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ON PLANAR HARMONIC LIPSCHITZ AND PLANAR HARMONIC HARDY CLASSES

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Abstract. In this paper, we investigate the properties of two classes of planar harmonic mappings. First, we discuss the equivalent norms on Lipschitz-type spaces of harmonic K-quasiregular mappings and then we study the relationship between the harmonic area functions and the harmonic Hardy classes. We also establish Landau's theorem for a class of harmonic Hardy mappings.

1. Introduction and preliminaries

A complex-valued function $f(z) = u(z) + iv(z)$ defined on a simply connected domain D of $\mathbf{C}$ is called a *harmonic mapping* if and only if it is twice continuously differentiable and $\Delta f = 0$. That is the components u and v are real harmonic in D, where Δ represents the complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z\partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Every harmonic mapping f defined in D has a canonical decomposition $f = h+\overline{g}$, where h and g are analytic in D (see [3] or [4]). Since the Jacobian $J_f(z)$ of f is given by

$$J_f(z) = |f_z(z)|^2 - |f_{\overline{z}}(z)|^2 = |h'(z)|^2 - |g'(z)|^2,$$

f is locally univalent and orientation-preserving if and only if $|g'(z)| < |h'(z)|$ in D; or equivalently $h'(z) \neq 0$ and the dilatation $\omega = g'/h'$ has the property that $|\omega(z)| < 1$ in D. For $a \in \mathbf{C}$, let $\mathbf{D}(a, r) = \{z\colon |z - a| < r\}$. In special, we denote $\mathbf{D}(0, r) = \mathbf{D}_r$ and $\mathbf{D} = \mathbf{D}(0, 1)$. Because a composition $f\circ g$ with an analytic function g remains harmonic, the Riemann mapping theorem allows us to assume that $D = \mathbf{D}$. Throughout this paper, we consider harmonic mappings in $\mathbf{D}$ unless specially stated.

A continuous increasing function $\omega\colon [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is non-increasing for $t > 0$. Given a subset Ω of $\mathbf{C}$, a function $f\colon \Omega \to \mathbf{C}$ is said to belong to the *Lipschitz space* $\Lambda_\omega(\Omega)$ if there is a positive constant C such that

$$(1)\qquad |f(z) - f(w)| \le C\omega(|z - w|) \quad \text{for all } z, w \in \Omega.$$

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For $\delta_0 > 0$, let

$$\int_0^\delta \frac{\omega(t)}{t}\,dt \le C\cdot\omega(\delta), \quad 0<\delta<\delta_0, \tag{2}$$

and

$$\delta\int_\delta^{+\infty} \frac{\omega(t)}{t^2}\,dt \le C\cdot\omega(\delta), \quad 0<\delta<\delta_0, \tag{3}$$

where ω is a majorant and C is a positive constant. A majorant ω is said to be *regular* if it satisfies the conditions (2) and (3) (see [5, 16]).

Let G be a proper subdomain of $\mathbf{C}$ or $\mathbf{R}^2$. We say that a function f belongs to the *local Lipschitz space* $\operatorname{loc}\Lambda_\omega(G)$ if (1) holds, with a fixed positive constant C, whenever $z\in G$ and $|z-w|<\frac{1}{2}\,d(z,\partial G)$, where $d(\cdot,\cdot)$ denotes the Euclidean distance (cf. [6, 10]). Moreover, G is said to be a Λ_ω*-extension domain* if $\Lambda_\omega(G)=\operatorname{loc}\Lambda_\omega(G)$. The geometric characterization of Λ_ω-extension domains was first given by Gehring and Martio [6]. Then Lappalainen [10] extended it to the general case and proved that G is a Λ_ω-extension domain if and only if each pair of points $z,w\in G$ can be joined by a rectifiable curve $\gamma\subset G$ satisfying

$$\int_\gamma \frac{\omega(d(z,\partial G))}{d(z,\partial G)}\,ds(z) \le C\omega(|z-w|) \tag{4}$$

with some fixed positive constant $C=C(G,\omega)$, where ds stands for the arc length measure on γ. Furthermore, Lappalainen [10, Theorem 4.12] proved that Λ_ω-extension domains exist only for majorants ω satisfying (2).

Dyakonov [5] characterized the holomorphic functions of class Λ_ω in terms of their modulus. Later in [16, Theorems A], Pavlović came up with a relatively simple proof of the results of Dyakonov. Recently, many authors considered this topic and generalized Dyakonov's results to pseudo-holomorphic functions and real harmonic functions of several variables for some special majorant $\omega(t)=t^\alpha$, where $\alpha>0$ (see [9, 11, 12, 13, 14]). In this paper, we first extend [16, Theorems A and B] to planar K-quasiregular harmonic mappings as follows, where $K\ge 1$.

Theorem 1. *Let ω be a majorant satisfying* (2), *and let G be a Λ_ω-extension domain. If f is a planar K-quasiregular harmonic mapping of G and continuous up to the boundary ∂G, then*

$$f\in\Lambda_\omega(G) \Longleftrightarrow |f|\in\Lambda_\omega(G) \Longleftrightarrow |f|\in\Lambda_\omega(G,\partial G),$$

where $\Lambda_\omega(G,\partial G)$ denotes the class of continuous functions f on $G\cup\partial G$ which satisfy (1) *with some positive constant C, whenever $z\in G$ and $w\in\partial G$.*

For any $z_1,z_2\in G\subset\mathbf{C}$, let

$$d_{\omega,G}(z_1,z_2) := \inf\int_\gamma \frac{\omega(d(z,\partial G))}{d(z,\partial G)}\,ds(z),$$

where the infimum is taken over all rectifiable curves $\gamma\subset G$ joining z_1 to z_2. We say that $f\in\Lambda_{\omega,\inf}(G)$ whenever for any $z_1,z_2\in G$,

$$|f(z_1)-f(z_2)|\le Cd_{\omega,G}(z_1,z_2),$$

where C is a positive constant which depends only on f (see [8]).

Theorem 2. *Let ω be a majorant satisfying* (2). *If f is a planar K-quasiregular harmonic mapping in G, then*

$$f \in \Lambda_{\omega,\inf}(G) \Longleftrightarrow |f| \in \Lambda_{\omega,\inf}(G).$$

Many authors discussed the relationships between Hardy classes of holomorphic functions and integral means (see [7, 15]). In order to derive an analogous result of [7, Theorem 1] for the setting of harmonic mappings, we need to introduce some notation.

For a harmonic function f in $\mathbf{D}$, for $p > 0$ and $0 \leq r < 1$, we define

$$I_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \tag{5}$$

and say that f belongs to the *harmonic Hardy class* $\mathscr{H}_h^p$ if

$$\|f\|_p = \sup_{0<r<1} (I_p(r, f))^{1/p} < +\infty.$$

For a harmonic mapping f in $\mathbf{D}$, the *generalized harmonic area function* $A_h(r, f)$ is defined by

$$A_h(r, f) = \int_{\mathbf{D}_r} |\nabla f(z)|^2 \, dA(z),$$

where dA denotes the normalized Lebesgue measure on $\mathbf{D}$ and

$$|\nabla f| = (|f_z|^2 + |f_{\overline{z}}|^2)^{1/2}.$$

The following theorem is an analogous result of [7, Theorem 1].

Theorem 3. *Let f be harmonic in $\mathbf{D}$ and $\delta > 0$. Then, if $1 < p \leq 2$,*

$$f \in \mathscr{H}_h^p(\mathbf{D}) \Rightarrow \int_0^1 A_h^{\frac{p}{2}}(r, f)(1 - r)^{\frac{\delta(2-p)}{2}} \, dr < +\infty, \tag{6}$$

while if $p > 2$,

$$\int_0^1 A_h^{\frac{p}{2}}(r, f)(1 - r)^{\frac{\delta(2-p)}{2}} \, dr < +\infty \Rightarrow f \in \mathscr{H}_h^p(\mathbf{D}). \tag{7}$$

Theorem 4. *Let $f \in \mathscr{H}_h^p(\mathbf{D})$ and $\delta > 0$. If $1 < p \leq 2$, then*

$$\lim_{r \to 1-} (1 - r)^{\frac{\delta(2-p)+2}{p}} A_h(r, f) = 0.$$

Finally, we prove a Landau's theorem for a class of harmonic Hardy mappings.

Theorem 5. *Let f be a harmonic in $\mathbf{D}$ with $\|f\|_p \leq M$ and $f(0) = \lambda_f(0) - 1 = 0$, where M is a positive constant, $\lambda_f(z) = \big||f_z(z)| - |f_{\overline{z}}(z)|\big|$ and $p \geq 1$. Then f is univalent in $\mathbf{D}_{\rho_0}$, where*

$$\rho_0 = \varphi(r_0) = \max_{0<r<1} \varphi(r), \quad \varphi(r) = r\left(1 - \sqrt{\frac{t}{1+t}}\right),$$

with

$$t = \frac{4}{\pi} \cdot \frac{2^{\frac{1}{p}} M}{r(1 - r)^{\frac{1}{p}}}.$$

Moreover, $f(\mathbf{D}_{\rho_0})$ *contains a univalent disk* $\mathbf{D}_{R_0}$ *with*

$$R_0 = \frac{r_0\varphi(r_0)}{2r_0 - \varphi(r_0)}.$$

In Theorem 5, we remark that $\max_{0<r<1} \varphi(r)$ does exist, since

$$\lim_{r\to 0+} \varphi(r) = \lim_{r\to 1-} \varphi(r) = 0.$$

The proofs of these theorems are presented in the following sections. We end this section with the following problem which is suggested by the referee: *Does Theorem* 1 *still hold if the hypothesis "mappings being harmonic" is dropped?*

2. Harmonic Lipschitz classes

In order to prove our main results, we need the following result.

Theorem A. [1, Theorem 7] *If* f *is a* K*-quasiregular harmonic mapping of* $\mathbf{D}$ *into itself, then*

$$|f_z(z)| + |f_{\overline{z}}(z)| \le \frac{4K}{\pi}\frac{\cos(|f(z)\pi/2|)}{1-|z|^2}$$

holds for $z \in \mathbf{D}$.

Proof of Theorem 1. The implications $f \in \Lambda_\omega(G) \Rightarrow |f| \in \Lambda_\omega(G) \Rightarrow |f| \in \Lambda_\omega(G, \partial G)$ are obvious. We only need to prove $|f| \in \Lambda_\omega(G, \partial G) \Rightarrow f \in \Lambda_\omega(G)$. For a fixed point $z \in G$, let

$$F(\eta) = f(z + d(z)\eta)/M_z, \quad \eta \in \mathbf{D},$$

where $d(z) := d(z, \partial G)$ and $M_z := \sup\{|f(\zeta)|\colon |\zeta - z| < d(z)\}$. By a simple calculation, we obtain that

$$\frac{|F_\eta(\eta)| + |F_{\overline{\eta}}(\eta)|}{|F_\eta(\eta)| - |F_{\overline{\eta}}(\eta)|} = \frac{|f_\xi(\xi)| + |f_{\overline{\xi}}(\xi)|}{|f_\xi(\xi)| - |f_{\overline{\xi}}(\xi)|} \le K,$$

where $\xi = z + d(z)\eta$. Then F is a K-quasiregular harmonic mapping of $\mathbf{D}$ into itself. By Theorem A, we have

$$|F_\eta(0)| + |F_{\overline{\eta}}(0)| \le \frac{4K(1-|F(0)|^2)}{\pi}$$

which in turn gives

$$d(z)(|f_\xi(z)| + |f_{\overline{\xi}}(z)|) \le \frac{8K}{\pi}(M_z - |f(z)|). \tag{8}$$

Without loss of generality, we let $\zeta \in \partial G$ with $|\zeta - z| = d(z)$, and let $w \in \mathbf{D}(z, d(z))$. Then

$$\begin{aligned} |f(w)| - |f(z)| &\le \big||f(w)| - |f(\zeta)|\big| + \big||f(\zeta)| - |f(z)|\big| \\ &\le C\omega(d(z)) + C\omega(2d(z)) \le 3C\omega(d(z)) \end{aligned}$$

and so,

$$\sup_{w\in\mathbf{D}(z,d(z))} (|f(w)| - |f(z)|) \le 3C\omega(d(z)),$$

which implies that

$$M_z - |f(z)| \le 3C\omega(d(z)). \tag{9}$$

Thus, by (8) and (9), we have

$$|f_\xi(z)| + |f_{\bar\xi}(z)| \le \frac{24CK}{\pi} \cdot \frac{\omega(d(z))}{d(z)}, \quad z \in G. \tag{10}$$

Finally, given any two points $z_1, z_2 \in G$, let $\gamma \subset G$ be a curve which joins z_1, z_2 satisfying (4). Integrating (10) along γ, we obtain that

$$|f(z_1) - f(z_2)| \le \int_\gamma (|f_z(z)| + |f_{\bar z}(z)|)\, ds(z) \le \frac{24CK}{\pi} \int_\gamma \frac{\omega(d(z))}{d(z)}\, ds(z). \tag{11}$$

Therefore, (4) and (11) yield

$$|f(z_1) - f(z_2)| \le C_1 \cdot \omega(|z_1 - z_2|),$$

where C_1 is a positive constant. This completes the proof. □

Proof of Theorem 2. The implication $f \in \Lambda_{\omega,\inf}(G) \Rightarrow |f| \in \Lambda_{\omega,\inf}(G)$ is obvious. We need only to prove that $|f| \in \Lambda_{\omega,\inf}(G) \Rightarrow f \in \Lambda_{\omega,\inf}(G)$. For a fixed point $z \in G$, let

$$F(\eta) = f(z + d(z)\eta)/M_z, \quad \eta \in \mathbf{D},$$

where $d(z) := d(z, \partial G)$ and $M_z := \sup\{|f(\zeta)| \colon |\zeta - z| < d(z)\}$. From the proof of Theorem 1, it follows that

$$d(z)(|f_\xi(z)| + |f_{\bar\xi}(z)|) \le \frac{8K}{\pi}(M_z - |f(z)|), \tag{12}$$

where $\xi = z + d(z)\eta$. For $w \in \mathbf{D}(z, d(z))$, there exists a positive constant C such that

$$|f(w)| - |f(z)| \le C d_{\omega,G}(w, z) \le C \int_{[w,z]} \frac{\omega(d(\zeta, \partial G))}{d(\zeta, \partial G)}\, ds(\zeta), \tag{13}$$

where $[w, z]$ denotes the straight segment with endpoints w and z. We observe that if $\zeta \in [w, z]$, then one has $[w, z] \subset \mathbf{D}(z, d(z)) \subset G$ and therefore

$$d(\zeta, \partial G) \ge d(\zeta, \partial \mathbf{D}(z, d(z))),$$

which gives that

$$\frac{\omega(d(\zeta, \partial G))}{d(\zeta, \partial G)} \le \frac{\omega(d(\zeta, \partial \mathbf{D}(z, d(z))))}{d(\zeta, \partial \mathbf{D}(z, d(z)))}. \tag{14}$$

For any $w \in \mathbf{D}(z, d(z))$, (13) and (14) imply that

$$\begin{aligned} |f(w)| - |f(z)| &\le C \int_{[w,z]} \frac{\omega(d(\zeta, \partial G))}{d(\zeta, \partial G)}\, ds(\zeta) \le C \int_{[w,z]} \frac{\omega(d(\zeta, \partial \mathbf{D}(z, d(z))))}{d(\zeta, \partial \mathbf{D}(z, d(z)))}\, ds(\zeta) \\ &= C \int_{[w,z]} \frac{\omega(d(z) - |\zeta - z|)}{d(z) - |\zeta - z|}\, ds(\zeta) \le C \int_0^{d(z)} \frac{\omega(t)}{t}\, dt \le C\omega(d(z)). \end{aligned}$$

From this we obtain that

$$M_z - |f(z)| \le C\omega(d(z)). \tag{15}$$

Again, for any $z_1, z_2 \in G$, by (12) and (15), there exists a positive constant C_1 such that

$$|f(z_1) - f(z_2)| \le C_1 d_{\omega,G}(z_1, z_2)$$

and the proof of this theorem is completed. □

3. Harmonic Hardy classes

It is worth to remark that the standard technologies of analytic functions are not useful to prove Theorem 3 and therefore, we use Green's Theorem in its proof. Green's theorem states that if $g \in C^2(\mathbf{D})$, then

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\,d\theta = g(0) + \frac{1}{2}\int_{\mathbf{D}_r} \Delta g(z)\log\left(\frac{r}{|z|}\right)dA(z), \quad 0 \le r < 1. \tag{16}$$

Proof of Theorem 3. First we prove the implication (7). Let $f \in \mathscr{H}_h^p(\mathbf{D})$ and $\delta > 0$. For $0 \le r < 1$, by the Poisson integral formula, we have

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} f(re^{i\theta})\,d\theta, \quad z \in \mathbf{D}_r.$$

Using Jensen's inequality, we have

$$|f(z)|^p \le \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2}|f(re^{i\theta})|^p\,d\theta \le \frac{2rI_p(r,f)}{r - |z|},$$

and so

$$|f(z)|^p(r - |z|) \le 2rI_p(r,f), \tag{17}$$

where $I_p(r,f)$ is defined by (5). It follows that

$$\int_0^r (r-\rho)^\delta M^p(\rho,f)\,d\rho \le 2rI_p(r,f)\int_0^r \frac{d\rho}{(r-\rho)^{1-\delta}} \le \frac{2r^{1+\delta}I_p(r,f)}{\delta}, \tag{18}$$

where $M(r,f) = \sup\{|f(z)| : |z| = r\}$. By (16), we have

$$I_p(r,f) = |f(0)|^p + \frac{1}{2}\int_{\mathbf{D}_r} \Delta(|f(z)|^p)\log\frac{r}{|z|}\,dA(z)$$

and therefore

$$\begin{aligned} r\frac{d}{dr}I_p(r,f) &= \frac{1}{2}\int_{\mathbf{D}_r} \Delta\big(|f(z)|^p\big)\,dA(z) \\ &= p\int_{\mathbf{D}_r} \Big[\big(\frac{p}{2}-1\big)|f(z)|^{p-4}|f_z(z)\overline{f(z)} + f(z)\overline{f_{\bar z}(z)}|^2 \\ &\qquad + |f(z)|^{p-2}|\nabla f(z)|^2\Big]\,dA(z) \\ &\le p(p-1)\int_{\mathbf{D}_r} |f(z)|^{p-2}|\nabla f(z)|^2\,dA(z) \\ &\le p(p-1)M^{p-2}(r,f)\int_{\mathbf{D}_r} |\nabla f(z)|^2\,dA(z) \\ &= p(p-1)A_h(r,f)M^{p-2}(r,f). \end{aligned}$$

By integration, Hölder's inequality and (18), we have

$$\begin{aligned}
I_p(r,f) &\le |f(0)|^p + p(p-1)\int_0^r \frac{A_h(\rho,f)}{\rho} M^{p-2}(\rho,f)\,d\rho \\
&\le |f(0)|^p + p(p-1)\left[\int_0^r (r-\rho)^\delta M^p(\rho,f)\,d\rho\right]^{\frac{p-2}{p}} \\
&\quad \cdot \left[\int_0^r \Big(\frac{A_h(\rho,f)}{\rho}\Big)^{\frac{p}{2}} (r-\rho)^{\frac{\delta(2-p)}{2}}\,d\rho\right]^{\frac{2}{p}} \\
&\le |f(0)|^p + p(p-1)\Big(\frac{2}{\delta}\Big)^{\frac{p-2}{p}} I_p^{\frac{p-2}{p}}(r,f) \\
&\quad \cdot \left[\int_0^r \Big(\frac{A_h(\rho,f)}{\rho}\Big)^{\frac{p}{2}} (r-\rho)^{\frac{\delta(2-p)}{2}}\,d\rho\right]^{\frac{2}{p}}.
\end{aligned}$$

Without loss of generality, we may now assume that $f(0)\neq 0$. Then we have

$$I_p^{\frac{2}{p}}(r,f) \le |f(0)|^{\frac{p^2-p+2}{p}} + p(p-1)\Big(\frac{2}{\delta}\Big)^{\frac{p-2}{p}} \left[\int_0^r \Big(\frac{A_h(\rho,f)}{\rho}\Big)^{\frac{p}{2}} (r-\rho)^{\frac{\delta(2-p)}{2}}\,d\rho\right]^{\frac{2}{p}}$$

which shows that

$$\int_0^1 A_h^{\frac{p}{2}}(r,f)(1-r)^{\frac{\delta(2-p)}{2}}\,dr < +\infty \Rightarrow f\in \mathscr{H}_h^p(\mathbf{D}).$$

Next, we prove the implication (6). By a simple calculation, we get

$$\begin{aligned}
r\frac{d}{dr}I_p(r,f) &= p\int_{\mathbf{D}_r}\Big[\Big(\frac{p}{2}-1\Big)|f(z)|^{p-4}|f_z(z)\overline{f(z)}+f(z)\overline{f_{\overline{z}}(z)}|^2 \\
&\quad + |f(z)|^{p-2}|\nabla f(z)|^2\Big]\,dA(z) \\
&\ge p(p-1)\int_{\mathbf{D}_r} |f(z)|^{p-2}|\nabla f(z)|^2\,dA(z)
\end{aligned}$$

and

$$A_h(r,f) = \int_{\mathbf{D}_r} |\nabla f(z)|^2\,dA(z) \le M^{2-p}(r,f)\int_{\mathbf{D}_r} |f(z)|^{p-2}|\nabla f(z)|^2\,dA(z),$$

which implies that

$$p(p-1)A_h(r,f) \le r\frac{d}{dr}I_p(r,f)M^{2-p}(r,f).$$

By (18) and Hölder's inequality, we see that

$$\begin{aligned}
&\Big[p(p-1)\Big]^{\frac{p}{2}}\int_0^r \left[\frac{A_h(\rho,f)}{\rho}(r-\rho)^{\frac{\delta(2-p)}{p}}\right]^{\frac{p}{2}}\,d\rho \\
&\le \int_0^r (r-\rho)^{\frac{\delta(2-p)}{2}} M^{\frac{p(2-p)}{2}}(r,f)\left(\frac{d}{dr}I_p(r,f)\right)^{\frac{p}{2}}\,d\rho \\
&\le \left[\int_0^r (r-\rho)^\delta M^p(r,f)d\rho\right]^{\frac{2-p}{2}} \left(I_p(r,f)-I_p(0,f)\right)^{\frac{p}{2}} \\
&\le \Big(\frac{2}{\delta}\Big)^{\frac{2-p}{2}} I_p(r,f),
\end{aligned}$$

which yields

$$f \in \mathscr{H}_h^p(\mathbf{D}) \Rightarrow \int_0^1 A_h^{\frac{p}{2}}(r,f)(1-r)^{\frac{\delta(2-p)}{2}}\,dr < +\infty.$$

The proof of the theorem is completed. □

Proof of Theorem 4. It is not difficult to see that

$$A_h^{\frac{p}{2}}(r,f)\int_r^1 (1-\rho)^{\frac{\delta(2-p)}{2}}\,d\rho \le \int_r^1 (1-\rho)^{\frac{\delta(2-p)}{2}} A_h^{\frac{p}{2}}(\rho,f)\,d\rho$$

which implies that

$$\frac{2}{2+\delta(2-p)}(1-r)^{\frac{2+\delta(2-p)}{2}} A_h^{\frac{p}{2}}(r,f) \le \int_r^1 (1-\rho)^{\frac{\delta(2-p)}{2}} A_h^{\frac{p}{2}}(\rho,f)\,d\rho.$$

By Theorem 3, we conclude that

$$\int_0^1 (1-\rho)^{\frac{\delta(2-p)}{2}} A_h^{\frac{p}{2}}(\rho,f)\,d\rho < +\infty$$

which gives

$$\lim_{r\to 1-}(1-r)^{\frac{\delta(2-p)+2}{p}} A_h(r,f) = 0$$

and the proof of the theorem is completed. □

4. Landau's theorem

Proof of Theorem 5. By assumption and the inequaltiy (17), we have

$$|f(z)| \le \frac{2^{\frac{1}{p}}M}{(1-|z|)^{\frac{1}{p}}}, \quad z \in \mathbf{D}.$$

Set $f(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$, and, for $\zeta \in \mathbf{D}$, let $F(\zeta) = f(r\zeta)/r$ so that

$$F(\zeta) = \sum_{n=1}^{\infty} A_n \zeta^n + \sum_{n=1}^{\infty} \overline{B}_n \overline{\zeta}^n,$$

where $A_n = a_n r^{n-1}$ and $B_n = b_n r^{n-1}$. Then $F(0) = \lambda_F(0) - 1 = 0$ and

$$|F(\zeta)| \le \frac{2^{\frac{1}{p}}M}{r(1-r)^{\frac{1}{p}}} = M(r) \ \text{ for } \zeta \in \mathbf{D}.$$

By [2, Lemma 1], for $n \in \{2,3,\cdots\}$, we have

$$|A_n| + |B_n| \le \frac{4M(r)}{\pi}. \tag{19}$$

To prove the univalence of F, we choose two distinct points ζ_1, $\zeta_2 \in \mathbf{D}_{\rho_1(r)}$ and let $\zeta_1 - \zeta_2 = |\zeta_1 - \zeta_2|e^{i\theta}$, where

$$\rho_1(r) = 1 - \sqrt{\frac{t}{1+t}},$$

with $t = \frac{4M(r)}{\pi}$. Then (19) yields that

$$\begin{aligned}|F(\zeta_2) - F(\zeta_1)| &= \left|\int_{[\zeta_1,\zeta_2]} F_\zeta(\zeta)\, d\zeta + F_{\overline{\zeta}}(\zeta)\, d\overline{\zeta}\right| \\ &\geq \left|\int_{[\zeta_1,\zeta_2]} F_\zeta(0)\, d\zeta + F_{\overline{\zeta}}(0)\, d\overline{\zeta}\right| \\ &\quad - \left|\int_{[\zeta_1,\zeta_2]} (F_\zeta(\zeta) - F_\zeta(0))\, d\zeta + (F_{\overline{\zeta}}(\zeta) - F_{\overline{\zeta}}(0))\, d\overline{\zeta}\right| \\ &> |\zeta_1 - \zeta_2| \big[\lambda_F(0) - \sum_{n=2}^{\infty}(|A_n| + |B_n|) n \rho_1^{n-1}(r)\big] \\ &\geq |\zeta_1 - \zeta_2| \left[1 - \frac{4M(r)}{\pi} \cdot \frac{\rho_1(r)(2-\rho_1(r))}{(1-\rho_1(r))^2}\right] \geq 0.\end{aligned}$$

Here in the last step we use the fact that

$$1 - \frac{4M(r)}{\pi} \cdot \frac{\rho_1(r)(2-\rho_1(r))}{(1-\rho_1(r))^2} = 0, \ \text{ i.e. } \ \rho_1(r) = \frac{\varphi(r)}{r}.$$

Thus, $F(\zeta_2) \neq F(\zeta_1)$. The univalence of F follows from the arbitrariness of ζ_1 and ζ_2. This implies that f is univalent in $\mathbf{D}_{r\rho_1(r)}$.

Now, for any $\zeta' = \rho_1(r) e^{i\theta} \in \partial \mathbf{D}_{\rho_1(r)}$, we easily obtain that

$$\begin{aligned}|F(\zeta')| &\geq \rho_1(r) - \sum_{n=2}^{\infty}(|A_n| + |B_n|)\rho_1^n(r) \geq \rho_1(r) - \sum_{n=2}^{\infty} \frac{4M(r)}{\pi} \rho_1^n(r) \\ &= \rho_1(r)\left[1 - \frac{4M(r)}{\pi} \frac{\rho_1(r)}{1-\rho_1(r)}\right] = \frac{R_0}{r},\end{aligned}$$

where the last step is a consequence of the expression for ρ_1 given by $\rho_1(r) = r^{-1}\varphi(r)$. Therefore, $f(\mathbf{D}_{r\rho_1(r)})$ contains a univalent disk of radius R_0. □

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