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PERTURBATION CLASSES FOR SEMI-FREDHOLM OPERATORS ON SUBPROJECTIVE AND SUPERPROJECTIVE SPACES

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Abstract. We prove that the component $P\Phi_+(X,Y)$ of the perturbation class for the upper semi-Fredholm operators between Banach spaces X and Y coincide with the strictly singular operators when every closed infinite dimensional subspace of X contains an infinite dimensional complemented subspace whose complement is isomorphic to X. Similarly, we prove that the component $P\Phi_-(X,Y)$ of the perturbation class for the lower semi-Fredholm operators coincide with the strictly cosingular operators when every infinite codimensional subspace of Y is contained in an infinite codimensional complemented subspace isomorphic to Y. We also give examples of Banach spaces satisfying the aforementioned conditions.

1. Introduction

The perturbation classes problem arises in the study of the stability of Fredholm and semi-Fredholm operators under additive perturbations. Let $\mathscr{L}(X,Y)$ denote the (continuous linear) operators between the Banach spaces X and Y. An operator $T \in \mathscr{L}(X,Y)$ is said to be *upper semi-Fredholm* ($T \in \Phi_+$) if its kernel $N(T)$ is finite dimensional and its range $R(T)$ is closed; T is said to be *lower semi-Fredholm* ($T \in \Phi_-$) if its range is closed and finite codimensional, and T is said to be *Fredholm* ($T \in \Phi$) if it is both upper and lower semi-Fredholm. Let $\mathscr{A}$ be any of the classes Φ_+, Φ_- or Φ. The *perturbation class* of $\mathscr{A}$ is defined by its components in $\mathscr{L}(X,Y)$, when $\mathscr{A}(X,Y)$ is non-empty:

$$P\mathscr{A}(X,Y) := \{K \in \mathscr{L}(X,Y) \colon K + T \in \mathscr{A}(X,Y) \text{ for all } T \in \mathscr{A}(X,Y)\}.$$

The components $P\mathscr{A}(X,Y)$ were studied in [17] in the case $X = Y$ and in [2] in the general case. It was proved in [22] that $P\Phi$ coincides with the inessential operators $\mathscr{I}n$ when it is defined, but the perturbation classes $P\Phi_+$ and $P\Phi_-$ have been identified only in a few cases. Kato showed that the strictly singular operators $\mathscr{S}\mathscr{S}$ are contained in $P\Phi_+$ [16, Theorem 5.2], Vladimirskii proved that the strictly

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cosingular operators $\mathscr{SC}$ are contained in $P\Phi_-$ [24, Corollary 1], and it is a consequence of the continuity of the index for semi-Fredholm operators that both $P\Phi_+$ and $P\Phi_-$ are contained in $\mathscr{I}n$ (see [5, Theorem 5.6.9]).

Recall that an operator $T\colon X \longrightarrow Y$ is in $\mathscr{SS}$ if its restriction $T|_E$ is an isomorphism for no infinite dimensional subspace E; $T \in \mathscr{SC}$ if $Q_F T$ is surjective for no infinite codimensional closed subspace F of Y, where $Q_F\colon Y \longrightarrow Y/F$ is the quotient operator, and $T \in \mathscr{I}n$ if for every $A \in \mathscr{L}(Y,X)$, $I_X - AT \in \Phi$.

The *perturbation classes problem* asks whether $\mathscr{SS}$ and $\mathscr{SC}$ coincide with $P\Phi_+$ and $P\Phi_-$ respectively. This problem was formulated by Gohberg, Markus and Feldman [11, p. 74]) for the upper semi-Fredholm operators. Later, it was explicitly stated in [5, page 101], [22, 26.6.12], [23, Section 3] and [3]. Finally, it was proved in [12] that there exists a complex separable Banach space Z for which $P\Phi_+(Z) \neq \mathscr{SS}(Z)$ and $P\Phi_-(Z^*) \neq \mathscr{SC}(Z^*)$. However, there is still interest in finding spaces X and Y for which $P\Phi_+(X,Y) = \mathscr{SS}(X,Y)$ or $P\Phi_-(X,Y) = \mathscr{SC}(X,Y)$ because these results provide intrinsic characterizations of the operators K in the respective classes; i.e., characterizations involving the action of K instead of the properties of the sums of K with all the operators in $\Phi_+(X,Y)$ or $\Phi_-(X,Y)$. Moreover, the aforementioned space Z of [12] is certainly special: it is a finite product of hereditarily indecomposable spaces. The existence of hereditarily indecomposable Banach spaces was only recently proved in [15]. So the perturbation classes problem still remains open for many classical Banach spaces.

Provided $\Phi_+(X,Y) \neq \emptyset$, we have $P\Phi_+(X,Y) = \mathscr{SS}(X,Y)$ in the following cases:

(1) Y subprojective [17, 1];
(2) $X = Y = L_p(\mu)$, $1 \le p \le \infty$ [25];
(3) X hereditarily indecomposable [1, Theorem 3.14];
(4) X is separable and Y contains a complemented copy of $C[0,1]$ [3];
(5) $X = L_p(0,1)$ when $1 < p < 2$ and Y satisfies the Orlicz property [14];
(6) $X = L_1(0,1)$ and Y is weakly sequentially complete [14];
(7) $X = L_p(0,1)$ with $2 \le p \le \infty$ [14].

Also, provided $\Phi_-(X,Y) \neq \emptyset$, we have $P\Phi_-(X,Y) = \mathscr{SC}(X,Y)$ in the following cases:

(1') X superprojective [17, 1];
(2') $X = Y = L_p(\mu)$, $1 \le p \le \infty$ [25];
(3') Y quotient indecomposable [1, Theorem 3.14];
(4') X contains a complemented copy of ℓ_1 and Y is separable [3];
(5') $Y = L_p(0,1)$ when $2 < p < \infty$ and X^* satisfies the Orlicz property [14];
(6') $Y = L_p(0,1)$ with $1 \le p \le 2$ [14].

In this paper, we introduce the notions of *strongly subprojective* and *strongly superprojective* Banach space, which strengthen those of subprojective and superprojective Banach space introduced in [26]. We remark that all known examples of subprojective spaces and superprojective spaces are respectively strongly subprojective and strongly superprojective. Next, we prove that if X is strongly superprojective, then $P\Phi_+(X,Y) = \mathscr{SS}(X,Y)$ for all spaces Y (Theorem 2.6), and if Y is strongly superprojective, then $P\Phi_-(X,Y) = \mathscr{SC}(X,Y)$ for all spaces X (Theorem 3.7).

We point out that although Theorem 3.7 is a certain dual form of Theorem 2.6, its proof does not follow by duality from it. This is because, given $T \in \mathscr{L}(X,Y)$, the implications $T^* \in \mathscr{SS} \Rightarrow T \in \mathscr{SC}$ and $T^* \in \mathscr{SC} \Rightarrow T \in \mathscr{SS}$ hold but their converses fail. See [21, Examples 1 and 2]. Moreover, the proof of Theorem 3.7 is technically more complicated than that of Theorem 2.6 because the former one involves quotients instead of subspaces.

2. Operators on strongly subprojective spaces

A Banach space X is said to be *subprojective* if every infinite dimensional closed subspace M of X contains an infinite dimensional subspace N complemented in X. Clearly, a closed subspace of a subprojective space is also subprojective. This concept was introduced by Whitley [26]. Here we consider a strengthening of it.

Definition 2.1. A Banach space X is said to be *strongly subprojective* if every infinite dimensional closed subspace M of X contains an infinite dimensional subspace N complemented in X with complement isomorphic to X.

The following remark will allow us to show that all the known examples of subprojective spaces (see Proposition 2.4) are strongly subprojective.

Remark 2.2. If the subspace N in the definition of subprojective space can be taken isomorphic to its square ($N \simeq N \times N$) then X is strongly subprojective.

Proof. Let M be an infinite dimensional closed subspace of a subprojective space X. Then there exist closed subspaces N and H of X such that $X = N \oplus H$ and $N \subset M$. By hypothesis, N contains two closed subspaces N_1 and N_2 such that $N \simeq N_1 \simeq N_2$ and $N = N_1 \oplus N_2$. Therefore, N_1 is a subspace of M, $N_2 \oplus H \simeq X$ and

$$X = N_1 \oplus (N_2 \oplus H),$$

which proves that X is strongly subprojective. □

Remark 2.3. Recall that a compact space K is said to be *scattered* (or dispersed) if every non-empty subset of K has an isolated point. As examples, we mention:

(1) Let κ be any ordinal. The interval $[0, \kappa] = \{\alpha \text{ ordinal } : 0 \leq \alpha \leq \kappa\}$, endowed with the order topology, is a scattered compact.
(2) Let Γ be a set, endowed with the discrete topology. The one-point compactification Γ_∞ is a scattered compact and $C(\Gamma_\infty)$ is isomorphic to $c_0(\Gamma)$.

Note that $\ell_p \simeq \ell_p \times \ell_p$ for $1 \leq p < \infty$ and $c_0 \simeq c_0 \times c_0$. Therefore, Remark 2.2 can be applied to obtain the following result.

Proposition 2.4. *The following Banach spaces are strongly subprojective:*

(1) *The sequence spaces ℓ_p for $1 \leq p < \infty$ and c_0.*
(2) *The James space J.*
(3) *The Lorentz sequence spaces $d(w, p)$ for $1 \leq p < \infty$ and $w = (w_n)$ a non-increasing null sequence with $\sum_{n=1}^{\infty} w_n$ divergent. This applies to $\ell_{p,q}$ for $1 \leq p, q < \infty$.*
(4) *The Baernstein spaces B_p for $1 < p < \infty$.*
(5) *The Tsirelson space T.*
(6) *The function spaces $L_p(0, 1)$ for $2 \leq p < \infty$.*
(7) *The function spaces $L_p(0, \infty) \cap L_2(0, \infty)$ for $1 \leq p \leq 2$.*
(8) *The Lorentz spaces $\Lambda_{W,p}(0, 1)$, $L_{p,q}(0, \infty)$ and $L_{p,q}(0, 1)$ for $2 < p < \infty$ and $1 \leq q < \infty$.*

(9) *The spaces of continuous functions $C(K)$, with K a scattered compact.*

(10) *Closed subspaces of the previous examples.*

Proof. (1) Denoting by X any of these spaces, every infinite dimensional closed subspace of X contains a subspace isomorphic to X and complemented in X [4, Proposition 2.2.1].

(2) Every infinite dimensional closed subspace of J contains a subspace isomorphic to ℓ_2 and complemented in J [9, Corollary 2.d.4].

(3) Every infinite dimensional closed subspace of $d(w,p)$ contains a subspace isomorphic to ℓ_p and complemented in $d(w,p)$ [18, Proposition 4.e.3].

(4) Every infinite dimensional closed subspace of B_p contains a subspace isomorphic to ℓ_p and complemented in B_p [7, Theorem 0.15].

(5) Let $\{t_n\}$ denote the unit basis of T. By [7, Proposition II.7], every closed subspace of T contains a subspace N complemented in T and isomorphic to the closed subspace generated by a subsequence of the basis $\{t_n\}$. Moreover, [7, Proposition I.12] ensures that $N \simeq N \times N$.

(6) Every infinite dimensional closed subspace of $L_p(0,1)$ is either isomorphic to ℓ_2 and complemented, or contains a subspace isomorphic to ℓ_p and complemented in $L_p(0,1)$ [4, Corollary 6.4.9].

(7) The argument given in (6) applies in this case [8, Theorem 4.1].

(8) The argument given in (6) applies for $\Lambda_{W,p}(0,1)$ and $L_{p,q}(0,\infty)$. See [10, Remark 5.7] and [6, Theorem 2.5]. For $L_{p,q}(0,1)$, the result follows from (10), since $L_{p,q}(0,1)$ is a closed subspace of $L_{p,q}(0,\infty)$.

(9) Every infinite dimensional closed subspace of $C(K)$ contains a subspace isomorphic to c_0 and complemented in $C(K)$ [19, Theorem 11].

(10) Given a pair of closed subspaces M and Z of X with $M \subset Z$, if M is complemented in X, then it is also complemented in Z. □

The next result will be useful later.

Proposition 2.5. *Let X be a strongly subprojective Banach space. Then every finite codimensional closed subspace of X contains a subspace isomorphic to X. Consequently, $\Phi_+(X,Y)$ is non-empty if and only if Y contains a subspace isomorphic to X.*

Proof. Let Z be a closed subspace of X with $\dim X/Z = n$. Since X is strongly subprojective, X contains an infinite codimensional subspace X_0 isomorphic to X. Let Z_0 be a closed n-codimensional subspace of X containing X_0. Since Z and Z_0 are isomorphic, the first assertion is clear.

For the second assertion, note that $\Phi_+(X,Y)$ is non-empty if and only if Y contains a closed subspace isomorphic to a finite codimensional subspace of X. □

Let us give the main result of this section.

Theorem 2.6. *Let X be a strongly subprojective space and let Y be a Banach space. If $\Phi_+(X,Y) \neq \emptyset$ then $P\Phi_+(X,Y) = \mathscr{S}\mathscr{S}(X,Y)$.*

Proof. It is enough to show that, given $K \in \mathscr{L}(X,Y) \setminus \mathscr{S}\mathscr{S}(X,Y)$, there exists $T \in \Phi_+(X,Y)$ such that $T + K \notin \Phi_+$.

Since K is not strictly singular, there exists an infinite dimensional closed subspace V of X such that $K|_V$ is an isomorphism; hence $K|_V \in \Phi_+(V,Y)$. As X is strongly subprojective, we may assume that

$$X = V \oplus X_1 \text{ with } \ X_1 \simeq X.$$

By Proposition 2.5, Y has a closed subspace L isomorphic to X. Taking into consideration the relative positions of the subspaces $K(V)$ and L inside Y, three cases may happen:

(a) $K(V) \cap L$ finite dimensional and $K(V) + L$ closed;
(b) $K(V) \cap L$ is infinite dimensional;
(c) $K(V) \cap L$ finite dimensional and $K(V) + L$ not closed.

(a) As L is strongly subprojective, by Proposition 2.5, the closed complement of $K(V) \cap L$ in L contains a subspace isomorphic to L. Thus we can assume $K(V) \cap L = \{0\}$.

Let $S\colon X_1 \longrightarrow L$ be a bijective isomorphism. We consider the operator

$$T\colon X = V \oplus X_1 \longrightarrow K(V) \oplus L \subset Y$$

that maps $v + x_1$ to $-K(v) + S(x_1)$, where $v \in V$ and $x_1 \in X_1$. Clearly $T \in \Phi_+$. However $(T + K)|_V = 0$, so $T + K \notin \Phi_+$, and we are done.

(b) Assume $K(V) \cap L$ is infinite dimensional. Since L is strongly subprojective, there exists a closed subspace W contained in V and a closed subspace L_3 in L such that $L_1 := K(W) \cap L$ is infinite dimensional, L_3 is isomorphic to L and $L = L_1 \oplus L_3$. Let $V_1 := (K|_V)^{-1}(L_1)$. By the strong subprojectivity of X, there exist an infinite dimensional closed subspace V_2 of V_1 and a closed subspace X_2 of X such that X_2 is isomorphic to X and $X = V_2 \oplus X_2$. Since $K|_{V_2}$ is an isomorphism and $K(V_2) + L_3$ is closed, we are in the conditions of case (a).

(c) As in case (a), we can assume that $K(V) \cap L = \{0\}$ and $K(V) + L$ not closed. In order to prove that $K \notin P\Phi_+(X, Y)$, it is enough to find a compact operator $K_1 \in \mathscr{L}(X, Y)$ so that $\dim\,(K + K_1)(V) \cap L = \infty$; indeed, once the operator K_1 has been found, since $(K + K_1)|_V \in \Phi_+(V, Y)$, the operator $K + K_1$ satisfies the conditions of case (b), which leads to $K + K_1 \notin P\Phi_+(X, Y)$, and therefore $K \notin P\Phi_+(X, Y)$.

In order to find that operator K_1, since $K(V) + L$ is not closed, there exists a normalized sequence (y_n) in $K(V)$ with $\operatorname{dist}(y_n, L) \underset{n}{\longrightarrow} 0$. If (y_n) has a subsequence weakly convergent to some $y \in Y$, since $y \in K(V)$, we may choose a sequence $(u_n) \subset L$ so that $\|u_n - y_n\| \underset{n}{\longrightarrow} 0$, so $u_n \overset{w}{\underset{n}{\longrightarrow}} y \in L$, hence $y = 0$. Therefore, [4, Theorem 1.5.6] implies that (y_n) contains a basic subsequence, and taking a bounded sequence $(v_n) \subset V$ such that $y_n = K(v_n)$ and passing to a subsequence if necessary, we may assume that both (y_n) and (v_n) are basic sequences.

Since the sequence (v_n) is basic and $\inf_n \|v_n\| > 0$, there exists a bounded sequence $(f_n) \subset X^*$ such that $\langle f_i, v_j\rangle = \delta_{ij}$. But $\operatorname{dist}(y_n, L) \underset{n}{\longrightarrow} 0$, so we can pick a sequence $(z_n) \subset L$ and a subsequence (y_{k_n}) of (y_n) so that $\sum_{n=1}^{\infty} \|y_{k_n} - z_n\| < \infty$. Hence, the expression

$$K_1(x) := \sum_{n=1}^{\infty} \langle f_{k_n}, x\rangle (z_n - y_{k_n})$$

defines a compact operator $K_1 \in \mathscr{L}(X, Y)$ that satisfies $(K + K_1)(v_{k_n}) = z_n$. Since $(K + K_1)|_V$ is upper semi-Fredholm and $z_n \in (K + K_1)(V) \cap L$ for every n, $(K + K_1)(V) \cap L$ is infinite dimensional, as we wanted to prove. □

Since all known examples of subprojective spaces are strongly subprojective, the following result implies that, in most of the cases, Theorem 2.6 is not a consequence of assertion (1) in the introduction.

Proposition 2.7. *Suppose that Y is subprojective and $\Phi_+(X,Y)$ is not empty. Then X is subprojective.*

Proof. It is enough to observe that $\Phi_+(X,Y) \neq \emptyset$ implies that a finite codimensional closed subspace of X is isomorphic to a subspace of Y. □

3. Operators into strongly superprojective spaces

Superprojectivity is the dual notion to subprojectivity. A Banach space X is said to be *superprojective* if every infinite codimensional closed subspace H of X is contained in an infinite codimensional complemented subspace E of X.

Definition 3.1. A Banach space X is said to be *strongly superprojective* if every infinite codimensional closed subspace H of X is contained in an infinite codimensional closed subspace E isomorphic to X and complemented in X.

The proof of the following result is similar to that of Remark 2.2.

Remark 3.2. If the complement of the subspace E in the definition of superprojective space can be taken isomorphic to its square, then X is strongly superprojective.

Some examples of strongly superprojective Banach spaces are obtained through duality:

Proposition 3.3. *Let X be a reflexive Banach space. Then X is strongly subprojective if and only if X^* is strongly superprojective.*

Proof. Assume X is a reflexive strongly subprojective space and let M be an infinite codimensional closed subspace of X^*. Thus, as $M^\perp$ is an infinite dimensional subspace of X, it contains an infinite dimensional complemented subspace N with $X/N \simeq X$. Hence $N^\perp$ is an infinite codimensional complemented subspace of X^* isomorphic to X^* that contains M. Therefore, X^* is strongly superprojective.
The proof of the converse implication is similar. □

Observe that Proposition 3.3 is also true for superprojective and subprojective spaces.
In the following result, we list some examples of strongly superprojective spaces. Given $1 < p < \infty$, p^* denotes the only real number satisfying $1/p + 1/p^* = 1$.

Proposition 3.4. *The following Banach spaces are strongly superprojective:*

1. *The sequence spaces ℓ_p for $1 < p < \infty$ and c_0.*
2. *The dual J^* of James' space.*
3. *The dual spaces $d(w,p)^*$ of $d(w,p)$ for $1 < p < \infty$ and $w = (w_n)$ a non-increasing null sequence with $\sum_{n=1}^\infty w_n$ divergent. This applies to $\ell_{p,q}^*$ for $1 < p, q < \infty$.*
4. *The dual spaces B_p^* of Baernstein's spaces for $1 < p < \infty$.*
5. *The dual T^* of Tsirelson's space.*
6. *The function spaces $L_p(0,1)$ for $1 < p \le 2$.*
7. *The function spaces $L_p(0,\infty) + L_2(0,\infty)$ for $2 \le p < \infty$.*

(8) *The dual spaces* $\Lambda_{W,p}(0,1)^*$, $L_{p,q}(0,\infty)^*$ *and* $L_{p,q}(0,1)^*$ *for* $2 < p < \infty$ *and* $1 < q < \infty$.

(9) *The spaces of continuous functions* $C(K)$, *with* K *a scattered compact.*

(10) *Quotients of the previous examples.*

Proof. (1) The result for ℓ_p follows from Propositions 2.4 and 3.3 and, by Remark 2.3, the result for c_0 is a special case of (9).

(2) Although J is non-reflexive, since $J \simeq J^{**}$ and $\dim J^{**}/J = 1$, the arguments in the proof of Proposition 3.3 allow us to show that J strongly subprojective implies J^* strongly superprojective.

(3) to (8) In these cases we consider dual spaces of reflexive strongly subprojective spaces (see Proposition 2.4); therefore they are strongly superprojective by Proposition 3.3. Note that

- $d(w,p)$ is reflexive if and only if $1 < p < \infty$ [18, page 178];
- B_p is reflexive for $1 < p < \infty$ [7, Theorem 0.15];
- Tsirelson's space T is reflexive [18, Theorem 1.c.12] and [7, Theorem I.8];
- for $2 \le p < \infty$, $L_p(0,\infty) + L_2(0,\infty)$ is the dual of $L_{p^*}(0,\infty) \cap L_2(0,\infty)$ and these spaces are reflexive [8, Theorem 3.1];
- the spaces $\Lambda_{W,p}(0,1)$ and $L_{p,q}(0,\infty)$ are reflexive for $1 < p, q < \infty$ [10, p. 406].

(9) Let K be a scattered compact and let M be a closed infinite codimensional subspace M of $C(K)$. By [20, Theorem 4.2], $C(K)/M$ has a quotient isomorphic to c_0 or to ℓ_2. In other words, $C(K)$ has a closed subspace A with $M \subset A$ such that $C(K)/A$ is isomorphic to c_0 or to ℓ_2. But K is scattered, so $C(K)^*$ has no copy of ℓ_2 because $C(K)^* \equiv \ell_1(K)$; therefore, $C(K)/A$ must be isomorphic to c_0.

Consider the quotient operator $Q_A\colon C(K) \longrightarrow C(K)/A$. Since $C(K)$ has the Pełczyński property, there exists a subspace F of $C(K)$ isomorphic to c_0 such that $Q_A|_F$ is an isomorphism. Observe that $Q_A(F)$ is complemented in $C(K)/A \simeq c_0$. So we can write $C(K)/A = Q_A(F) \oplus N$ for some closed subspace N. Hence $C(K) = F \oplus Q_A^{-1}(N)$. We have proved that M is contained in a complemented infinite codimensional subspace. Thus $C(K)$ is superprojective. Since F is isomorphic to c_0 we have $F \simeq F \times F$, so Remark 3.2 shows that $C(K)$ is strongly superprojective.

(10) It is enough to prove that quotients of superprojective spaces are superprojective. Let M be a closed subspace of X and let $Q_M\colon X \longrightarrow X/M$ be the quotient map. Given an infinite codimensional closed subspace A of X/M, $Q_M^{-1}(A)$ is closed an infinite codimensional in X and $A \simeq Q_M^{-1}(A)/M$. Moreover, if B is an infinite codimensional complemented subspace of X containing $Q_M^{-1}(A)$, then $Q_M(B)$ is an infinite codimensional complemented subspace of X/M containing A. □

The next two results will be needed later.

Lemma 3.5. *Let* $K \in \mathscr{L}(X,Y)$ *be an operator and* Y_0 *be a closed subspace of* Y *such that* $Q_{Y_0}K$ *is surjective. If* E *is a closed subspace of* X *such that* $K^{-1}(Y_0) \subset E$, *then* Y *contains a closed subspace* F *such that* $Y_0 \subset F$ *and* $E = K^{-1}(F)$. *Moreover, if* E *is infinite codimensional in* X *then* F *is infinite codimensional in* Y.

Proof. Consider the surjective isomorphism $U\colon X/N(Q_{Y_0}K) \longrightarrow Y/Y_0$ induced by $Q_{Y_0}K$.

Let E be any closed subspace of X such that $E \supset K^{-1}(Y_0)$. The desired subspace F is $Q_{Y_0}^{-1}\, U\, Q_{K^{-1}(Y_0)}(E)$. Indeed, the facts that F is closed, F contains Y_0 and $E =$

$K^{-1}(F)$ are straightforward. Moreover, if E is infinite codimensional in X, then $Q_{Y_0}^{-1} U Q_{K^{-1}(Y_0)}(E)$ is infinite codimensional in Y. □

Proposition 3.6. *Let Y be a strongly superprojective space. Then every quotient of Y by a finite dimensional subspace has a quotient isomorphic to Y. Therefore, $\Phi_-(X,Y)$ is not empty if and only if X has a quotient isomorphic to Y.*

Proof. Let Z be a finite dimensional subspace of Y. As Y is strongly superprojective, there is a closed infinite dimensional subspace Y_0 of Y such that $Y/Y_0 \simeq Y$. Let F be any subspace of Y_0 with $\dim F = \dim Z$. Then

$$\frac{Y/F}{Y_0/F} \simeq \frac{Y}{Y_0} \simeq Y,$$

and as $Y/Z \simeq Y/F$, the first assertion follows easily.

For the second assertion, the 'if' part is trivial. For the reverse, given $T \in \Phi_-(X,Y)$, there exists a finite dimensional subspace N of Y such that

$$\frac{Y}{N} \simeq R(T) \simeq \frac{X}{N(T)},$$

thus, an application of the first assertion finishes the proof. □

Next theorem is the main result of this section.

Theorem 3.7. *Let Y be a strongly superprojective space and let X be a Banach space such that $\Phi_-(X,Y) \neq \emptyset$. Then $P\Phi_-(X,Y) = \mathscr{SC}(X,Y)$.*

Proof. It is enough to show that, given $K \in \mathscr{L}(X,Y) \setminus \mathscr{SC}(X,Y)$, there exists $T \in \Phi_-(X,Y)$ such that $T + K \notin \Phi_-$. In order to do that, let $K\colon X \longrightarrow Y$ be a non-strictly cosingular operator. Thus there exists a closed subspace $Y_0 \subset Y$ with $\dim Y/Y_0 = \infty$ such that $Q_{Y_0}K$ is surjective, where Q_{Y_0} is the quotient operator onto Y/Y_0. Obviously, $R(K) + Y_0 = Y$.

Since Y is strongly superprojective, the space Y_0 can be assumed to be isomorphic to Y and complemented in Y. Thus $Y = Y_0 \oplus N$ with $\dim N = \infty$.

Let $P\colon Y \longrightarrow Y$ be the projection with $N(P) = Y_0$ and $R(P) = N$. Thus

$$R(PK) = P\big(R(K) + Y_0\big) = N,$$
$$N(PK) = K^{-1}(Y_0).$$

As $\Phi_-(X,Y) \neq \emptyset$, Proposition 3.6 provides a closed subspace M of X such that $X/M \simeq Y$. Hence, as $Y \simeq Y_0$, there exists $S \in \mathscr{L}(X,Y)$ such that $N(S) = M$ and $R(S) = Y_0$.

Taking into account the relative positions of $K^{-1}(Y_0)$ and M in X, three cases occur:

(a) $K^{-1}(Y_0) + M$ is closed and finite codimensional in X,
(b) $\overline{K^{-1}(Y_0) + M}$ is infinite codimensional in X,
(c) $\overline{K^{-1}(Y_0) + M}$ is finite codimensional in X but $K^{-1}(Y_0) + M$ is not closed.

(a) Let M_1 be a subspace of X containing M such that $\dim M_1/M < \infty$ and $K^{-1}(Y_0) + M_1 = X$. Since X/M is superprojective, by Proposition 3.6, there exists a closed subspace M_2 containing M_1 such that $X/M_2 \simeq Y$. Thus we can assume $K^{-1}(Y_0) + M = X$.

Let $T := S - PK$. Observe that $R(T) = Y$. Indeed, given $y \in Y$, decompose

$$y = y_0 + y_1 \text{ with } y_0 \in Y_0 \text{ and } y_1 \in N.$$

We will show that

$$y_0 = S(x_0) \text{ for some } x_0 \in K^{-1}(Y_0), \tag{1}$$

$$y_1 = PK(x_1) \text{ for some } x_1 \in M. \tag{2}$$

In order to prove (1), take $x \in X$ such that $y_0 = S(x)$ and consider any decomposition $x = x_0 + x_0'$ with $x_0 \in K^{-1}(Y_0)$ and $x_0' \in M$. Thus $x_0' \in N(S)$, hence $y_0 = S(x_0)$.

For (2), as $N = R(PK)$, there exists $x \in X$ such that $y_1 = PK(x)$. Take any decomposition $x = x_1 + x_1'$ with $x_1 \in M$ and $x_1' \in K^{-1}(Y_0)$. Since $K(x_1') \in Y_0 = N(P)$, it follows that $y_1 = PK(x_1)$, and (2) is proved.

Finally, formulas (1) and (2) yield that $x_0 \in N(PK)$ and $x_1 \in N(S)$, hence $y = (S - PK)(x_0 - x_1)$. We have just proved that T is surjective, hence, $T \in \Phi_-(X, Y)$.

However, $R(T + K) = R(S + (I_X - P)K) \subset Y_0$, so $T + K \notin \Phi_-(X, Y)$.

(b) Assume $\overline{K^{-1}(Y_0) + M}$ is infinite codimensional. Thus $(\overline{K^{-1}(Y_0) + M})/M$ is an infinite codimensional subspace of X/M. But X/M is isomorphic to Y, so it is strongly superprojective, hence there exists a closed infinite codimensional subspace X_1 of X such that $\overline{K^{-1}(Y_0) + M} \subset X_1$, $X_1/M \simeq Y$ and

$$\frac{X}{M} = \frac{X_1}{M} \oplus \frac{E}{M}$$

for some subspace $M \subset E \subset X$ with $\dim E/M = \infty$. Lemma 3.5 provides a closed subspace Y_1 of Y such that $\dim Y/Y_1 = \infty$, $Y_0 \subset Y_1$ and $X_1 = K^{-1}(Y_1)$. Thus

$$K^{-1}(Y_1) + E = X_1 + E = X.$$

Moreover,

$$X/E \simeq \frac{X/M}{E/M} \simeq X_1/M \simeq Y.$$

Therefore, using again that Y is strongly superprojective, there exists a complemented infinite codimensional subspace Y_2 of Y such that $Y_1 \subset Y_2 \subset Y$ and $Y_2 \simeq Y$.

Obviously, $Q_{Y_2}K$ is surjective, $X/E \simeq Y$ and $K^{-1}(Y_2) + E = X$ so we are in the conditions of case (a) (using an operator $S_1 \in \mathscr{L}(X, Y)$ with $N(S_1) = E$ and $R(S_1) = Y_2$, instead of S).

(c) As in the case (a), we can assume that $K^{-1}(Y_0) + M$ is dense but not closed in X. We will find a compact operator $K_1 \in \mathscr{L}(X, Y)$ such that $\overline{(K + K_1)^{-1}(Y_0) + M}$ is infinite codimensional in X. Once K_1 has been found, as $R\big(Q_{Y_0}(K + K_1)\big)$ is finite codimensional in Y/Y_0, there exists a finite rank operator $K_2 \in \mathscr{L}(X, Y)$ such that $Q_{Y_0}(K + K_1 + K_2)$ is surjective and $\overline{(K + K_1 + K_2)^{-1}(Y_0) + M}$ is infinite codimensional in X yet. Hence, applying the argument of (b), we get that $K + K_1 + K_2 \notin P\Phi_-(X, Y)$, and as $K_1 + K_2$ is compact, we can conclude that $K \notin P\Phi_-(X, Y)$.

In order to find K_1, since $K^{-1}(Y_0) + M$ is not closed, it follows $K^{-1}(Y_0)^\perp + M^\perp$ is not closed either; but $X = \overline{K^{-1}(Y_0) + M}$, so

$$\{0\} = K^{-1}(Y_0)^\perp \cap M^\perp. \tag{3}$$

Thus, we may take a normalized sequence $(f_n) \subset K^{-1}(Y_0)^\perp$ such that $\operatorname{dist}(f_n, M^\perp) < 1/2^n$. Take also a sequence (h_n) in $M^\perp$ so that $\|f_n - h_n\| < 1/2^n$.

Note that (f_n) does not have any convergent subsequence; otherwise, if $f_{k_n} \underset{n}{\longrightarrow} f$, then $h_{k_n} \underset{n}{\longrightarrow} f$ too, so $f = 0$ because of (3), a contradiction.

Let f be a weak* cluster point of (f_n). As both subspaces $K^{-1}(Y_0)^\perp$ and $M^\perp$ are weak* closed, (3) yields that $f = 0$. Thus, by [13, Lemma 3.1.19], there is a bounded sequence (x_n) in X and a basic subsequence (f_{k_n}) of (f_n) such that $\langle f_{k_n}, x_m\rangle = \delta_{nm}$.

As $K^{-1}(Y_0)^\perp = K^*(Y_0^\perp)$, we may pick a sequence (g_n) in $Y_0^\perp$ such that $K^*(g_n) = f_{k_n}$; note that (g_n) is bounded because $K^*|_{Y_0^\perp}$ is an isomorphism. Let $y_n := K(x_n)$. Obviously,

$$\langle g_i, y_j\rangle = \langle K^*(g_i), x_j\rangle = \delta_{ij}.$$

Consider the compact operator $K_1\colon X \longrightarrow Y$ given by the expression

$$K_1(x) := \sum_{n=1}^{\infty} \langle h_{k_n} - f_{k_n}, x\rangle y_n.$$

Its conjugate operator is given by $K_1^*(g) = \sum_{n=1}^{\infty} \langle g, y_n\rangle (h_{k_n} - f_{k_n})$. Thus $(K^* + K_1^*)(g_n) = h_{k_n} \in (K^* + K_1^*)(Y_0^\perp) \cap M^\perp$ for all n, which proves that $(K^* + K_1^*)(Y_0^\perp) \cap M^\perp$ is infinite dimensional. But

$$(K^* + K_1^*)(Y_0^\perp) \cap M^\perp = (K + K_1)^{-1}(Y_0)^\perp \cap M^\perp = \overline{(K + K_1)^{-1}(Y_0) + M}^\perp$$

hence $\overline{(K + K_1)^{-1}(Y_0) + M}$ is an infinite codimensional subspace of X, as we wanted to prove. The proof is done. □

Since all known examples of superprojective spaces are strongly superprojective, the following result implies that, in most of the cases, Theorem 3.7 is not a consequence of assertion (1') in the introduction.

Proposition 3.8. *Assume that X is superprojective and $\Phi_-(X, Y)$ is not empty. Then Y is superprojective.*

Proof. It is enough to note that $\Phi_-(X, Y) \neq \emptyset$ implies that a quotient of Y by a finite dimensional subspace is isomorphic to a quotient of X. □

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