc. In particular, we extend Diamantopoulos–Siskakis result by proving that H maps H^p_α into H^p_α if and only if $\alpha+1/p < 1$. A criterion for Hf to belong to H^1 is given provided the coefficients of f are nonnegative. Also, H maps the A^2-space with weight $\log^\alpha(2/(1-|z|^2))$ into the ordinary Bergman space A^2 if $\alpha > 3$. Similarly, the Bloch space with logarithmic weight is mapped by H into the ordinary Bloch space.

1. Introduction

The Hilbert matrix is an infinite matrix H whose entries are $a_{n,k} = (n+k+1)^{-1}$. This matrix induces a linear operator on sequences:

$$H\colon (a_k)_{k\in\mathbf{N}_0} \longmapsto \left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1}\right)_{n\in\mathbf{N}_0}.$$

The following Hilbert's inequality implies that this operator is well defined and bounded on the space l^p of all p-summable sequences ($p > 1$).

Theorem 1.1. (Hilbert's inequality [5, Chapter IX]) *Suppose* $1 < p < \infty$. *If* $(a_k)_{k\in\mathbf{N}_0} \in l^p$, *then*

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right|^p\right)^{\frac{1}{p}} \leq \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{k=0}^{\infty}|a_k|^p\right)^{\frac{1}{p}}. \tag{1.1}$$

Moreover, the constant $\frac{\pi}{\sin\frac{\pi}{p}}$ *is best possible.*

Apart from sequence spaces, the Hilbert matrix can be viewed as an operator on spaces of analytic functions by its action on their Taylor coefficients. If

$$f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$$

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is a holomorphic function in the unit disk $\mathbf{D} = \{z \in \mathbf{C}\colon |z| < 1\}$, then we define a transformation H by

$$Hf(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^n. \tag{1.2}$$

Let $H(\mathbf{D})$ be the algebra of holomorphic functions in $\mathbf{D}$. For $0 < p \leq \infty$ Hardy space H^p is the space of all holomorphic functions $f \in H(\mathbf{D})$ for which

$$\|f\|_p = \sup_{0 \leq r < 1} M_p(r, f) < \infty,$$

where

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}, \quad 0 < p < \infty;$$
$$M_\infty(r,f) = \sup_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.$$

It follows from the Hardy's inequality ([4], p. 48)

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(k)|}{k+1} \leq \pi \|f\|_1$$

that H is well defined for each $f \in H^p$, $p \geq 1$. It was proved by Diamantopoulos and Siskakis ([1]) that the operator H is bounded on H^p, $1 < p < \infty$, and not bounded on H^1 and H^∞. In [3] the following formula for H acting on H^p, $p \geq 1$, was noticed

$$Hf = P_+(M_b C f),$$

where $Cf(e^{it}) = f(e^{-it})$ is an isometry from H^p into $L^p(\mathbf{T})$, $M_b(u) = bu$, $b(t) = ie^{-it}(\pi - t), 0 \leq t < 2\pi$ and P_+ is the Szegö projection given by

$$P_+ u(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{u(t)}{(1 - ze^{-it})} \, dt, \quad z \in \mathbf{D}.$$

Recall that the space BMOA consists of the functions $f \in H^1$ whose boundary values $f(e^{it})$ are of bounded mean oscillation on $\mathbf{T}$, that is

$$\sup_I \int_I |f(e^{it}) - I(f)| dt < \infty,$$

where supremum is taken over all intervals $I \subset \mathbf{T}$ and

$$I(f) = \frac{1}{|I|} \int_I f(e^{it}) \, dt.$$

If

$$\lim_{|I| \to 0} \int_I |f(e^{it}) - I(f)| \, dt = 0,$$

then we say that $f \in VMOA$.

Since the space BMOA is the Szegö projection of $L^\infty(\mathbf{T})$, we have also the following

Theorem 1.2. *The Hilbert matrix operator H acts as a bounded operator from H^∞ into* BMOA.

The next theorem describes the polynomials that are mapped by H into $VMOA$.

Theorem 1.3. *Let w be a polynomial of degree at least 1. Then $Hw \in VMOA$ if and only if $w(1) = 0$.*

Proof. We know that the operator $Hw = P_+(w(e^{-i\theta})b(\theta))$, where $b(\theta) = ie^{-i\theta}(\pi - \theta)$ for $0 \le \theta < 2\pi$. The function b is continuous on the unit circle $\mathbf{T}$ except for 1. If $w(1) = 0$, then the function $w(e^{-i\theta})b(\theta)$ can be continuously extended on the whole unit circle and Hw is the Szegö projection of this continuous function which means that $Hw \in VMOA$. It is also clear that if the function $w(e^{-i\theta})b(\theta)$ is continuous on $\mathbf{T}$ then $w(1) = 0$. □

In the next section we show that if $f \in H^1$, then Hf extends to a continuous function on $\overline{\mathbf{D}} \setminus \{1\}$ and give a sufficient condition for $Hf \in H^1$. In the case of positive Taylor coefficients we obtain a sufficient and necessary condition for $Hf \in H^1$. Section 3 is devoted to the weighted Hardy spaces H^p_α, $0 < p \le \infty$, $\alpha > 0$, consisting of those $f \in H(\mathbf{D})$ for which $M_p(r, f) = O((1-r)^{-\alpha})$. We prove that the Hilbert matrix operator is bounded on H^p_α if and only if $\alpha + 1/p < 1$. It is known that the operator H cannot be defined on the Bergman space A^2 of analytic functions that are square integrable over the unit disk with respect to the Lebesgue area measure. Here we find the subspace of A^2 which is mapped by H boundedly into A^2. Finally, we study the acting of the operator H on the Bloch space and Besov spaces.

Throughout the paper the notion $A \asymp B$ means that there exists a positive constant C such that $B/C \le A \le CB$.

2. Hilbert matrix operator acting on H^1

This section contains results on the Hilbert matrix operator acting on H^1 that are analogous to that obtained for the Libera operator in [12]. The proofs presented here are slightly different from the proofs given in [12].

We start with the following

Lemma 2.1. *If $f \in H^1$, then Hf extends to a continuous function on $\overline{\mathbf{D}} \setminus \{1\}$.*

Proof. By (1.2),

$$Hf(z) = \frac{1}{1-z} F_f(z),$$

where

$$F_f(z) = (1-z) \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^n.$$

We will show that the function F_f can be continuously extended to $\overline{\mathbf{D}}$. For $z \in \mathbf{D}$ we have

$$\begin{aligned} F_f(z) &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^n - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^{n+1} \\ &= \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{k+1} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} z^n - \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k} z^n \\ &= \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{k+1} - \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{(n+k)(n+k+1)} z^n. \end{aligned}$$

To see that the last double series converges absolutely and uniformly on $\overline{\mathbf{D}}$ it is enough to note that

$$\sum_{k=0}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{(n+k)(n+k+1)}\right)|\hat{f}(k)| = \sum_{k=0}^{\infty}\frac{|\hat{f}(k)|}{k+1}. \qquad \square$$

Consequently, we also get the following

Corollary 2.2. *The operator H acts as a bounded operator from H^1 into H^p, $0<p<1$.*

Theorem 2.3. *If $f\in H^1$ is such that*

$$\int_{-\pi}^{\pi}|f(e^{it})|\log\frac{\pi}{|t|}\,dt<\infty, \tag{2.1}$$

then $Hf\in H^1$.

Proof. We first show that if f satisfies the assumptions, then the function $g(z)=f(z)\log\frac{2}{1-z}$ is in H^1. To this end, we note that

$$\int_{-\pi}^{\pi}|f(e^{it})|\log\frac{2}{|1-e^{it}|}\,dt=\int_{-\pi}^{\pi}|f(e^{it})|\log\frac{1}{|\sin\frac{t}{2}|}\,dt\le\int_{-\pi}^{\pi}|f(e^{it})|\log\frac{\pi}{|t|}\,dt, \tag{2.2}$$

which implies that $g(e^{it})$ is in $L^1(\partial\mathbf{D})$. Since $g\in H^p$, $0<p<1$, the Smirnov theorem (see, e.g., [9] p. 74) implies that g is in H^1. Now using the formula (see [2]),

$$Hf(z)=\int_0^1\frac{f(r)}{1-rz}\,dr,\quad z\in\mathbf{D}, \tag{2.3}$$

and Fubini theorem we get

$$\frac{1}{2\pi}\int_0^{2\pi}|Hf(e^{it})|\,dt\le\frac{1}{2\pi}\int_0^{2\pi}\int_0^1\frac{|f(r)|\,dr}{|1-re^{it}|}\,dt=\int_0^1|f(r)|\frac{1}{2\pi}\int_0^{2\pi}\frac{dt}{|1-re^{it}|}\,dr$$
$$\le C\int_0^1|f(r)|\log\frac{2}{1-r}\,dr.$$

Applying the Fejér–Riesz inequality to g, we see that $Hf\in L^1(\partial\mathbf{D})$. Since Hf is in H^p for $0<p<1$, the Smirnov theorem implies that Hf is in H^1. $\square$

2.1. The case of positive coefficients. If $\hat{f}(k)\ge 0$ for all k, then Hf is well defined by (1.2) or by (2.3) if and only if

$$\sum_{k=0}^{\infty}\frac{\hat{f}(k)}{k+1}<\infty. \tag{2.4}$$

To see the "only if" part it is enough to take $z=0$. Furthermore, it is shown in [14] that if $\hat{f}(k)\downarrow 0$, then f is in H^1 if and only if (2.4) holds. We use this fact to prove:

Theorem 2.4. *If $\hat{f}(k)\ge 0$, then $Hf\in H^1$ if and only if*

$$\sum_{n=0}^{\infty}\frac{\hat{f}(n)\log(n+2)}{n+1}<\infty. \tag{2.5}$$

Proof. The coefficients of $h = Hf$ are given by

$$\hat{h}(n) = \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{n+k+1} \tag{2.6}$$

and obviously $\hat{h}(n) \downarrow 0$ as $n \to \infty$. Hence, by what we mentioned above, $h \in H^1$ if and only if

$$\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{\infty} \frac{\hat{f}(k)}{(n+k+1)} < \infty. \tag{2.7}$$

Now note that this double sum is equal to

$$\begin{aligned}
&\sum_{k=0}^{\infty} \hat{f}(k) \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+k+1)} \\
&= \hat{f}(0) \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} + \sum_{k=1}^{\infty} \frac{\hat{f}(k)}{k} \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+k+1} \right) \\
&= \hat{f}(0) \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} + \sum_{k=1}^{\infty} \frac{\hat{f}(k)}{k} \sum_{n=0}^{k-1} \frac{1}{n+1},
\end{aligned}$$

which implies the result. □

Now let us consider the space $\mathfrak{B}^1 \subsetneqq H^1$ defined by

$$\mathfrak{B}^1 = \left\{ f \in H(\mathbf{D})\colon \int_{\mathbf{D}} |f'(z)|\, dA(z) < \infty \right\}.$$

It was also shown in [14] that if $\hat{f}(k) \downarrow 0$ then f belongs to $\mathfrak{B}^1$ if and only if (2.4) holds. This can be used to strengthen the statement that H does not map H^1 into itself. More exactly we have

Proposition 2.5. *The operator H does not map $\mathfrak{B}^1$ into H^1.*

Proof. By the above , the function

$$f(z) = \sum_{n=2}^{\infty} \frac{z^n}{\log^{3/2} n}$$

belongs to $\mathfrak{B}^1$, while Hf, by Theorem 2.4, does not belong to H^1. □

3. Weighted Hardy spaces

For $\alpha > 0$ and $0 < p \le \infty$, we define the weighted Hardy spaces H^p_α as follows.

$$H^p_\alpha = \{ f \in H(\mathbf{D})\colon M_p(r,f) = O(1-r)^{-\alpha} \},$$

The norm in these spaces is defined by

$$\|f\|_{p,\alpha} = \sup_{0<r<1} (1-r)^\alpha M_p(r,f).$$

We start with the following

Theorem 3.1. *If $\alpha + 1/p < 1$, then the operator H maps H^p_α into H^p_α .*

Proof. Let $h = Hf$, $f \in H^p_\alpha$. Then we have

$$h'(z) = \int_0^1 \frac{rf(r)\,dr}{(1-rz)^2}.$$

Using Minkowski's inequality, the inequality

$$\int_0^{2\pi} |1-\rho e^{it}|^{-2p}\,dt \asymp (1-\rho)^{1-2p},$$

and the inequality

$$|f(r)| \le C(1-r)^{-\alpha-1/p} \quad \text{(implied by } f \in H^p_\alpha)$$

we get

$$M_p(\rho, h') \le C\int_0^1 |f(r)|(1-\rho r)^{1/p-2}\,dr \le C\int_0^1 (1-r)^{-\alpha-1/p}(1-\rho r)^{1/p-2}\,dr$$
$$\le C(1-\rho)^{-\alpha-1/p}\int_0^\rho (1-r)^{1/p-2}\,dr + C(1-\rho)^{1/p-2}\int_\rho^1 (1-r)^{-\alpha-1/p}\,dr.$$

Now the desired result is obtained by simple computation. It is enough to observe that $1/p-2<-1$ and that $-\alpha-1/p>-1$. □

3.1. The case of monotone coefficients. Now our aim is to prove the following

Theorem 3.2. *If $\{\widehat{f}(k)\}$ is a positive monotone sequence, then $f = \sum_{k=0}^\infty \widehat{f}(k)z^k \in H^p_\alpha$ $(1<p<\infty)$ if and only if*

$$\widehat{f}(k) \le C(k+1)^{\alpha+1/p-1}. \tag{3.1}$$

Let

$$\Delta_n(z) = \sum_{k\in I_n} z^k, \quad n \ge 0,$$

where

$$I_0 = \{0,1\}, \quad I_n = \{2^n \le k \le 2^{n+1}-1\}, \quad n \ge 1.$$

For $f \in H(\mathbf{D})$, let

$$\Delta_n f(z) = \sum_{k\in I_n} \widehat{f}(k)z^k.$$

The following fact was proved in [11].

Lemma 3.3. *Let $1<p<\infty$. A function $f \in H(\mathbf{D})$ is in H^p_α if and only if*

$$K(f) := \sup_n 2^{-n\alpha}\|\Delta_n f\|_p < \infty,$$

and we have $K(f) \asymp \|f\|_{p,\alpha}$.

Lemma 3.4. *If $1<p<\infty$ and $\{\lambda_n\}$ is a positive monotone sequence, then*

$$C^{-1}\lambda_{2^n}\|\Delta_n\|_p \le \Big\|\sum_{k\in I_n}\lambda_k z^k\Big\|_p \le C\lambda_{2^{n+1}}\|\Delta_n\|_p \quad \text{if } \{\lambda_n\} \text{ is increasing,}$$

$$C^{-1}\lambda_{2^{n+1}}\|\Delta_n\|_p \le \Big\|\sum_{k\in I_n}\lambda_k z^k\Big\|_p \le C\lambda_{2^n}\|\Delta_n\|_p \quad \text{if } \{\lambda_n\} \text{ is decreasing.}$$

Proof. Since $\{z^n\}$ is a Schauder basis in H^p, $1 < p < \infty$, by Proposition 1.a.3 in [10], for any sequence $\{a_k\}$ and $0 \le m \le j < n$,

$$\left\| \sum_{k=m}^{j} a_k z^k \right\|_p \le C \left\| \sum_{k=m}^{n} a_k z^k \right\|_p ,$$

where the constant C is independent of $\{a_k\}$, m, n and j.

By summation by parts,

$$\sum_{k=m}^{n} \lambda_k a_k z^k = \sum_{k=m}^{n-1} (\lambda_k - \lambda_{k+1}) s_k + \lambda_n s_n,$$

where

$$s_k = \sum_{j=m}^{k} a_j z^j .$$

Consequently,

$$\left\| \sum_{k=m}^{n} \lambda_k a_k z^k \right\|_p \le C \left(\sum_{k=m}^{n-1} |\lambda_k - \lambda_{k+1}| + \lambda_n \right) \left\| \sum_{k=m}^{n} a_k z^k \right\|_p .$$

If $\{\lambda_k\}$ is monotonically decreasing, then the sum in brackets is λ_m, if $\{\lambda_k\}$ is monotonically increasing, this sum is $(\lambda_n - \lambda_m) + \lambda_n \le 2\lambda_n$. This proves the right-hand side inequalities. To prove the left inequalities we observe that if, for example, $\{\lambda_k\}$ increases, then $1/\lambda_k$ decreases and by what we have already proved,

$$\left\| \sum_{k=m}^{n} a_k z^k \right\|_p = \left\| \sum_{k=m}^{n} \frac{1}{\lambda_k} (\lambda_k a_k z^k) \right\|_p \le C \frac{1}{\lambda_m} \left\| \sum_{k=m}^{n} \lambda_k a_k z^k \right\|_p . \qquad \square$$

Proof of Theorem 3.2. Assume first that $1 < p < \infty$. Since

$$\|\Delta_n\|_p = \|1 + z + \cdots + z^{2^n - 1}\|_p \asymp 2^{n(1-1/p)},$$

Lemmas 3.3 and 3.4 imply that $f \in H^p_\alpha$ if and only if

$$\widehat{f}(2^n) \le C 2^{n(\alpha + 1/p - 1)},$$

and our claim follows from the monotonicity of $\{\widehat{f}(k)\}$. $\square$

3.2. Necessity of the condition $\boldsymbol{\alpha + 1/p < 1}$. To include in our considerations the cases $p = 1, \infty$, we use polynomials W_n (instead of Δ_n) constructed in [7] (see also [13]). Let φ be a C^∞-function on $\mathbf{R}$ such that $\varphi(t) = 1$ for $t \le 1$, $\varphi(t) = 0$ for $t \ge 2$, and $\varphi(t)$ is positive and decreasing on $(1, 2)$. We set

$$W_0(z) = 1 + z, \quad W_n(z) = \sum_{k \in J_n} \omega(k/2^{n-1}) z^k, \quad n \ge 1,$$

where

$$J_n = \{k \in \mathbf{N} \colon 2^{n-1} \le k \le 2^{n+1}\}$$

and

$$\omega(t) = \varphi(t/2) - \varphi(t).$$

The convolution $f * g$ of two functions $f, g \in H(\mathbf{D})$ is defined by

$$f * g(z) = \sum_{n=0}^{\infty} \widehat{f}(n)\,\widehat{g}(n) z^n,$$

where $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ and $g(z) = \sum_{n=0}^{\infty} \widehat{g}(n) z^n$.

The following inequality was proved in [6] and [7].

$$\|W_n * f\|_p \le C\|f\|_p, \quad n = 0, 1, 2, \ldots, \quad 0 < p \le \infty. \tag{3.2}$$

We will also need the following lemmas.

Lemma 3.5. [8] *Let $0 < p \le \infty$. A function $f \in H(\mathbf{D})$ is in H^p_α if and only if $\|W_n * f\|_p = O(2^{n\alpha})$, and we have*

$$\|f\|_{p,\alpha} \asymp \sup_n 2^{-n\alpha} \|W_n * f\|_p.$$

Lemma 3.6. [13, Exercise 7.3.5] *Let $p \in (0, \infty]$, $P(z) = \sum_{k=m}^{4m} a_k z^k$, $Q(z) = \sum_{k=m}^{4m} (k+1)^\beta a_k z^k$, where m is a positive integer and $\beta \in \mathbf{R}$. Then there is a constant $C = C(p, \beta)$ such that*

$$C^{-1} m^\beta \|P\|_p \le \|Q\|_p \le C m^\beta \|P\|_p.$$

Moreover, for $|\beta| < \frac{1}{2}$ the constants $C(p, \beta)$ are uniformly bounded with respect to β.

Proof. Let W_m be a trigonometric polynomial such as in Lemma 7.3.2 in [13] with $\psi(x) = (x + 1/m)^\beta \varphi(x)$, where φ is a C^∞-function such that $\operatorname{supp}(\varphi) \subset (\frac{1}{2}, 5)$ and $\varphi(x) = 1$ for $x \in [1, 4]$. Then

$$W_m * P(z) = m^{-\beta} Q(z).$$

Our claim will follow from Theorem 7.3.4 in [13] if we can find the constant C_N in Lemma 7.3.2 [13] independent of β and m. But, since $\operatorname{supp}(\varphi) \subset (\frac{1}{2}, 5)$, the Leibniz formula

$$\psi^{(N)}(x) = \sum_{j=0}^{N} \binom{N}{j} \beta(\beta-1)\ldots(\beta-j+1) \left(x + \frac{1}{m}\right)^{\beta-j} \varphi^{(N-j)}(x)$$

implies that $|\psi^{(N)}(x)|$ is bounded uniformly with respect to β and m and the claim follows. □

To prove the last statement it is enough to show that

Lemma 3.7. *For $p \in [1, \infty]$ we have*

$$\|W_n\|_p \asymp 2^{n(1-1/p)}.$$

Proof. The case $p = \infty$ is easy. Assume that $1 \le p < \infty$. Since

$$M_\infty(r, W_n) \le C(1-r)^{-1/p} \|W_n\|_p$$

and $M_\infty(r, W_n) \ge r^{2^{n+1}} \|W_n\|_\infty$, taking $r = 1 - 2^{-n-1}$, we obtain

$$C\|W_n\|_p \ge 2^{n(1-1/p)}.$$

To prove the reverse inequality, we take $f(z) = (1-z)^{-2}$ and use Lemma 3.1 in [11] and (3.2) to obtain

$$r^{2^{n+1}} \left\| \sum_{k\in J_n} (k+1)\widehat{W_n}(k) z^k \right\|_p \le M_p(r, W_n * f) = \|W_n * f_r\|_p \le C(1-r)^{-2+1/p}.$$

Taking $r = 1 - 2^{-n-1}$ we get

$$\left\| \sum_{k\in J_n} (k+1)\widehat{W_n}(k) z^k \right\|_p \le C2^{(2-1/p)n},$$

and the result follows from Lemma 3.6 □

We will show that if the condition $\alpha + 1/p < 1$ is not satisfied, then the operator H cannot be extended as a continuous operator even into the space $H(\mathbf{D})$. More exactly, we have

Theorem 3.8. *If $\alpha + 1/p \ge 1$, $\alpha > 0$ and $p \ge 1$, then the operator H cannot be extended to a continuous operator from H^p_α into $H(\mathbf{D})$.*

Proof. For $\beta \in (0, \frac{1}{2})$ set

$$f_\beta(z) = \sum_{k=0}^{\infty} (k+1)^{-\beta} z^k.$$

Then Lemmas 3.6 and 3.7, and the assumption $\alpha \ge 1 - 1/p$ imply that

$$\|W_n * f_\beta\|_p \le C2^{-n\beta} \|W_n\|_p \le C2^{n(-\beta+1-1/p)} \le C2^{n(-\beta+\alpha)} \le C2^{n\alpha}.$$

By Lemma 3.5 this means that

$$\sup_{0<\beta<\frac{1}{2}} \|f_\beta\|_{p,\alpha} < \infty.$$

If H could be extended to a bounded operator from H^p_α into $H(\mathbf{D})$, then we would have $\sup_{0<\beta<\frac{1}{2}} |Hf_\beta(0)| < \infty$, because $f \mapsto f(0)$ is a continuous linear operator on $H(\mathbf{D})$. However,

$$Hf_\beta(0) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^{\beta+1}}$$

and $\lim_{\beta\to 0} Hf_\beta(0) = \infty$. This contradiction proves the result. □

4. Logarithmically weighted Bergman spaces

It is known that the Hilbert matrix does not act on A^2 (see [3]). For $\alpha > 0$ we define the logarithmically weighted Bergman space $A^2_{\log_\alpha} \subset A^2$ as follows.

$$A^2_{\log^\alpha} = \{f \in H(\mathbf{D})\colon \|f\|^2_{\log^\alpha} = \int_{\mathbf{D}} |f(z)|^2 \left(\log \frac{2}{1-|z|^2}\right)^\alpha dA(z) < \infty\},$$

where $dA(z)$ is the area measure on $\mathbf{D}$ normalized so that $\int_{\mathbf{D}} dA(z) = 1$. The following lemma can be proved in a standard way.

Lemma 4.1. *If $f \in A^2_{\log^\alpha}$, $\alpha > 0$, then there exists a constant $C > 0$ such that*

$$|f(z)| \leq \frac{C\|f\|_{\log^\alpha}}{(1-|z|^2)\left(\log \frac{2}{1-|z|^2}\right)^{\frac{\alpha}{2}}}$$

for every $z \in \mathbf{D}$.

We claim that H is well defined on $A^2_{\log^\alpha}$ for $\alpha > 2$. This follows from the following

Lemma 4.2. *Let $\alpha > 2$. If $f \in A^2_{\log^\alpha}$, then*

$$\sum_{k=0}^{\infty} \frac{|\hat{f}(k)|}{k+1} \leq C\|f\|_{\log^\alpha}.$$

Proof. Since the function $s \mapsto M_2(s, f)$ is increasing on $[0, 1)$, the Chebyshev inequality implies

$$\begin{aligned}\|f\|^2_{\log^\alpha} &= \int_0^1 M_2^2(s,f)\left(\log \frac{2}{1-s^2}\right)^\alpha s\,ds \geq \frac{1}{2}\int_r^1 M_2^2(s,f)\left(\log \frac{2}{1-s^2}\right)^\alpha ds \\ &\geq \frac{1}{2}(1-r)\left(\log \frac{2}{1-r^2}\right)^\alpha M_2^2(r,f).\end{aligned}$$

This means that for $r \in [0, 1)$,

$$\sum_{k=0}^{\infty} |\hat{f}(k)|^2 r^{2k} \leq C\|f\|^2_{\log^\alpha}(1-r)^{-1}\left(\log \frac{2}{1-r}\right)^{-\alpha}.$$

Taking $r = 1 - 1/m$, we get

$$\sum_{k=m}^{2m} |\hat{f}(k)|^2 \leq C\|f\|^2_{\log^\alpha} m \left(\log 2m\right)^{-\alpha}.$$

Consequently, for $\alpha > 2$, we have

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{|\hat{f}(k)|}{k+1} &= \sum_{k=1}^{\infty} \sum_{j=2^{k-1}}^{2^k-1} \frac{|\hat{f}(j)|}{j+1} \leq \sum_{k=1}^{\infty} 2^{1-k} \sum_{j=2^{k-1}}^{2^k-1} |\hat{f}(j)| \\ &\leq \sum_{k=1}^{\infty} 2^{1-k} 2^{\frac{k-1}{2}} \left(C\|f\|^2_{\log^\alpha} 2^{k-1}\left(\log 2^k\right)^{-\alpha}\right)^{\frac{1}{2}} \\ &= C\|f\|_{\log^\alpha} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{\alpha}{2}}} \leq C\|f\|_{\log^\alpha}.\end{aligned}$$

□

Moreover, for $\alpha > 2$ the Hilbert matrix operator acting on $A^2_{\log^\alpha}$ can also be expressed in the integral form (2.3). Furthermore, we have

Theorem 4.3. *If $\alpha > 3$, then H acts as a bounded operator from $A^2_{\log^\alpha}$ to A^2.*

Proof. From (2.3) and the integral form of Minkowski's inequality we obtain

$$\|Hf\|_{A^2} = \left(\int_{\mathbf{D}} |Hf(z)|^2\, dA(z)\right)^{\frac{1}{2}} \le \left(\int_{\mathbf{D}} \left(\int_0^1 \frac{|f(r)|}{|1-rz|}\, dr\right)^2 dA(z)\right)^{\frac{1}{2}}$$

$$\le \int_0^1 |f(r)| \left(\int_{\mathbf{D}} \frac{dA(z)}{|1-rz|^2}\right)^{\frac{1}{2}} dr \le C \int_0^1 |f(r)| \left(\log \frac{2}{1-r^2}\right)^{\frac{1}{2}} dr.$$

By Lemma 4.1,

$$\|Hf\|_{A^2} \le C \int_0^1 \frac{dr}{(1-r^2)\left(\log \frac{2}{1-r^2}\right)^{\frac{\alpha-1}{2}}} \|f\|_{\log^\alpha},$$

and the last integral converges for $\alpha > 3$. □

5. The Bloch and Besov spaces

For $1 < p \le \infty$, let $\mathcal{B}_p$ denote the analytic Besov space consisting of functions $f \in H(\mathbf{D})$ for which

$$\|f\|_{\mathcal{B}_p} := |f(0)| + \left(\int_{\mathbf{D}} |f'(z)|^p (1-|z|^2)^{p-2}\, dA(z)\right)^{1/p} < \infty.$$

In the case $p = \infty$ this is understood as

$$\|f\|_{\mathcal{B}_\infty} = |f(0)| + \sup_{z\in\mathbf{D}} (1-|z|^2)|f'(z)| < \infty,$$

and hence $\mathcal{B}_\infty = \mathcal{B}$ is the Bloch space. The reader is referred to, e.g., [15] for results on these spaces.

It is easy to check that if $f(z) = \log \frac{1}{1-z}$, then

$$Hf(z) = \sum_{n=0}^\infty \frac{1}{n+1}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right) z^n.$$

This shows that Bloch space $\mathcal{B}$ is not mapped into itself.

The following lemma describes a space of analytic functions in $\mathbf{D}$ that are mapped by H into the Bloch space.

Proposition 5.1. *If $f \in H(\mathbf{D})$ satisfies the condition*

$$\sup_{z\in\mathbf{D}} |f'(z)|(1-|z|)\left(\log \frac{2}{1-|z|}\right)^{1+\varepsilon} < \infty \tag{5.1}$$

for an $\varepsilon > 0$, then $Hf \in \mathcal{B}$.

Proof. Assume that $f \in H(\mathbf{D})$ satisfies (5.1) and set

$$F(z) = f(z) - f(0).$$

It is enough to show that $HF \in \mathcal{B}$. Clearly, F also satisfies (5.1). Then by Lemma 4.2.8 in [15] we can write

$$F(z) = \int_{\mathbf{D}} \frac{F'(w)(1-|w|^2)}{\bar{w}(1-\bar{w}z)^2}\, dA(w).$$

Consequently,

$$\begin{aligned}|(HF)'(z)| &\le C\int_0^1\int_{\mathbf{D}}\log^{-1-\varepsilon}\left(\frac{2}{1-|w|}\right)\frac{1}{|1-\bar{w}r|^2|1-rz|^2}\,dA(w)\,dr\\ &\le C\int_0^1\int_0^1\log^{-1-\varepsilon}\left(\frac{2}{1-s}\right)\frac{ds\,dr}{(1-sr)(1-r|z|)^2}\\ &\le \frac{C}{1-|z|}\int_0^1\log^{-1-\varepsilon}\left(\frac{2}{1-s}\right)\frac{1}{1-s}\,ds.\end{aligned}$$

Since the last integral is finite, our claim is proved. □

On the other hand, we have

Proposition 5.2. *If $f\in\mathcal{B}$, then*

$$|(Hf)'(z)|\le C\frac{1}{1-|z|}\log\frac{2}{1-|z|}.$$

Proof. For $f\in\mathcal{B}$ set

$$A_n(f)=\sum_{k=0}^{\infty}\frac{\widehat{f}(k)}{n+k+1}=\int_{\mathbf{D}}\frac{f(z)}{1-\bar{z}}|z|^{2n}\,dA(z).$$

Assuming additionally that $f(0)=0$ and using the Fubini theorem, we obtain

$$\begin{aligned}|A_n(f)| &\le \left|\int_{\mathbf{D}}\int_{\mathbf{D}}\frac{f'(w)(1-|w|^2)}{\bar{w}(1-\bar{w}z)^2}\,dA(w)\frac{|z|^{2n}}{1-\bar{z}}\,dA(z)\right|\\ &= \left|\int_{\mathbf{D}}\frac{f'(w)(1-|w|^2)}{\bar{w}}\int_0^1\left(\frac{1}{\pi}\int_0^{2\pi}\frac{d\theta}{(1-\bar{w}re^{i\theta})^2(1-re^{-i\theta})}\right)r^{2n+1}\,dr\,dA(w)\right|\\ &\le C\int_0^1 r^{2n+1}\int_{\mathbf{D}}\frac{dA(w)}{|1-r^2\bar{w}|^2}\,dr\le C\int_0^1\log\frac{2}{(1-r^2)}r^{2n+1}\,dr\\ &\le C\int_0^1 r^n\left(\log 2+\sum_{k=1}^{\infty}\frac{r^k}{k}\right)dr\le C\frac{1}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n+1}\right)\end{aligned}$$

Hence

$$\begin{aligned}|(Hf)'(z)| &= \left|\sum_{n=1}^{\infty}nA_n(f)z^{n-1}\right|\le\sum_{n=1}^{\infty}n|A_n(f)||z|^{n-1}\\ &\le C\sum_{n=0}^{\infty}\left(1+\frac{1}{2}+\ldots\frac{1}{n+1}\right)|z|^n\le C\frac{1}{1-|z|}\log\frac{2}{1-|z|}.\end{aligned}$$ □

The example of $f(z)=\log\frac{1}{1-z}$ shows that the inequality in the last lemma cannot be improved.

A little bit more complicated calculations give the following

Proposition 5.3. *If $f\in\mathcal{B}_p$, $1<p<\infty$, then*

$$|(Hf)'(z)|\le C\frac{1}{1-|z|}\left(\log\frac{2}{1-|z|}\right)^{\frac{1}{p'}},$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. Using the notation from the proof of Proposition 5.2 under the assumption that $f(0) = 0$ we get, in much the same way as above,

$$|A_n(f)| \le C \int_0^1 r^{2n+1} \int_{\mathbf{D}} \frac{|f'(w)|(1-|w|^2)}{|1-r^2\bar{w}|^2}\, dA(w)\, dr \le C\|f\|_{\mathcal{B}_p} \int_0^1 r^{2n+1} \left(\log \frac{2}{1-r^2}\right)^{\frac{1}{p'}} dr.$$

Hence

$$|A_n(f)|^{p'} \le C\|f\|_{\mathcal{B}_p}^{p'} \left(\int_0^1 r^{2n+1}\, dr\right)^{p'-1} \int_0^1 r^{2n+1} \log \frac{2}{1-r^2}\, dr \le C\|f\|_{\mathcal{B}_p}^{p'} \left(\frac{1}{n+1}\right)^{p'} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right).$$

Consequently,

$$\begin{aligned} |(Hf)'(z)|^{p'} &\le \left(\sum_{n=0}^{\infty} (n+1)|A_{n+1}(f)||z|^n\right)^{p'} \\ &\le \left(\sum_{n=0}^{\infty} (n+1)^{p'}|A_{n+1}(f)|^{p'}|z|^n\right) \left(\sum_{n=0}^{\infty} |z|^n\right)^{p'-1} \\ &\le C\|f\|_{\mathcal{B}_p}^{p'} \left(\frac{1}{1-|z|}\right)^{p'-1} \left(\sum_{n=0}^{\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+1}\right)|z|^n\right) \\ &\le C\|f\|_{\mathcal{B}_p}^{p'} \left(\frac{1}{1-|z|}\right)^{p'} \log \frac{2}{1-|z|}, \end{aligned}$$

which proves our claim. □

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