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POSITIVE HARMONIC FUNCTIONS ON COMB-LIKE DOMAINS

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Abstract. This paper investigates positive harmonic functions on a domain which contains an infinite cylinder, and whose boundary is contained in the union of parallel hyperplanes. (In the plane its boundary consists of two sets of vertical semi-infinite lines.) It characterizes, in terms of the spacing between the hyperplanes, those domains for which there exist minimal harmonic functions with a certain exponential growth.

1. Introduction

The subtle relationship between the structure of positive harmonic functions on a domain Ω in $\mathbf{R}^N$ $(N \geq 2)$ and boundary geometry has been much studied. One avenue of investigation has been to examine the effect of modifying the boundary of a familiar domain such as a half-space, cone or cylinder. Thus many authors have been led to investigate the case of Denjoy domains Ω, where the complement $\mathbf{R}^N \setminus \Omega$ is contained in a hyperplane, say $\mathbf{R}^{N-1} \times \{0\}$ (see [6, 11, 14, 1, 24, 25, 8, 10, 2, 21]). For example, Benedicks [6] has established a harmonic measure criterion that describes when the cone of positive harmonic functions on Ω that vanish on the boundary $\partial\Omega$ is generated by two linearly independent minimal harmonic functions. (We recall that a positive harmonic function h on a domain Ω is called *minimal* if any non-negative harmonic minorant of h on Ω is proportional to h.) Benedicks' criterion is also equivalent to the existence of a harmonic function u on Ω vanishing on $\partial\Omega$ and satisfying $u(x) \geq |x_N|$ on Ω, and thus describes when a Denjoy domain behaves like the union of two half-spaces from the point of view of potential theory. Related results, based on sectors, cones or cylinders, may be found in [12, 21, 18]. The purpose of this paper is to describe what happens in the case of another relative of the infinite cylinder. More precisely, let (a_n) be a strictly increasing sequence of non-negative numbers such that $a_n \to +\infty$ and $a_{n+1} - a_n \to 0$ as $n \to \infty$, and let B' be the unit ball in $\mathbf{R}^{N-1}$. We define

$$E = \bigcup_{n \in \mathbf{N}} (\mathbf{R}^{N-1} \setminus B') \times \{a_n\}$$

and investigate when the domain $\Omega = \mathbf{R}^N \setminus E$ inherits the potential theoretic character of the cylinder $U = B' \times \mathbf{R}$; that is, when the set E imitates ∂U in terms of its effect on the asymptotic behaviour of positive harmonic functions on Ω. We call

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such domains Ω *comb-like* because they are a generalization of comb domains in the plane.

Let $x = (x', x_N)$ denote a typical point of Euclidean space $\mathbf{R}^N = \mathbf{R}^{N-1} \times \mathbf{R}$. It is known (see [15], for example) that the cone of positive harmonic functions on U that vanish on ∂U is generated by two minimal harmonic functions $h_\pm(x', x_N) = e^{\pm\alpha x_N}\phi(x')$, where α is the square root of the first eigenvalue of the operator $-\Delta = -\sum_{j=1}^{N-1} \partial^2/\partial x_j^2$ on B' and ϕ is the corresponding eigenfunction, normalized by $\phi(0) = 1$. We describe below when a comb-like domain admits a minimal harmonic function u that vanishes on $\partial\Omega$ and satisfies $u \geq h_+$ on U.

Theorem 1.1. *Let $\nu > 1$. Assume that (a_n) satisfies the following condition*

$$\frac{1}{\nu} \leq \frac{a_{k+1} - a_k}{a_{j+1} - a_j} \leq \nu \tag{1.1}$$

whenever $|a_k - a_j| < 4$. The following statements are equivalent:

(a) *there exists a positive harmonic function u on Ω that satisfies $u \geq h_+$ on U and u vanishes continuously on E;*
(b) $\sum_{n=1}^{\infty}(a_{n+1} - a_n)^2 < +\infty$.

Moreover, if (b) holds, then u can be chosen to be minimal in part (a).

We will prove Theorem 1.1 by combining methods from [14], [12] and [18] with some new ideas. It is known (see [7, 9, 16]) that the behaviour of minimal harmonic functions on simply connected domains is intimately related to the classical angular derivative problem. We note that when $N = 2$, condition (b) of Theorem 1.1 is necessary and sufficient for the comb domain Ω to have an angular derivative at $+\infty$ (see [22, 23, 20]).

2. Notation and preliminary results

We use $\partial^\infty D$ to denote the boundary of a domain D in compactified space $\mathbf{R}^N \cup \{\infty\}$. Let $B_\rho(x)$ denote the open ball in $\mathbf{R}^N$ of centre x and radius $\rho > 0$. We write B'_ρ (resp. B_ρ) for the open ball in $\mathbf{R}^{N-1}$ (resp. $\mathbf{R}^N$) of centre 0 and radius ρ, and $V(\rho) = \partial B'_\rho \times \mathbf{R}$. If $\rho = 1$, we write B' instead of B'_1. For $0 < \rho_1 < \rho_2$ let $A(\rho_1, \rho_2) = \left(B'_{\rho_2} \setminus \overline{B'_{\rho_1}}\right) \times \mathbf{R}$. We denote by μ_x^D the harmonic measure for an open set $D \subset \mathbf{R}^N$ evaluated at $x \in D$. If f is a function defined on $\partial^\infty D$, we use $\overline{H}_f^D$ to denote the upper Perron–Wiener–Brelot solution to the Dirichlet problem on D and H_f^D for the PWB solution of the Dirichlet problem on D when it exists. We denote by $P_D(\cdot, y)$ the Poisson kernel for D with pole $y \in \partial D$, where ∂D is smooth enough for it to be defined. If $W \subseteq D$ and u is a superharmonic function on D, we denote by R_u^W (resp. $\widehat{R}_u^W$) the reduced function (resp. the regularized reduced function) of u relative to W in D. We denote surface area measure on a given surface by σ. We use $C(a, b, \ldots)$ to denote a constant depending at most on $a, b, \ldots$, the value of which may change from line to line.

For the remainder of the paper, we fix $0 < r < 1 < R$ and for $x \in \partial U$ we define $F_x = \partial B'_r \times [x_N - 1, x_N + 1]$ and

$$T_x = (B'_R \setminus \overline{B'_r}) \times (x_N - 1, x_N + 1).$$

We note that the first eigenfunction ϕ of $-\Delta$ in B' is comparable with the distance to $\partial B'$, that is

$$C_1(N)(1-|x'|) \le \phi(x') \le C_2(N)(1-|x'|) \quad (x' \in B'). \tag{2.1}$$

A simple proof of (2.1) can be found in [17, pp. 419–420]. The following estimate for the Poisson kernel (see [18, Section 2.1], for example) will prove useful. For $|x'| = s < 1,\ x_N \in \mathbf{R},\ y \in \partial U$

$$C_1(N,s)e^{-\alpha|x_N-y_N|} \le P_U(x,y) \le C_2(N,s)e^{-\alpha|x_N-y_N|}. \tag{2.2}$$

If $0 < r_1 < s < r_2$, similar estimates hold for $P_{A(r_1,r_2)}$ with α replaced by the square root of the first eigenvalue of $-\Delta$ in $B'_{r_2}\backslash\overline{B'_{r_1}}$ and constants C_1, C_2 depending on N, r_1, r_2 and s.

Proposition 2.1. *Assume there exists a positive harmonic function u on Ω such that $u \ge h_+$ on U and u vanishes on E. Then*

$$\sum_{n=1}^{\infty}(a_{n+1}-a_n)^2 < +\infty. \tag{2.3}$$

Proof. By (2.2) we have

$$\begin{aligned} +\infty > u(0) &\ge \int_{\partial U} u(y)P_U(0,y)\,d\sigma(y) \\ &\ge C(N)\sum_{n=1}^{\infty}\int_{\partial B'\times(a_n,a_{n+1})} u(y)e^{-\alpha y_N}d\sigma(y). \end{aligned} \tag{2.4}$$

We use Harnack's inequalities and (2.1) to see that for $y \in \partial U$ with

$$y_N \in \big(a_n + (a_{n+1}-a_n)/4, a_{n+1} - (a_{n+1}-a_n)/4\big)$$

the following holds

$$\begin{aligned} u(y) &\ge C(N)u\left((1-(a_{n+1}-a_n)/8)y', y_N\right) \\ &\ge C(N)e^{\alpha y_N}\phi\left((1-(a_{n+1}-a_n)/8)y'\right) \\ &\ge C(N)e^{\alpha y_N}(a_{n+1}-a_n). \end{aligned} \tag{2.5}$$

We deduce from (2.4) and (2.5) that (2.3) holds. □

Assume now that $\sum_{n=1}^{\infty}(a_{n+1}-a_n)^2 < +\infty$. Let $J \in \mathbf{N}$ be large enough so that $a_{n+1} - a_n \le 1/2$ for $n \ge J$. For ease of exposition we rename the sequence $(a_n)_{n=J}^{\infty}$ as $(b_n)_{n=1}^{\infty}$. We also define $\rho_n = (b_{n+1}-b_n)/2$ for $n \in \mathbf{N}$. We introduce $b_0 = b_1 - 1$ and $\rho_0 = 1/2$. For technical reasons, we will work with

$$E'' = \bigcup_{n=1}^{\infty}\left(\mathbf{R}^{N-1}\setminus B'\right)\times\{b_n\} \quad \text{and} \quad E' = (\partial B' \times (-\infty, b_1]) \cup E'',$$

and at the end we will dispense with these additional requirements.

Lemma 2.1. *There exists a positive constant c_1, depending on N, R and r, such that for any $x \in \partial U$ we have*

$$\mu_x^{T_x\backslash E''}(F_x) \le \mu_x^{T_x\backslash E''}(\partial T_x) \le c_1\, \mu_x^{T_x\backslash E''}(F_x). \tag{2.6}$$

Proof. Let $x \in \partial U$. The left hand inequality in (2.6) is obvious since $F_x \subset \partial T_x$. Let $h = H^{T_x}_{\chi_{F_x}}$ on T_x and $h = \chi_{F_x}$ on ∂T_x. In order to establish the right hand inequality, it is enough to prove that

$$h \le h(x) \quad \text{on } E'' \cap T_x. \tag{2.7}$$

We will borrow an argument from [18, Lemma 2.1]. Using reflection in $\mathbf{R}^{N-1} \times \{x_N + 1\}$ to extend h to $(\overline{B'_R} \setminus B'_r) \times [x_N - 1, x_N + 3]$, and translation, for $y \in \partial B' \times (x_N, x_N + 1)$ we obtain

$$\begin{aligned} h(y) &= H_h^{T_x + (0', y_N - x_N)}(y) \\ &= \mu_x^{T_x}(\partial B'_r \times [x_N - 1, 2x_N + 1 - y_N]) - \mu_x^{T_x}(\partial B'_r \times [2x_N + 1 - y_N, x_N + 1]) \\ &\le \mu_x^{T_x}(F_x) = h(x). \end{aligned}$$

By symmetry, $h(y) \le h(x)$ for $y \in \partial B' \times (x_N - 1, x_N + 1)$. Since

$$h(y) = 0 \le h(x) \ \text{ for } \ y \in [\partial B'_R \times (x_N - 1, x_N + 1)] \cup [(B'_R \setminus \overline{B'}) \times \{x_N - 1, x_N + 1\}],$$

using the maximum principle, we see that $h \le h(x)$ on $(B'_R \setminus \overline{B'}) \times (x_N - 1, x_N + 1)$, which proves (2.7). □

We note that Lemma 2.1 holds in a more general context when E'' is a closed subset of $\mathbf{R}^N \setminus U$.

To prove Theorem 1.1 we shall need the following estimate.

Lemma 2.2. *Let $\nu > 1$. Assume that (b_n) satisfies*

$$\frac{1}{\nu} \le \frac{b_{k+1} - b_k}{b_{j+1} - b_j} \le \nu \tag{2.8}$$

whenever $|b_k - b_j| < 4$. Then there exists a constant c_2, depending only on N, r and ν, such that

$$\mu_x^{T_x \setminus E''}(F_x) \le c_2(b_{n+1} - b_n)$$

whenever $x \in \partial B' \times (b_n, b_{n+1})$ and $n \in \mathbf{N}$.

Proof. We suppose that $x \in \partial B' \times (b_n, b_{n+1})$ for some $n \in \mathbf{N}$. We define $\omega = (B'_R \setminus \overline{B'_r}) \times (b_{j_0}, b_{k_0})$, where $j_0 = \max\{j \colon b_j \le x_N - 1\}$ and $k_0 = \min\{j \colon b_j \ge x_N + 1\}$. Let $g = H^{\omega \setminus E''}_{\chi_{V(r)}}$ on $\omega \setminus E''$ and define $g = \chi_{V(r)}$ elsewhere. Let $m = \sup_{\partial U} g$. We note that

$$\mu_x^{T_x \setminus E''}(F_x) \le \mu_x^{\omega \setminus E''}(V(r)) \le m.$$

We will obtain an upper bound for m in terms of ρ_n. To do this, we define an open set Z as follows

$$Z = \omega \setminus \bigcup_{k=0}^{\infty} \bigcup_{p \in [b_k, b_{k+1}]} \{z \in \overline{B'_s} \times \{p\} \colon s = (1 - r)(|p - (b_k + \rho_k)| - \rho_k) + 1\}.$$

We estimate g on ∂Z in terms of m and ρ_n. Since $g = 0$ on $\partial Z \setminus U$, we estimate g on $\partial Z \cap U$, noting that, for $y \in \omega \cap U$, we have

$$g(y) = H_g^{\omega \cap U}(y) = H^{\omega \cap U}_{\chi_{V(r)}}(y) + \int_{\partial U \cap \omega} g \, d\mu_y^{\omega \cap U}. \tag{2.9}$$

Let $g_1(y) = H^{\omega\cap U}_{\chi_{V(r)}}(y)$ and $g_2(y) = \int_{\partial U\cap\omega} g\, d\mu^{\omega\cap U}_y$ for $y \in \omega\cap U$. Using the function

$$f_N(y) = \begin{cases} |y'|^{3-N} - 1 & (N \geq 4) \\ -\log|y'| & (N = 3) \\ 1 - |y'| & (N = 2) \end{cases}$$

and the maximum principle, we find that for $y \in \partial Z \cap U$

$$g_1(y) \leq f_N(y)/f_N(rx) \leq C_1(N,r)(1 - |y'|) \leq C_1(N,r,\nu)\rho_n. \tag{2.10}$$

We now wish to show that there exists a constant $C_2(N,\nu) \in (0,1)$ such that

$$g_2 \leq C_2(N,\nu)m \quad \text{on} \quad \partial Z \cap U. \tag{2.11}$$

Let $l = (1-r)\min_{j_0\leq k\leq k_0-1}\rho_k$ and let $t = (1,0,\ldots,0,t_N)$ with $t_N \in \{b_k : k = j_0+1,\ldots,k_0-1\}$. By [5, Lemma 8.5.1], for $x \in B_{l/2}(1+l,0,\ldots,0,t_N)$ we have $g(x) \leq C(N)(g(p_+)+g(p_-))$, where $p_\pm = (1+l,0,\ldots,0,t_N \pm l/2)$. Using a Harnack chain to cover the longer arc joining p_+ and p_- along the circle $\partial B_{\sqrt{5}l/2}(t)\cap(\mathbf{R}\times\{0\}^{N-2}\times\mathbf{R})$, we deduce that $g \leq C_3(N)m$ on that circle. By the invariance of g under rotations around the x_N-axis and the maximum principle, this inequality holds on a torus-shaped set enclosing the edge of $(\mathbf{R}^{N-1}\setminus B')\times\{t_N\}$; more precisely on every closed ball centred at a point of $\partial B'\times\{t_N\}$ and having radius $\sqrt{5}l/2$. When $t_N = b_{j_0}$ or $t_N = b_{k_0}$, this inequality follows directly from [5, Lemma 8.5.1], with a perhaps different constant, $C_4(N)$ say. In particular, for $y \in B^t\setminus E^t$, where $B^t = B_{\sqrt{5}l/2}(t)$, $E^t = [1,+\infty)\times\mathbf{R}^{N-2}\times\{t_N\}$ and $t_N \in \{b_k\colon k = j_0,\ldots,k_0\}$, we have

$$g(y) \leq H^{B^t\setminus E^t}_g(y) = \int_{\partial B^t} g\, d\mu^{B^t\setminus E^t}_y \leq \max\{C_3(N),C_4(N)\}mH^{B^t\setminus E^t}_{\chi_{\partial B^t}}(y). \tag{2.12}$$

Since t is a regular boundary point for $B^t\setminus E^t$, there exists $\delta = \delta(N) > 0$ such that

$$H^{B^t\setminus E^t}_{\chi_{\partial B^t}}(y) \leq \frac{1}{2\max\{C_3(N),C_4(N)\}} \quad (y \in B_{\delta l}(t)\setminus E^t). \tag{2.13}$$

Let $K_{\delta l} = \bigcup_{k=j_0}^{k_0}\{y \in \partial U\colon |y_N - b_k| < \delta l\}$. In view of (2.12) and (2.13), and the invariance of g under rotations around the x_N-axis, we conclude that $g \leq m/2$ on $K_{\delta l}$.

Hence, for $y \in \partial Z\cap U$, we have

$$g_2(y) \leq \int_{\partial U} g\, d\mu^U_y \leq \frac{m}{2}\mu^U_y(K_{\delta l}) + m\mu^U_y(\partial U\setminus K_{\delta l}) \leq m\left(1 - \frac{1}{2}\mu^U_y(K_{\delta l})\right). \tag{2.14}$$

We now show that there exists a constant $C_5(N,\nu) \in (0,1)$ such that $\mu^U_y(K_{\delta l}) \geq C_5(N,\nu)$ for $y \in \partial Z\cap U$. We first estimate $\mu^U_y(K_{\delta l})$ on some ball centred at t and then join other points of $\partial Z\cap U$ by a Harnack chain.

Let $W_{\delta l} = B'\times(t_N - \delta l, t_N + \delta l)$. We use a dilation $\psi(y) = t + (y-t)/(\delta l)$ and note that, by continuity, there exists an absolute positive constant γ such that for $y \in \psi(W_{\delta l})\cap B_\gamma(t)$ the following inequalities hold

$$H^{W_{\delta l}}_{\chi_{K_{\delta l}}}(\psi^{-1}(y)) = H^{\psi(W_{\delta l})}_{\chi_{\psi(K_{\delta l})}}(y) \geq H^{(-\infty,1)\times\mathbf{R}^{N-2}\times(t_N-1,t_N+1)}_{\chi_{\{1\}\times\mathbf{R}^{N-2}\times[t_N-1,t_N+1]}}(y) \geq 1/2.$$

Hence

$$\mu^U_y(K_{\delta l}) \geq \mu^{W_{\delta l}}_y(K_{\delta l}) \geq 1/2 \quad (y \in B_{\gamma\delta l}(t)\cap U),$$

and by Harnack's inequalities

$$\mu_y^U(K_{\delta l}) \geq C_5(N,\nu) \quad \text{for all} \quad y \in \partial Z \cap U.$$

Let $C_2(N,\nu) = 1 - C_5(N,\nu)/2$. Then (2.11) holds in view of (2.14), and by (2.9) and (2.10) we have

$$g \leq C_1(N,r,\nu)\rho_n + C_2(N,\nu)m \quad \text{on} \quad \partial Z.$$

By the maximum principle this inequality holds on Z and implies that

$$m \leq \frac{C_1(N,r,\nu)}{1 - C_2(N,\nu)}\rho_n.$$

This finishes the proof of lemma. □

We define $\beta_{E'}(x)$ to be the harmonic measure of ∂T_x in $T_x \backslash E'$ evaluated at x. If $x \in E'$, then $\beta_{E'}(x)$ is interpreted as 0. We observe that, if (b_n) satisfies the ratio condition (2.8), then, in view of Lemmas 2.1 and 2.2, we have

$$\tag{2.15} \begin{aligned} \int_{\partial B' \times (b_1,+\infty)} \beta_{E'}(y)\, d\sigma(y) &\leq c_1 \int_{\partial B' \times (b_1,+\infty)} \mu_y^{T_y \backslash E''}(F_y)\, d\sigma(y) \\ &\leq c_1 c_2 \sigma_{N-1} \sum_{n=1}^{\infty} (b_{n+1} - b_n)^2, \end{aligned}$$

where σ_{N-1} denotes the surface measure of $\partial B'$ in $\mathbf{R}^{N-1}$.

Henceforth let (b_n) satisfy (2.8) and let

$$\Lambda = \sum_{n=1}^{\infty} (b_{n+1} - b_n)^2 < +\infty.$$

Before we prove the next lemma, we collect together some facts about certain Bessel functions (see [4, Section 4]). Let $K = K_{(N-3)/2} \colon (0,\infty) \to (0,\infty)$ denote the Bessel function of the third kind, of order $(N-3)/2$. Then the function

$$\tag{2.16} h_0(x', x_N) = |x'|^{(3-N)/2} K(\pi |x'|) \sin(\pi x_N)$$

is positive and superharmonic on the strip $\mathbf{R}^{N-1} \times (0,1)$, harmonic on $(\mathbf{R}^{N-1} \backslash \{0'\}) \times (0,1)$ and vanishes on $\mathbf{R}^{N-1} \times \{0,1\} \backslash \{(0',0),(0',1)\}$. Moreover, there exists $c(N) \geq 1$ such that

$$\tag{2.17} c(N)^{-1} \leq (2t/\pi)^{1/2} e^t K(t) \leq c(N) \quad \text{for } t \in [1,+\infty).$$

We also recall a result of Domar ([13, Theorem 2]). Suppose that D is a domain in $\mathbf{R}^N$ and $F \colon D \to [0,+\infty]$ is a given upper semicontinuous function on D. Let $\mathcal{F}$ be the collection of all subharmonic functions u, such that $u \leq F$ on D. Domar's result says that if

$$\tag{2.18} \int_D [\log^+ F(x)]^{N-1+\varepsilon} dx < \infty,$$

for some $\varepsilon > 0$, then the function $M(x) = \sup_{u \in \mathcal{F}} u(x)$ is bounded on every compact subset of D.

Let $0 < r' < \min\{r, 1/2\}$. Define $V = A(r',\infty) \setminus E'$ and $U_n = (\mathbf{R}^{N-1} \setminus \overline{B'}) \times (b_n, b_{n+1})$ for $n \in \mathbf{N}$.

Lemma 2.3. *There exists a positive constant c_3, depending on N, R, r and r', such that, for any positive harmonic function u on V that is bounded on each U_n and vanishes on E',*

$$u(y) \le c_3 u(rx', x_N) H^{T_x \setminus E'}_{\chi \partial T_x}(y) \quad (x \in \partial U, y \in T_x \setminus E').$$

In particular,

$$u(x', x_N) \le c_3 \beta_{E'}(x) u(rx', x_N) \quad (x \in \partial U).$$

Proof. Let $x \in \partial U$, $l = (1 + r')/3$ and $L = 2R$. Define $A_x = \{y\colon l < |y'| < L, |x_N - y_N| < 2\}$. We will show that

$$\frac{u(y)}{C(N, r, r')u(rx', x_N)} \le F(y) \quad (y \in A_x), \tag{2.19}$$

where

$$F(y) = \begin{cases} |1 - |y'||^{1-N}, & |y'| \ne 1, \\ +\infty, & |y'| = 1. \end{cases}$$

Step 1. Let $(y', y_N) \in A_x \cap U$. Harnack's inequalities yield that

$$u(y) \le C(N, r, r')u(rx', x_N)(1 - |y'|)^{1-N}.$$

Step 2. If $y \in A_x \cap U_n$ and $|y'| - 1 \le \min\{y_N - b_n, b_{n+1} - y_N\}$, then there is a Harnack chain of fixed length joining (y', y_N) with $((2 - |y'|)y'/|y'|, y_N) \in A_x \cap U$. By Step 1, we have

$$u(y) \le C(N)u((2 - |y'|)y'/|y'|, y_N) \le C(N, r, r')u(rx', x_N)(|y'| - 1)^{1-N}.$$

Step 3. If $y \in A_x \cap U_n$ and $\rho_n \ge |y'| - 1 > \min\{y_N - b_n, b_{n+1} - y_N\}$, we apply [5, Lemma 8.5.1] and Harnack's inequalities to see that

$$u(y) \le C(N)u(y', \widetilde{y}_N),$$

where $\widetilde{y}_N$ is such that $|\widetilde{y}_N - y_N| < |b_n + \rho_n - y_N|$ and $|y'| - 1 = \min\{\widetilde{y}_N - b_n, b_{n+1} - \widetilde{y}_N\}$. By Step 2,

$$u(y) \le C(N, r, r')u(rx', x_N)(|y'| - 1)^{1-N}.$$

Step 4. If $y \in A_x \cap U_n$ and $|y'| \ge 1 + \rho_n$, let $V_n = \{(z', z_N)\colon 1 + \rho_n < |z'|, z_N \in (b_n, b_{n+1})\}$. For $z \in U_n$ we define a function

$$h_n(z) = \frac{h_0((z', z_N - b_n)/(2\rho_n))}{K(\pi(1 + \rho_n)/(2\rho_n))} \left(\frac{1 + \rho_n}{2\rho_n}\right)^{(N-3)/2}$$

which is harmonic on U_n and vanishes on $\partial U_n \setminus \partial U$. Applying [5, Lemma 8.5.1] and Harnack's inequalities to u and h_n, by Step 3, we get

$$u(z) \le C(N, r, r')u(rx', x_N)\rho_n^{1-N} h_n(z) \quad \text{for} \quad z \in \partial V_n.$$

Since u is bounded on V_n and ∞ has zero harmonic measure for V_n,

$$u(y) \le C(N, r, r')u(rx', x_N)\rho_n^{1-N} h_n(y). \tag{2.20}$$

Furthermore, by (2.16) and (2.17)

$$\begin{aligned} h_n(y) &\le \left(\frac{1 + \rho_n}{|y'|}\right)^{\frac{N-3}{2}} K\left(\frac{\pi|y'|}{2\rho_n}\right) \left(K\left(\frac{\pi(1 + \rho_n)}{2\rho_n}\right)\right)^{-1} \\ &\le C(N)e^{-\frac{\pi}{2\rho_n}(|y'|-1)} \left(\frac{1 + \rho_n}{|y'|}\right)^{\frac{N-2}{2}} \le C(N)e^{-\frac{\pi}{2\rho_n}(|y'|-1)} \le C(N)\left(\frac{|y'| - 1}{\rho_n}\right)^{1-N}. \end{aligned}$$

Hence we see from (2.20) that

$$u(y) \le C(N,r,r')u(rx',x_N)(|y'|-1)^{1-N}.$$

We conclude that (2.19) follows from Steps 1–4. Since

$$\int_{A_x} (\log^+ F(y))^N dy \le C(N,R),$$

Domar's result and Harnack's inequalities (if $r<l$) yield

$$u(y) \le C(N,R,r,r')u(rx',x_N) \quad (y \in \overline{T_x}).$$

Therefore

$$u(y) = H_u^{T_x \setminus E'}(y) \le C(N,R,r,r')u(rx',x_N)H_{\chi_{\partial T_x}}^{T_x\setminus E'}(y) \quad (y \in T_x \setminus E').$$

In particular,

$$u(x) \le C(N,R,r,r')u(rx',x_N)\beta_{E'}(x). \qquad \square$$

Lemma 2.4. *Let $v\colon \mathbf{R}^N \cup \{\infty\} \to [0,+\infty]$ be a Borel measurable function such that $v(x) \le e^{\alpha x_N}\chi_{V(r')}(x)$ on $\mathbf{R}^N$. There exist positive constants c_4 and c_5, depending on N,R,r and r', such that, if $\Lambda \le c_4$, then H_v^V exists and*

$$H_v^V(x) \le H_v^{A(r',1)}(x) + c_5\Lambda e^{\alpha x_N} \quad (|x'|=r).$$

Proof. Let $h_n = H^V_{\min\{v,n\}}$ on V and $h_n = \min\{v,n\}$ on $\partial^\infty V$, and let

$$m_n = \sup\{e^{-\alpha x_N}h_n(x',x_N)\colon |x'|=r,\ x_N > -n\}.$$

Then

$$h_n = H_{h_n}^{A(r',1)} = H_{h_n\chi_{\partial U}}^{A(r',1)} + H_{h_n\chi_{V(r')}}^{A(r',1)} \quad \text{in} \ A(r',1). \tag{2.21}$$

Let $\alpha_{r'} > 0$ denote the square root of the first eigenvalue of $-\Delta$ in $B' \setminus \overline{B'_{r'}}$. Then $\alpha < \alpha_{r'}$ because the complement of $B' \setminus \overline{B'_{r'}}$ in B' is non-polar (see [19, Section 1.3.2]). Since $d\mu_x^{A(r',1)} = P_{A(r',1)}(x,\cdot)\,d\sigma$ on ∂U, the Poisson kernel estimates yield, for $|x'|=r$, that

$$\begin{aligned} e^{-\alpha x_N}H_{h_n\chi_{\partial U}}^{A(r',1)}(x) &\le C(N,r,r')e^{-\alpha x_N}\int_{\partial U} h_n(y)e^{-\alpha_{r'}|x_N-y_N|}d\sigma(y) \\ &\le C(N,r,r')\int_{\partial U} h_n(y)e^{-\alpha y_N}d\sigma(y). \end{aligned}$$

Noting that h_n satisfies the hypotheses of Lemma 2.3, we see from (2.15) that, when $|x'|=r$ we have

$$\begin{aligned} e^{-\alpha x_N}H_{h_n\chi_{\partial U}}^{A(r',1)}(x) &\le C(N,R,r,r')\int_{\partial U} e^{-\alpha y_N}h_n(ry',y_N)\beta_{E'}(y)\,d\sigma(y) \\ &\le C_1 m_n\Lambda, \end{aligned} \tag{2.22}$$

where C_1 is a constant depending on $N,\ R,\ r,\ r'$ and ν.

Moreover, for $|x'| = r$ we have

$$e^{-\alpha x_N} H^{A(r',1)}_{h_n\chi_{V(r')}}(x) \le e^{-\alpha x_N} \int_{V(r')} e^{\alpha y_N} d\mu_x^{A(r',1)}(y)$$

(2.23)
$$\le C(N,r,r') \int_{V(r')} e^{\alpha(y_N - x_N)} e^{-\alpha_{r'}|y_N - x_N|} d\sigma(y)$$

$$\le C(N,r,r') \int_{-\infty}^{+\infty} e^{(\alpha-\alpha_{r'})|y_N - x_N|} dy_N \le C_2(N,r,r').$$

By (2.21)–(2.23) we obtain

$$e^{-\alpha x_N} h_n(x) = e^{-\alpha x_N} H^{A(r',1)}_{h_n\chi_{\partial U}}(x) + e^{-\alpha x_N} H^{A(r',1)}_{h_n\chi_{V(r')}}(x) \le C_1 m_n \Lambda + C_2 \quad (|x'| = r).$$

Taking $c = \max\{C_1, C_2\}$ we arrive at

$$m_n \le c(1 + m_n\Lambda).$$

We choose $c_4 = (2c)^{-1}$ and suppose that $\Lambda \le c_4$. Then

$$m_n \le c + m_n c c_4 = c + m_n/2,$$

which implies that $m_n \le 2c$.

It follows from (2.21) and (2.22) that for $|x'| = r$ we have

(2.24)
$$e^{-\alpha x_N} h_n(x) \le 2c^2\Lambda + e^{-\alpha x_N} H^{A(r',1)}_{h_n\chi_{V(r')}}(x).$$

We choose $c_5 = 2c^2$ and let $n \to \infty$. By (2.23) the limit of the latter term on the right hand side of (2.24) is finite and so H_v^V exists and satisfies

$$H_v^V(x) \le c_5 \Lambda e^{\alpha x_N} + H_v^{A(r',1)}(x) \quad (|x'| = r). \qquad \square$$

Lemma 2.5. *Let $w\colon \partial^\infty U \to [0, +\infty)$ be a Borel measurable function such that*

(2.25)
$$w(y) \le \beta_{E'}(y) e^{\alpha y_N} \quad (y \in \partial U) \quad \text{and} \quad w(\infty) = 0.$$

Then, there exists a positive constant c_6, depending on N, R, r and ν, such that

$$H_w^U(x', x_N) \le c_6 e^{\alpha x_N} \Lambda \quad (|x'| = r).$$

Proof. Using (2.2), in view of (2.25) and (2.15), for $|x'| = r$ we have

$$H_w^U(x', x_N) \le C(N,r) \int_{\partial U} w(y) e^{-\alpha|y_N - x_N|} d\sigma(y)$$

$$\le C(N,r) e^{\alpha x_N} \int_{\partial U} \beta_{E'}(y)\, d\sigma(y) \le C(N,R,r,\nu) e^{\alpha x_N} \Lambda. \qquad \square$$

We extend h_+ to be 0 outside U and recall that V stands for $A(r', \infty) \setminus E'$. We define inductively a sequence (s_k) as follows

$$s_{-2} = s_{-1} = 0, \quad s_0 = h_+,$$

$$s_{2k+1} = \begin{cases} \overline{H}^V_{s_{2k}} & \text{on } V, \\ s_{2k} & \text{on } \mathbf{R}^N \setminus V, \end{cases} \qquad s_{2k+2} = \begin{cases} \overline{H}^U_{s_{2k+1}} + h_+ & \text{on } U, \\ s_{2k+1} & \text{on } \mathbf{R}^N \setminus U. \end{cases}$$

We put $s_k(\infty) = 0$ for all k.

Lemma 2.6. *There is a positive constant c_7, depending on N, R, r, r' and ν, such that, if $\Lambda \le c_7\lambda$ for some $\lambda \in (0,1)$, then:*

(a) *(s_k) is an increasing sequence of continuous functions on $\mathbf{R}^N$;*

(b) *each* s_k *is bounded on* $\mathbf{R}^{N-1}\times(-\infty,b_n)$ *for each* $n\in\mathbf{N}$*;*
(c) *for all* $k=0,1,\dots$ *we have*

$$(s_{2k}-s_{2k-2})(x)\le\lambda^k e^{\alpha x_N},\quad |x'|=r.$$

Proof. We will use ideas from [18, Lemma 3.1]. Suppose that $\Lambda\le c_7\lambda$, where c_7 is to be determined later. Assume that $s_0\le s_1\le\dots\le s_{2k}$ on $\mathbf{R}^N$ for some $k\ge 0$, that all the functions $s_{k'}$ are continuous on $\mathbf{R}^N$ for $0\le k'\le 2k$, and that for $0\le k'\le k$

$$(s_{2k'}-s_{2k'-2})(x',x_N)\le\lambda^{k'}e^{\alpha x_N}\quad(|x'|=r).\tag{2.26}$$

We also fix $n\in\mathbf{N}$ and assume that s_{2k} is bounded on $\mathbf{R}^{N-1}\times(-\infty,b_n)$. Once the terms of (s_k) are seen to be finite, it is clear that the upper PWB solutions appearing in their definitions are actually well defined PWB solutions. The induction hypotheses clearly hold for $k=0$. We split the proof of Lemma 2.6 into three steps.

Step 1. We show that s_{2k+1} is a finite-valued continuous function on $\mathbf{R}^N$ which is bounded on $\mathbf{R}^{N-1}\times(-\infty,b_n)$. Harnack's inequalities and (2.26) yield the existence of a constant $c_8=c_8(N,r,r')>0$ such that

$$(s_{2k}-s_{2k-2})(y)\le c_8\lambda^k e^{\alpha y_N}\quad(|y'|=r').\tag{2.27}$$

Now, for $|x'|=r$, by (2.27) and Lemma 2.4 we have

$$\begin{aligned}(s_{2k+1}-s_{2k-1})(x)&\le\overline{H}^V_{s_{2k}-s_{2k-2}}(x)=\overline{H}^V_{(s_{2k}-s_{2k-2})\chi_{V(r')}}(x)\\&\le c_5c_8\lambda^k\Lambda e^{\alpha x_N}+H^{A(r',1)}_{(s_{2k}-s_{2k-2})\chi_{V(r')}}(x).\end{aligned}$$

Since $s_{2k}-s_{2k-1}=0$ on ∂U and $s_{2k}-s_{2k-1}=s_{2k}-s_{2k-2}$ on $V(r')$, it follows that $s_{2k}-s_{2k-1}$ belongs to the upper class for $H^{A(r',1)}_{(s_{2k}-s_{2k-2})\chi_{V(r')}}$. Hence

$$(s_{2k+1}-s_{2k-1})(x)\le c_5c_8\lambda^k\Lambda e^{\alpha x_N}+(s_{2k}-s_{2k-1})(x),$$

and so

$$(s_{2k+1}-s_{2k})(x)\le c_5c_8\lambda^k\Lambda e^{\alpha x_N}\quad(|x'|=r).\tag{2.28}$$

This proves finiteness of s_{2k+1}.

A result of Armitage concerning a strong type of regularity for the PWB solution of the Dirichlet problem (see [3, Theorem 2]) implies that s_{2k+1} is continuous at points of $\partial V\setminus\bigcup_{n=1}^\infty(\partial B'\times\{b_n\})$. Applying Lemma 2.3 to $v_j=H^V_{\min\{s_{2k},j\}}$ and $x\in\bigcup_{n=1}^\infty(\partial B'\times\{b_n\})$ we obtain

$$v_j(y)\le c_3v_j(rx',x_N)H^{T_x\setminus E'}_{\chi_{\partial T_x}}(y)\quad(y\in T_x\setminus E').$$

Letting $j\to\infty$ we notice that the same inequality holds for s_{2k+1}, and hence the regularity of x for $T_x\setminus E'$ implies that s_{2k+1} vanishes at x. We conclude that s_{2k+1} is continuous on $\mathbf{R}^N$.

We also have $s_{2k+1}=H^{V\cap[\mathbf{R}^{N-1}\times(-\infty,b_n)]}_{s_{2k+1}}$ on $V\cap[\mathbf{R}^{N-1}\times(-\infty,b_n)]$. Further, since s_{2k+1} is continuous on $\overline{B'}\times\{b_n\}$, vanishes on E and is bounded on $(\mathbf{R}^N\setminus V)\cap[\mathbf{R}^{N-1}\times(-\infty,b_n)]$ in view of the induction hypothesis, we deduce that s_{2k+1} is bounded above on $\mathbf{R}^{N-1}\times(-\infty,b_n)$.

Step 2. We now prove that $s_{2k}\le s_{2k+1}\le s_{2k+2}$ on $\mathbf{R}^N$. We note that $s_{2k}=H^{A(r',1)}_{s_{2k}}$ on $A(r',1)$ (for a simple proof see Step 2 in the proof of [18, Lemma 3.1]).

It follows immediately from the induction hypothesis, that

$$s_{2k+1} = H^V_{s_{2k}} \geq H^V_{s_{2k-2}} = s_{2k-1} \quad \text{on } \; V.$$

In particular, this gives $s_{2k+1} \geq s_{2k}$ on $\mathbf{R}^N \setminus U$. Hence, $s_{2k+1} \geq s_{2k}$ on $\partial U \cup \partial V$. Using [5, Theorem 6.3.6], we obtain

$$s_{2k+1} = H^V_{s_{2k}} = H^{A(r',1)}_{s_{2k+1}} \geq H^{A(r',1)}_{s_{2k}} = s_{2k} \quad \text{on } \; A(r',1).$$

Therefore, $s_{2k+1} \geq s_{2k}$ on $\mathbf{R}^N$. We now deduce that

$$s_{2k+2} = \overline{H}^U_{s_{2k+1}} + h_+ \geq H^U_{s_{2k-1}} + h_+ = s_{2k} = s_{2k+1} \;\text{ on }\; \mathbf{R}^N \setminus V.$$

We finally note that, if s_{2k+2} belongs to the upper class for $\overline{H}^{A(r',1)}_{s_{2k+2}}$, we obtain

$$s_{2k+2} \geq \overline{H}^{A(r',1)}_{s_{2k+2}} \geq H^{A(r',1)}_{s_{2k+1}} = s_{2k+1} \quad \text{on } \; A(r',1),$$

and so $s_{2k+2} \geq s_{2k+1}$ on $\mathbf{R}^N$. To verify that s_{2k+2} belongs to the upper class for $\overline{H}^{A(r',1)}_{s_{2k+2}}$ it is enough to check that $\liminf_{x\to y} s_{2k+2}(x) \geq s_{2k+2}(y)$ for $y \in \partial U$. This is clear from regularity and the continuity of s_{2k+1}, as if $s_{2k+2} \not\equiv +\infty$, then for $y \in \partial U$ we have

$$\liminf_{x\to y} s_{2k+2}(x) = \liminf_{x\to y} H^U_{s_{2k+1}}(x) \geq \liminf_{x\to y, x\in\partial U} s_{2k+1}(x) = s_{2k+1}(y) = s_{2k+2}(y).$$

Step 3. In the final step we will prove that

$$(s_{2k+2} - s_{2k})(x) \leq \lambda^{k+1} e^{\alpha x_N} \quad (|x'| = r). \tag{2.29}$$

Then, using [3, Theorem 2], we can conclude that s_{2k+2} is continuous on $\mathbf{R}^N$. Further, $s_{2k+2} - h_+ = H^U_{s_{2k+1}} = H^{U\cap[\mathbf{R}^{N-1}\times(-\infty,b_n)]}_{s_{2k+2}-h_+}$ on $U \cap [\mathbf{R}^{N-1} \times (-\infty, b_n)]$. By continuity, s_{2k+2} is bounded on $\overline{B'} \times \{b_n\}$. On $\mathbf{R}^N \setminus U$ we have $s_{2k+2} = s_{2k+1}$, which is bounded on $(\mathbf{R}^N \setminus U) \cap [\mathbf{R}^{N-1} \times (-\infty, b_n)]$ by Step 1. Hence s_{2k+2} is bounded on the whole of $\mathbf{R}^{N-1} \times (-\infty, b_n)$.

To prove the desired inequality (2.29), we first recall that

$$U_m = (\mathbf{R}^{N-1} \setminus \overline{B'}) \times (b_m, b_{m+1}) \quad (m \in \mathbf{N}).$$

Noting that

$$s_{2k+1} = H^V_{s_{2k}} = H^{U_m}_{s_{2k+1}} = H^{U_m}_{s_{2k+1}\chi_{\partial B' \times (b_m, b_{m+1})}} \quad \text{on } \; U_m,$$

and that, by continuity, s_{2k+1} is bounded on $\partial B' \times (b_m, b_{m+1})$, we see that $s_{2k+1} - s_{2k-1}$ satisfies the hypotheses of Lemma 2.3. Hence, for $x \in \partial U$, we have

$$\begin{aligned}(s_{2k+1} - s_{2k-1})(x) &\leq c_3 \beta_{E'}(x)(s_{2k+1} - s_{2k-1})(rx', x_N) \\ &= c_3 \beta_{E'}(x)[(s_{2k+1} - s_{2k})(rx', x_N) + (s_{2k} - s_{2k-1})(rx', x_N)] \\ &\leq c_3 \beta_{E'}(x)[(s_{2k+1} - s_{2k})(rx', x_N) + (s_{2k} - s_{2k-2})(rx', x_N)].\end{aligned}$$

It follows from (2.28) and our induction hypothesis that

$$(s_{2k+1} - s_{2k-1})(x) \leq c_3(c_5 c_8 \Lambda + 1)\lambda^k e^{\alpha x_N} \beta_{E'}(x) \quad (x \in \partial U).$$

Assuming that $c_7 \leq 1$ and letting $c_9 = c_3(c_5 c_8 + 1)$ we obtain

$$(s_{2k+1} - s_{2k-1})(x) \leq c_9 \lambda^k e^{\alpha x_N} \beta_{E'}(x) \quad (x \in \partial U).$$

By Lemma 2.5, for $|x'| = r$, we have

$$(s_{2k+2} - s_{2k})(x) \le \overline{H}^U_{s_{2k+1}-s_{2k-1}}(x) \le c_9\lambda^k c_6 \Lambda e^{\alpha x_N} = c_6c_7c_9\lambda^{k+1}e^{\alpha x_N}.$$

Taking $c_7 = \min\{1, (c_6c_9)^{-1}\}$ we find that (2.29) holds, and the proof is complete. □

3. Proof of Theorem 1.1

Proposition 2.1 gives the implication $(a) \Rightarrow (b)$. To prove that $(b) \Rightarrow (a)$ we first observe that taking J large enough when setting $b_1 = a_J$, we can ensure that $\Lambda \le c_7\lambda$ for some $\lambda \in (0,1)$. Let $\Omega' = \mathbf{R}^N \setminus E'$ and $u' = \lim_{k\to\infty} s_k$. By Lemma 2.6, for $|x'| = r$ we obtain

$$s_{2k}(x) = \sum_{j=0}^{k}(s_{2j} - s_{2j-2})(x) \le \sum_{j=0}^{k} \lambda^j e^{\alpha x_N} \le \frac{1}{1-\lambda}e^{\alpha x_N}.$$

Hence $u' \not\equiv +\infty$. As a limit of an increasing sequence (s_{2k}) of harmonic functions on U, the function u' is harmonic on U. Since u' is the limit of an increasing sequence (s_{2k+1}) of harmonic functions on V, it is also harmonic on V. Hence u' is harmonic in Ω'. It follows from the monotonicity of (s_k) that $u' \ge h_+$ on U.

For $x \in E'$ we have $u'(x) = 0$. By the monotone convergence theorem applied to the equation $s_{2k+1} = H^V_{s_{2k}}$ we obtain $u' = H^V_{u'}$ on V. We can follow the reasoning from the second last paragraph of Step 1 in the proof of Lemma 2.6 to see that u' vanishes continuously on E'.

We next prove that u' is minimal on Ω' using an argument from [18, Theorem 1.1]. As a consequence of the monotone convergence theorem we find that

$$u'(x) = H^U_{u'}(x) + h_+(x) \quad (x \in U). \tag{3.1}$$

Let Δ_1 denote the minimal Martin boundary of Ω' and let M be the Martin kernel of Ω' relative to the origin. By the Martin representation theorem (see [5, Theorem 8.4.1]) we have

$$u'(x) = \int_{\Delta_1} M(x,z)\, d\nu_{u'}(z) \quad (x \in \Omega'), \tag{3.2}$$

where $\nu_{u'}$ is uniquely determined by u'.

We define $T = \{z \in \Delta_1 \colon \Omega'\backslash U \text{ is minimally thin at } z\}$ so that

$$R^{\Omega'\backslash U}_{M(\cdot,z)} = M(\cdot,z) \quad (z \in \Delta_1 \setminus T). \tag{3.3}$$

Changing the order of integration, and using (3.1)–(3.3) and [5, Theorem 6.9.1], we obtain

$$\begin{aligned} h_+(x) &= \int_{\Delta_1} \left(M(x,z) - \int_{\partial U} M(y,z)\, d\mu^U_x(y) \right) d\nu_{u'}(z) \\ &= \int_{\Delta_1} \left(M(x,z) - R^{\Omega'\backslash U}_{M(\cdot,z)}(x) \right) d\nu_{u'}(z) \\ &= \int_{T} \left(M(x,z) - R^{\Omega'\backslash U}_{M(\cdot,z)}(x) \right) d\nu_{u'}(z) \quad (x \in U). \end{aligned}$$

We now claim that $\nu_{u'}|_T$ is concentrated at a single point. For the sake of contradiction suppose that there are two distinct points $y_1, y_2 \in \Delta_1 \cap \operatorname{supp}(\nu_{u'}|_T)$ and let N_1, N_2 be disjoint neigbourhoods of y_1 and y_2 respectively. We define

$$h_j(x) = \int_{N_j \cap T} \left(M(x,y) - R^{\Omega' \setminus U}_{M(\cdot,y)}(x) \right) d\nu_{u'}(y) \quad (x \in \Omega',\ j = 1,2),$$

and note that $h_j \leq h_+$ on U. Minimality of h_+ on U implies that

$$h_j / h_j(0) = h_+ \quad \text{on } U \quad (j = 1,2). \tag{3.4}$$

We now define

$$v_j(x) = \int_{N_j \cap T} M(x,y)\, d\nu_{u'}(y) \quad (x \in \Omega', j = 1,2).$$

Then $h_j \leq v_j \leq u'$ on Ω', and by (3.4), $v_j/h_j(0) \geq h_+$ on Ω' $(j = 1,2)$. In view of the definition of s_k we have $v_j/h_j(0) \geq s_k$ on Ω' for all $k \in \mathbf{N}$ and so $v_j/h_j(0) \geq u'$ on Ω' $(j = 1,2)$. It follows that $h_1(0)v_2 \leq v_1$ on Ω'. This implies that $\nu_{u'}|_{T \cap N_1}$ is minorized by a multiple of $\nu_{u'}|_{T \cap N_2}$, which contradicts the fact that $N_1 \cap N_2 = \emptyset$. Hence $\nu_{u'}|_T = c\delta_{t'}$ for some $t' \in T$ and $c > 0$. Furthermore, the minimal harmonic function $v = cM(\cdot, t')$ on Ω' satisfies $u' \geq v$ on Ω' and $v \geq h_+$ on U. We observe that $v \geq s_k$ on Ω' for all $k \in \mathbf{N}$, and so $v \geq u'$. Hence $v \equiv u'$ and we conclude that u' is minimal on Ω'.

Let $\Omega'' = \mathbf{R}^N \setminus E''$. We define $g = H^{\Omega'}_{\chi_{\Omega'' \setminus \Omega'}}$ and $g = \chi_{\Omega'' \setminus \Omega'}$ on $\partial^\infty \Omega'$. By [5, Theorem 6.9.1] we have $g = R_1^{\Omega'' \setminus \Omega'}$ on Ω'' (reductions with respect to non-negative superharmonic functions on Ω''). Since $\Omega'' \setminus \Omega'$ is non-thin at each constituent point, it follows from [5, Theorem 7.3.1(i)] that $R_1^{\Omega'' \setminus \Omega'} = \widehat{R}_1^{\Omega'' \setminus \Omega'}$ on Ω'' and so g is superharmonic there. Let h be a non-negative harmonic minorant of g on Ω''. Then h is bounded on Ω'' and vanishes quasi-everywhere on $\partial \Omega''$. Since a polar subset of $\partial\Omega''$ and $\{\infty\}$ are both negligible for Ω'' (see [5, Theorems 6.5.5 and 7.6.5]), we deduce that $h \equiv 0$. Hence g is a potential on Ω''.

Let $W = [\mathbf{R}^{N-1} \times (-\infty, b_n)] \cap \Omega'$ for some $n > 1$. Since $1 - g$ is positive and continuous on $\overline{B'} \times \{b_n\}$, it follows that $1 - g$ is bounded below by a positive constant on this set while u' is bounded from above there. Hence there exists a positive constant c such that $c(1-g) \geq u'$ on $\overline{B'} \times \{b_n\}$, and thus on ∂W. By Lemma 2.6(b) each s_k is bounded on W and so it belongs to the lower class for $H^W_{s_k}$. These facts combined with monotonicity of (s_k) lead to the observation that

$$s_k \leq H^W_{s_k} \leq H^W_{u'} \leq cH^W_{1-g} = c(1-g) \ \text{ on } \ W.$$

Therefore, $u' \leq c(1-g)$ on W. Since $c(1-g) - u'$ is a non-negative harmonic function on W which vanishes on $\Omega'' \setminus \Omega'$, we conclude that $c(1-g) - u'$ is subharmonic on Ω'', so that $u' + cg$ is superharmonic on Ω''.

By the Riesz decomposition,

$$u' + cg = u'' + G_{\Omega''}\mu \quad \text{on } \ \Omega'', \tag{3.5}$$

where u'' is the greatest harmonic minorant of $u' + cg$ on Ω'' and $G_{\Omega''}\mu$ is the Green potential of the Riesz measure μ associated with $u' + cg$. Hence u'' vanishes on $E'' \setminus (\partial B' \times \{b_1\})$ and for each $n \in \mathbf{N}$ it is bounded on $\mathbf{R}^{N-1} \times (-\infty, b_n)$. It follows from a removable singularity result (see [5, Theorem 5.2.1]) that u'' extends to a subharmonic function on $\mathbf{R}^N$. This together with the non-thinness of E'' at points of $\partial B' \times \{b_1\}$ implies that u'' vanishes also on $\partial B' \times \{b_1\}$. Since h_+ is a subharmonic minorant of $u' + cg$ on Ω'', we deduce that $h_+ \le u''$ on Ω''.

It remains to show that u'' is minimal. Let h be a positive harmonic minorant of u'' on Ω''. We notice that h is bounded on $\Omega'' \setminus \Omega'$ and vanishes on $\partial\Omega''$. Hence the greatest harmonic minorant of $R_h^{\Omega''\setminus\Omega'}$ on Ω'' is bounded and vanishes on $\partial\Omega''$, and we see that $R_h^{\Omega''\setminus\Omega'}$ is a potential on Ω''. Since the upper-bounded harmonic function $h - R_h^{\Omega''\setminus\Omega'} - u'$ on Ω' satisfies

$$\limsup_{x\to y}(h - R_h^{\Omega''\setminus\Omega'} - u')(x) \le 0 \quad \text{for} \ \ y \in \partial\Omega',$$

and $\{\infty\}$ has zero harmonic measure for Ω', it follows that

$$h - R_h^{\Omega''\setminus\Omega'} - u' \le 0 \quad \text{on} \ \ \Omega'.$$

Now, since $h - R_h^{\Omega''\setminus\Omega'}$ is a positive harmonic minorant of the minimal function u' on Ω', we conclude that $h - R_h^{\Omega''\setminus\Omega'} = au'$ for some $a \in (0, 1]$. Substituting this into (3.5) we obtain

$$h + acg = au'' + aG_{\Omega''}\mu + R_h^{\Omega''\setminus\Omega'} \quad \text{on} \ \ \Omega''.$$

Taking the greatest harmonic minorant in Ω'' of both sides we get $h = au''$, which means that u'' is minimal.

Let $u = u'' - H_{u''}^{\Omega}$. Since $u'' - h_+ \ge 0$ is superharmonic on Ω'' and equals u'' on $\Omega'' \setminus \Omega$, we have

$$u = u'' - R_{u''}^{\Omega''\setminus\Omega} = u'' - R_{u''-h_+}^{\Omega''\setminus\Omega} \ge h_+.$$

Since the points of $\partial\Omega$ are regular for Ω and u'' is continuous, it follows that u vanishes on $\partial\Omega$. Further, [5, Theorem 9.5.5] shows that u is minimal.

Remark. The proof of the implication $(a) \Rightarrow (b)$ in Theorem 1.1 does not rely on condition (1.1). It is in the proof of the converse that our methods rely on such a condition. However, it is enough to assume merely that Ω is contained in a comb-like domain Ω_0 for which (1.1) holds. To see this, suppose that (b) holds. Theorem 1.1 applied to Ω_0 yields the existence of a minimal harmonic function u_0 on Ω_0 which vanishes on $\partial\Omega_0$ and satisfies $u_0 \ge h_+$. Let $u = u_0 - H_{u_0}^{\Omega}$ on Ω. The argument from the previous paragraph shows that u is as stated in (a).

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