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THE EXISTENCE OF A NONTRIVIAL SOLUTION TO A NONLINEAR ELLIPTIC PROBLEM OF LINKING TYPE WITHOUT THE AMBROSETTI–RABINOWITZ CONDITION

Gongbao Li† and Chunhua Wang*

Central China Normal University, School of Mathematics and Statistics
Wuhan 430079, P. R. China; ligb@mail.ccnu.edu.cn

Central China Normal University, School of Mathematics and Statistics
Wuhan 430079, P. R. China; wch5923@yahoo.com.cn

Abstract. In this paper, we study the existence of a nontrivial solution to the following nonlinear elliptic problem:

$$\begin{cases} -\Delta u - a(x)u = f(x,u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{0.1}$$

where Ω is a bounded domain of $\mathbf{R}^N$ and $a \in L^{\frac{N}{2}}(\Omega), N \geq 3$, $f \in C^0(\bar{\Omega} \times \mathbf{R}^1, \mathbf{R}^1)$ is superlinear at $t = 0$ and subcritical at $t = \infty$. Under suitable conditions, (0.1) possesses the so-called linking geometric structure. We prove that the problem (0.1) has at least one nontrivial solution without assuming the Ambrosetti–Rabinowitz condition. Our main result extends a recent result of Miyagaki and Souto given in [14] for (0.1) with $a(x) = 0$ and possessing the mountain-pass geometric structure.

1. Introduction and main result

In this paper, we study the existence of nontrivial solutions to the following problem:

$$\begin{cases} -\Delta u - a(x)u = f(x,u), & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain, $a(x) \in L^{\frac{N}{2}}(\Omega), N \geq 3$, $f \in C^0(\bar{\Omega} \times \mathbf{R}^1, \mathbf{R}^1)$ and (1.1) possesses the so-called linking geometric structure.

We first recall something about the eigenvalues of elliptic operators. According to the theory of spectrum of compact operators (see e.g. Ch. 4 of [3], or Lemma 2.13 in this paper), we let

$$-\infty < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$$

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†Corresponding author.

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be the sequence of all eigenvalues of the following eigenvalue problem

$$\begin{cases} -\Delta u - a(x)u = \lambda u, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where each eigenvalue is repeated according to its multiplicity, $\lim\limits_{j\to\infty} \lambda_j = +\infty$ and let $e_1, e_2, \ldots, e_n, \ldots$ be the corresponding eigenfunctions in $H_0^1(\Omega)$ normalized in the sense of $L^2(\Omega)$, that is,

$$\int_\Omega e_i e_j \, dx = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

hence for any i and j we have

$$\int_\Omega [\nabla e_j \cdot \nabla e_i - a(x) e_j e_i] \, dx = \lambda_j \int_\Omega e_i e_j \, dx = \lambda_j \delta_{ij}.$$

In this paper, we study the case when (1.1) possesses the so-called linking geometric structure, so we assume that $\lambda_1 \leq 0$, and there exists an $n \in N$ such that

$$\lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leq \lambda_n \leqslant 0 < \lambda_{n+1} \leqslant \cdots . \tag{1.3}$$

To recall the history, we list some conditions which may be imposed on $f(x,t)$.

(f_1) $f \in C^0(\bar{\Omega} \times \mathbf{R}^1, \mathbf{R}^1)$, $f(x,0) = 0$, $\lim\limits_{t\to 0} \dfrac{f(x,t)}{|t|^{p-2}t} = 0$ uniformly in $x \in \Omega$.

(f_2) There are positive constants a and b such that

$$|f(x,t)| \leq a + b|t|^{q-1}, \ \forall (x,t) \in \Omega \times \mathbf{R}^1,$$

where $q \in [1, p^*[$, $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = +\infty$ if $p \geq N$.

(f_3) $\lim\limits_{|t|\to+\infty} \dfrac{F(x,t)}{|t|^p} = +\infty$ uniformly in $x \in \Omega$, where $F(x,t) \triangleq \displaystyle\int_0^t f(x,s)\, ds$.

(f_4) There exists a constant $C_* > 0$ such that

$$H(x,t) \leq H(x,s) + C_*$$

for each $x \in \Omega$, $0 < t < s$ or $s < t < 0$ where $H(x,t) \triangleq tf(x,t) - pF(x,t)$ and $F(x,t) = \displaystyle\int_0^t f(x,s) ds$.

(f_4') There exist a positive constant $s_0 > 0$ such that $\dfrac{f(x,s)}{|s|^{p-2}s}$ is nondecreasing in $s \geq s_0$, and nonincreasing in $s \leq -s_0$ for any $x \in \Omega$.

(f_5) $\frac{\lambda_n}{2} t^2 \leq F(x,t), \ \forall (x,t) \in \bar{\Omega} \times \mathbf{R}^1$.

(f_5') $\lim\limits_{|t|\to+\infty} \dfrac{f(x,t)}{|t|^{p-2}t} = +\infty$ uniformly in $x \in \Omega$.

(f_5'') $\lim\limits_{t\to+\infty} \dfrac{f(x,t)}{t^{p-1}} = +\infty$ uniformly in $x \in \Omega$.

(f_6) $\dfrac{f(x,t)}{|t|^{p-2}t}$ is nondecreasing in $t \geq 0$ for any $x \in \Omega$.

(f_7) There exists a positive constant s_0 such that $H(x,t) \triangleq tf(x,t) - pF(x,t)$ is nondecreasing in $t \geq s_0$ and nonincreasing in $t \leq -s_0$.

If $p = 2, (f_1)$ and (f_2) hold, we can define weak solutions to (1.1). We say that $u \in H_0^1(\Omega)$ is a weak solution to (1.1) if

$$\int_\Omega [\nabla u \cdot \nabla v - a(x)uv]\,dx = \int_\Omega f(x,u)v\,dx,\ \forall v \in H_0^1(\Omega).$$

By hypothesis (f_1), we see that $f(x,0) = 0$, so $u \equiv 0$ is a trivial solution of (1.1). We are interested in getting nontrivial solutions to (1.1).

Let $g(x,t) = a(x)t + f(x,t)$, then problem (1.1) can be written as:

$$\begin{cases} -\Delta u = g(x,u), & x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1.4}$$

Problem (1.4) is a special case of the following p-Laplacian type problem:

$$\begin{cases} -\Delta_p u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

where $p > 1$, $\Omega \subset \mathbf{R}^N$ is a bounded domain and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u.

The problem (P) is one of the main nonlinear elliptic problems which has been studied extensively for many years. Since Ambrosetti and Rabinowitz proposed the mountain-pass theorem in 1973 (see [1]), critical point theory has become one of the main tools for finding solutions to elliptic equations of variational type. Clearly, weak solutions to (P) correspond to critical points of the functional

$$I(u) = \frac{1}{p}\int_\Omega |Du|^p\,dx - \int_\Omega F(x,u)\,dx, \tag{1.5}$$

defined on the Sobolev space $W_0^{1,p}(\Omega)$. A standard existence result for (P) is that (P) possesses at least a nontrivial solution if $f(x,t)$ satisfies (f_1) (f_2) together with the following Ambrosetti–Rabinowitz condition ((AR) for short): there are constants $\theta > 0$, $0 < M < +\infty$ such that

$$0 \le (p+\theta)F(x,s) \le sf(x,s), \tag{AR}$$

whenever $|s| \ge M$ and $x \in \Omega$. Ambrosetti and Rabinowitz solved the existence of a nontrivial weak solution to (P) when $f(x,t)$ is of super-linear at $t = 0$ and subcritical at $t = \infty$ such that it possesses the mountain-pass geometric structure.

Clearly, if the (AR) condition holds, then

$$F(x,t) \ge c_1|t|^{p+\theta} - c_2,\ \forall (x,t) \in \Omega \times \mathbf{R}^1, \tag{1.6}$$

where c_1, c_2 are two positive constants. The conditions (f_1) and (1.6) ensure that the functional $I(u)$ given by (1.5) possesses the so-called mountain-pass geometric structure near $u = 0$. The condition (AR) guarantees that every $(PS)_c$ sequence of $I(u)$ is bounded in $W_0^{1,p}(\Omega)$ and (f_2) guarantees that every bounded $(PS)_c$ sequence of $I(u)$ possesses a subsequence which converges strongly in $W_0^{1,p}(\Omega)$; hence $I(u)$ satisfies the $(PS)_c$ condition, and one can get a nontrivial solution to (P) by applying the mountain-pass theorem.

As the (AR) condition implies (1.6), one can not deal with (P) using the mountain-pass theorem directly if $f(x,t)$ is of p-asymptotically linear at ∞, i.e.

$$\lim_{|t|\to+\infty} \frac{f(x,t)}{|t|^{p-2}t} = l, \text{ uniformly in } x \in \Omega, \tag{1.7}$$

where l is a constant. During the past three decades, many results have been obtained for the existence of nontrivial solutions to (P) when $f(x,t)$ does not satisfy the (AR) condition (see e.g. [7] [12] [11] [13] and the references therein). We will mention several results for the case where $f(x,t)$ is p-superlinear at $t=0$ (i.e. (f_1) holds).

In [5], Costa and Magalhaes studied (P) for $p=2$ and replaced the (AR) condition by one of the following conditions:

$$(F_1)_q \qquad \limsup_{|t|\to+\infty} \frac{F(x,t)}{|t|^q} \le b < +\infty, \text{ uniformly in } x \in \Omega;$$

$$(F_2^+)_\mu \qquad \lim_{|t|\to+\infty} \frac{f(x,t)t - pF(x,t)}{|t|^\mu} \ge a > 0, \text{ uniformly in } x \in \Omega;$$

$$(F_2^-)_\mu \qquad \lim_{|t|\to+\infty} \frac{f(x,t)t - pF(x,t)}{|t|^\mu} \le -a < 0, \text{ uniformly in } x \in \Omega,$$

for some constants $a, b \in \mathbf{R}^1$ and $q > p$, $\mu > N/p/(q-p)$ if $N > p$ and $\mu > q-p$ if $1 \le N \le p$. Notice that from $(F_2^+)_\mu$, we have

$$(fF) \qquad \lim_{|t|\to+\infty} \{f(x,t)t - pF(x,t)\} = +\infty \text{ uniformly in } x \in \Omega.$$

In [19], Willem and Zou studied (P) for $p=2$ and replaced the (AR) condition by the following conditions: $H(x,s) \triangleq sf(x,s) - 2F(x,s)$ is nondecreasing in s for any $x \in \Omega$, $x \in \mathbf{R}$, $sf(x,s) \ge 0$ for $(x,s) \in \Omega \times \mathbf{R}^1$, and there exist constants $s_0 > 0$, $\mu > 2$, $c_0 > 0$ such that

$$sf(x,s) \ge c_0|s|^\mu$$

for $(x,s) \in \Omega \times \mathbf{R}^1$ with $|s| \ge s_0$.

In [17], Schechter and Zou proved that for $p=2$, (P) has at least one nontrivial weak solution if $f(x,t)$ satisfies $(f_1)(f_2)$ and either $H(x,s)$ is a convex function of s for each $x \in \Omega$ or there are constants $c > 0$, $\mu > 0$ and $r \ge 0$ such that

$$\mu F(x,t) - tf(x,t) \le C(1+t^2), |t| \ge r,$$

together with the following

$$(F)_\infty \qquad \text{either } \lim_{s\to\infty} \frac{F(x,s)}{s^2} = +\infty \text{ uniformly in } x \in \Omega,$$

$$\text{or } \lim_{s\to-\infty} \frac{F(x,s)}{s^2} = +\infty \text{ uniformly in } x \in \Omega.$$

In [13], Li and Zhou studied the problem (P) for the case of $p > 1$. One of the main results in [13] is that (P) has at least one positive solution if $f \in C^0(\bar{\Omega} \times \mathbf{R}^1, \mathbf{R}^1)$

satisfies (f_5') (f_6), $f(x,t)=0$ for $t\leq 0$ and $x\in\Omega$; $f(x,t)\geq 0$ for $x\in\Omega, t\geq 0$ and $\lim\limits_{t\to 0}\dfrac{f(x,t)}{|t|^{p-2}t}=P(x)$ uniformly in $x\in\Omega$ where $P(x)\in L^\infty(\Omega)$ with

$$\|P\|_\infty<\lambda_1=\inf_{u\in W_0^{1,p}(\Omega)\backslash\{0\}}\frac{\displaystyle\int_\Omega |Du|^p\,dx}{\displaystyle\int_\Omega |u|^p\,dx}.$$

In [4], Chen, Shen and Yao studied (P) and obtained the existence of a nontrivial solution. The assumption in [4] is slightly different from what given in [13]. They replace (f_6) by the following condition: there exist constants $s_0\geq 0,\ t_0>0$ and $c_1,c_2\geq 0$ such that

$$t^pf(x,s)s-pF(x,s)\leq c_1(f(x,s)s-pF(x,s))+c_2\ \text{for}|s|\geq s_0,\ 0\leq t\leq t_0.$$

Recently, Miyagaki and Souto studied

$$\begin{cases}-\Delta u=\lambda f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}\tag{1.8}$$

in [14], where $\Omega\subset\mathbf{R}^N$ is a bounded domain. They assumed that $f(x,t)$ satisfies (f_1)–(f_4) with $p=2$ and proved that (1.8) has at least one nontrivial solution for any $\lambda>0$ (see Theorem 1.1 in [14]). Theorem 1.1 of [14] generalizes the main results of [3, 8, 27] concerning (1.8). The approach in [14] is similar to that of [7]. The main idea is to use the mountain-pass theorem under the (PS) condition and to show that for any $\lambda>0$, there is a sequence $\{\lambda_n\}_{n=1}^{+\infty}\subset\mathbf{R}^1$ and a sequence $\{u_n\}_{n=1}^{+\infty}\subset W_0^{1,p}(\Omega)$ with

$$\lambda_n\to\lambda,\ c_{\lambda_n}\to c_\lambda,\ I_{\lambda_n}(u_n)=c_{\lambda_n},\ I'_{\lambda_n}(u_n)=0,$$

such that the norm of u_n in $W_0^{1,p}(\Omega)$ is uniformly bounded, where c_{λ_n} and c_λ are the so-called mountain-pass levels of I_{λ_n} and I_λ respectively, and then prove that the weak limit u of $\{u_n\}_{n=1}^{+\infty}$ is a critical point of I_λ with $I_\lambda(u)=c_\lambda$. In doing so, the main difficulty is to prove that if c_λ is differentiable at μ then there is a sequence $\{u_n\}_{n=1}^{+\infty}\subset W_0^{1,p}(\Omega)$ with

$$I_\mu(u_n)\to c_\mu,\ I'_\mu(u_n)\to 0,\ \|u_n\|^p\leq C_0,$$

where $C_0=pc_\mu+p\mu(2-c'(\mu))+1$ (see Lemma 2.3 in [14]).

Li and Yang in [10] studied the problem

$$\begin{cases}-\Delta_p u=\lambda f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}\tag{$(P)_\lambda$}$$

where $p>1$, $\lambda>0$, $\Omega\subset\mathbf{R}^N$ is a bounded domain. And the corresponding functional possesses the mountain-pass geometric structure. They proved that $(P)_\lambda$ has at least one nontrivial solution under the hypothesis (f_1)–(f_4) via the mountain-pass theorem under the $(C)_c$ condition.

In 1978, Rabinowitz proposed the so-called linking theorem in [15] which resulted in the existence of at least one nontrivial solution to (1.4) when it possesses the linking geometric structure together with the (AR) condition. A standard existence results for (1.1) when it possesses the linking geometric structure is that (1.1) possesses at least a nontrivial solution if f satisfies (f_1), (f_2), (f_5) together with the (AR)

condition (see e.g. [18]). However, in all the results mentioned above, the existence of a nontrivial solution for (1.1) when it possesses the linking geometric structure are obtained when either the (AR) condition holds or f is asymptotically linear at ∞ (i.e. (1.7) holds).

Our purpose in this paper is to study the existence of a nontrivial solution to problem (1.1) for the case where neither the (AR) condition holds nor f is asymptotically linear at ∞. Our main result is as follows:

Theorem 1.1. *Suppose that Ω is a bounded domain in R^N with $N \geq 3$ and $a \in L^{\frac{N}{2}}(\Omega)$. If $f(x,t)$ satisfies the assumptions (f_1)–(f_5) with $p = 2$, then problem* (1.1) *has at least one nontrivial weak solution.*

Our main result provides an existence result about (1.1) with linking geometric structure and extends the main result given in [14] where the mountain-pass geometric structure is assumed. However, we use a different approach which seems easier to handle compared to the techniques which are used in [14]. Instead of using the approximating process combining with the linking theorem under the (PS) condition, which might be possible to carry out, we use a linking theorem under the $(C)_c$ condition. To do so, we have to overcome some difficulties.

In order to prove Theorem 1.1, we first prove that the functional I possesses a $(C)_c$ sequence by a linking theorem without the $(C)_c$ condition. Note that the usual linking theorem under the $(PS)_c$ condition in [18] is not good enough to deal with the problem. The main difficulty consists in that one can not prove that a $(PS)_c$ sequence is bounded in $H_0^1(\Omega)$ without the (AR) condition. It seems that there is not an explicitly available linking theorem under the $(C)_c$ condition which can be directly used for our purpose. Although there is a linking theorem in [9] under the $(C)_c$ condition, it is not convenient for us to verify the assumptions which are required in the theorem. So we want to look for a linking theorem under the $(C)_c$ condition which we can apply directly. We believe that such a result may exist somewhere but it is hard for us to trace. So we state and prove it in Section 2 below. The idea to weaken the (PS) condition to the $(C)_c$ condition has existed in some papers (see e.g. [2, 22] and references therein). To obtain the linking theorem we need, we imitate the framework given in [18]. The deformation lemma (see e.g. Lemma 2.6 below) is very crucial in the process of the whole proof. This type of deformation lemma under the $(C)_c$ condition had appeared in [2], but the form given in [2] is not the form we need. The linking theorem given in [18] is obtained from a general minimax theorem. We follow the framework given in [18] to establish a general critical point theorem of minimax type under the $(C)_c$ condition first in Section 2 (see Corollary 2.9) and then obtain the linking theorem under the $(C)_c$ condition (see Proposition 2.10 below) as a direct application of the minimax theorem.

Another difficulty for the proof of Theorem 1.1 is to prove the boundedness of $(C)_c$ sequence without the (AR) condition. As the nonlinear function $f(x,t)$ is no longer asymptotically linear at ∞, the standard method using in [11] is not applicable directly. So we combine the method in both [11] and [14] to prove the boundedness of the $(C)_c$ sequence. Then, by a standard argument, we show that the $(C)_c$ sequence has a subsequence which converges strongly to a critical point of I (see Lemma 3.4 below).

The paper is organized as follows. In section 2 we present some definitions and preliminary results. In section 3 we give the proof of our main result Theorem 1.1.

2. Preliminary results

In this section we give some definitions and preliminary results which will be used in Section 3 for the proof of our main result.

Throughout this paper, we denote the norm of u in $H_0^1(\Omega)$, $L^p(\Omega)$, $1 \le p < +\infty$, and $(H_0^1(\Omega))^*$ (the dual space of $H_0^1(\Omega)$), respectively, by

$$\|u\| = \Big(\int_\Omega |\nabla u|^2\,dx\Big)^{\frac{1}{2}},\ |u|_p = \Big(\int_\Omega |u|^p\,dx\Big)^{\frac{1}{p}},\ \|u\|_* \triangleq \|u\|_{(H_0^1(\Omega))^*}.$$

We define the energy functional associated to problem (1.1), as

$$I(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - a(x)u^2)\,dx - \int_\Omega F(x,u)\,dx,\ u \in H_0^1(\Omega). \tag{2.1}$$

It is easy to see that the functional $I \in C^1(H_0^1(\Omega), \mathbf{R})$ and

$$\langle I'(u), v\rangle = \int_\Omega [\nabla u\cdot\nabla v - a(x)uv]\,dx - \int_\Omega f(x,u)v\,dx,\ \forall u,v \in H_0^1(\Omega),$$

where $I'(u)$ is the Fréchet derivative of I and $\langle\cdot,\cdot\rangle$ denotes the pairing between $H_0^1(\Omega)$ and its dual. The critical points of I are precisely the weak solutions of problem (1.1).

Definition 2.1. Let $(X, \|\cdot\|_X)$ be a real Banach space with its dual space $(X', \|\cdot\|_{X'})$ and $I \in C^1(X, \mathbf{R})$.

(i) For $c \in \mathbf{R}^1$, we say that I satisfies the $(PS)_c$ condition, if for any sequence $\{u_n\} \subset X$ with

$$I(u_n) \to c,\ I'(u_n) \to 0 \text{ in } X',$$

there is a subsequence $\{u_n\}$ such that $\{u_n\}$ converges strongly in X.

(ii) For $c \in \mathbf{R}^1$, we say that I satisfies the $(C)_c$ condition, if for any sequence $\{u_n\} \subset X$ with

$$I(u_n) \to c,\ \|I'(u_n)\|_{X'}(1 + \|u_n\|_X) \to 0,$$

there is a subsequence $\{u_n\}$ such that $\{u_n\}$ converges strongly in X.

Suppose that $\varphi\colon [0,+\infty) \times X \mapsto X$ is continuous and $\forall x_0 \in X,\ \forall \alpha > 0,\ \exists r > 0$ and $L = L(x_0, \alpha, r)$ such that

$$\|\varphi(t,x) - \varphi(t,y)\|_X \le L\|x - y\|_X,\ \forall x,y \in B(x_0, r),\ t \in [0,\alpha]. \tag{2.2}$$

Consider the following initial value problem

$$\begin{cases} \frac{dx}{dt} = \varphi(t,x), \\ x(0) = x_0 \in X. \end{cases} \tag{2.3}$$

Lemma 2.2. (Theorem 5.1 of [20]) *Suppose that φ satisfies the assumption (2.2). Then there exists a $\beta > 0$ such that (2.3) has a unique solution $x(t)$ in $[0,\beta]$ which continuously depends on x_0. More generally, if $\|\varphi(x,t) - \varphi(y,t)\|_X \le L\|x - y\|_X$, then*

$$\|x(t) - y(t)\|_X \le L\|x_0 - y_0\|_X e^{Lt},\ \forall x,y \in X,\ t \in [0,\beta],$$

where $x(t)$ and $y(t)$ are the solutions of (2.3) with initial values x_0 and y_0, respectively.

Lemma 2.3. (Theorem 5.3 of [20]) *Suppose that φ satisfies the assumption (2.2). If there exist $a, b > 0$ such that*

$$\|\varphi(t,x)\|_X \le a + b\|x\|_X,\ \forall (t,x) \in [0,+\infty) \times X \mapsto X, \tag{2.4}$$

then the unique local solution of (2.3) can be extended as a global solution for $t \in [0,+\infty)$.

Definition 2.4. Let X be a Banach space, $\varphi \in C^1(X, \mathbf{R})$ and $M = \{x \in X \colon \varphi'(x) \neq 0\}$. A pseudogradient vector field for φ on M is a locally Lipschitz continuous vector field $g \colon M \to X$ such that, for every $u \in M$,

$$\|g(u)\|_X \le 2||\varphi'(u)||_{X'}, \quad \langle \varphi'(u), g(u)\rangle \ge ||\varphi'(u)||^2_{X'},$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between X and its dual X'.

Lemma 2.5. (Theorem 2.1 of [20]) *Suppose that $\varphi \in C^1(X, \mathbf{R}^1)$. Then there exists a pseudogradient vector field for φ on M.*

Suppose that g is a pseudogradient vector field for φ on M, let $\Phi(u) = \frac{g(u)}{\|\varphi'(u)\|^2_{X'}}$, then for any $u \in M$ we have

$$\begin{cases} \Phi(u) \le \frac{2}{\|\varphi'(u)\|_{X'}}, \\ \langle \varphi'(u), \Phi(u)\rangle \ge 1. \end{cases} \tag{2.5}$$

We consider the following initial value problem

$$\begin{cases} \frac{d\sigma(t)}{dt} = -\Phi(\sigma(t)), \\ \sigma(0) = u_0. \end{cases} \tag{2.6}$$

Since Φ is locally Lipschitz continuous, for any $u_0 \in M$, there exists a unique local solution of (2.6). Moreover, φ decreases along $\sigma(t)$. In fact, we have

$$\frac{d}{dt}\varphi(\sigma(t)) = \langle \varphi'(\sigma(t)), \frac{d}{dt}\sigma(t)\rangle = -\langle \varphi'(\sigma(t)), \Phi(\sigma(t))\rangle \le -1.$$

To guarantee that $\sigma(t)$ exists on $[0,+\infty)$, by Lemma 2.3 it is enough to show that

$$\|\Phi(u)\|_X \le a + b\|u\|_X,\ a, b > 0,$$

which is a direct result of

$$\|\varphi'(u)\|_{X'}(a + b\|u\|_X) \ge 2,\ a, b > 0. \tag{2.7}$$

For $\varphi \in C^1(X, \mathbf{R})$ and $c \in \mathbf{R}$, we set

$$\varphi^c = \{u \in X \mid \varphi(u) \le c\}$$

and

$$S_{2\delta} = \{u \in X \colon ||u - v|| \le 2\delta,\ \forall v \in S\},$$

where $S \subset X$.

Lemma 2.6. (Deformation Lemma) *Let X be a real Banach space, $\varphi \in C^1(X, \mathbf{R})$, $S \subset X$, $c \in \mathbf{R}$, $\epsilon, \delta > 0$ such that*

$$(\forall u \in \varphi^{-1}([c - 2\epsilon, c + 2\epsilon]) \cap S_{2\delta}) \colon (1 + \|u\|_X)\|\varphi'(u)\|_{X'} \ge \frac{8\epsilon}{\delta}. \tag{2.8}$$

Then there exists $\eta \in C([0,1] \times X, X)$ such that

(i) *$\eta(t,u) = u$ if $t = 0$ or if $u \notin \varphi^{-1}([c - 2\epsilon, c + 2\epsilon]) \cap S_{2\delta}$,*

(ii) $\eta(1, \varphi^{c+\epsilon} \cap S) \subset \varphi^{c-\epsilon}$,
(iii) $\eta(t, \cdot)$ *is a homeomorphism of* X, $\forall t \in [0, 1]$,
(iv) $\varphi(\eta(t, u))$ *is nonincreasing*, $\forall u \in X$.

Remark 2.7. Lemma 2.6 extends Lemma 2.3 of [18], where the assumption was that $\forall u \in \varphi^{-1}([c-\epsilon, c+\epsilon]) \cap S_{2\delta})\colon \|\varphi'(u)\|_{X'} \geq \frac{8\epsilon}{\delta}$. However, we don't need all the conclusions as Lemma 2.3 of [18] states.

Proof. The proof is similar to that of Lemma 2.3 of [18]. By the preceding Lemma 2.5, there exists a pseudogradient vector field g for φ' on $M \triangleq \{u \in X \colon \varphi'(u) \neq 0\}$. Then by the definition of pseudogradient vector field, we know that

$$\|g(u)\| \leq 2\|\varphi'(u)\| \tag{2.9}$$

and

$$\langle \varphi'(u), g(u) \rangle \geq \|\varphi'(u)\|^2. \tag{2.10}$$

Let us define

$$A \triangleq \varphi^{-1}([c-2\epsilon, c+2\epsilon]) \cap S_{2\delta}, \quad B \triangleq \varphi^{-1}([c-\epsilon, c+\epsilon]),$$
$$\psi(u) \triangleq \operatorname{dist}(u, X\backslash A)(\operatorname{dist}(u, X\backslash A) + \operatorname{dist}(u, B))^{-1},$$

so that ψ is locally Lipschitz continuous, $\psi = 1$ on B and $\psi = 0$ on $X\backslash A$. Let us also define the locally continuous vector field

$$f(u) \triangleq \begin{cases} -\psi(u)\|\varphi'(u)\|^{-2} g(u), & x \in A, \\ 0, & x \in X\backslash A. \end{cases} \tag{2.11}$$

Then by (2.8), (2.9) and (2.11), we have

$$\|f(u)\| \leq \frac{|\psi(u)|\|g(u)\|}{\|\varphi'(u)\|^2} \leq \frac{2}{\|\varphi'(u)\|} \leq \frac{\delta(1+\|u\|)}{8\epsilon} \tag{2.12}$$

on X. For each $u \in X$, now we consider the following initial value problem

$$\begin{cases} \frac{d\sigma(t,u)}{dt} = f(\sigma(t, u)), \\ \sigma(0, u) = u. \end{cases} \tag{2.13}$$

Since f is locally Lipschitz continuous, for each initial value $u \in X$, (2.13) possesses a unique solution $\sigma(\cdot, u)$ which is defined on $\mathbf{R}^+ = \{\mathbf{R} \colon t \geq 0\}$ by virtue of Lemma 2.2, Lemma 2.3 and (2.12). Moreover, for every fixed t, $\sigma(t, \cdot)\colon X \mapsto X$ is an homeomorphism. Let us define η on $[0, 1] \times X$ by $\eta(t, u) = \sigma(8\epsilon t, u)$.

Obviously, $\eta(0, u) = \sigma(0, u) = u$. If $u \notin \varphi^{-1}([c-2\epsilon, c+2\epsilon]) \cap S_{2\delta}$, then by (2.11) and (2.13) we see that $\eta(t, u) = u$. So, (i) holds.

For $t > 0$, by (2.10), (2.11) and (2.13) we have

$$\begin{aligned} \frac{d}{dt}\varphi(\sigma(t, u)) &= \langle \varphi'(\sigma(t, u)), \frac{d}{dt}\sigma(t, u) \rangle = \langle \varphi'(\sigma(t, u)), f(\sigma(t, u)) \rangle \\ &= -\frac{\psi(\sigma(t, u))}{\|\varphi'(\sigma(t, u))\|^2} \langle \varphi'(\sigma(t, u)), g(\sigma(t, u)) \rangle \leq -\psi(\sigma(t, u)) \leq 0. \end{aligned}$$

Hence $\eta(\cdot, u)$ is nonincreasing, $\forall u \in X$, i.e., (iv) is true. We fix $t \in [0, 1]$, since $\sigma(t, \cdot)\colon X \to X$ is an homeomorphism, $\eta(t, \cdot)\colon X \to X$ is an homeomorphism.

Let $u \in \varphi^{c+\epsilon} \cap S$. If there is a $t \in [0, 8\epsilon]$ such that $\varphi(\sigma(t, u)) < c - \epsilon$, then $\varphi(\sigma(8\epsilon, u)) \leq \varphi(\sigma(t, u)) < c - \epsilon$ and (ii) is satisfied. If there exist $u \in \varphi^{c+\epsilon} \cap S$ and

$\sigma(t,u) \notin \varphi^{c-\epsilon}$, then $\sigma(t,u) \in \varphi^{-1}([c-\epsilon, c+\epsilon])$, $\forall t \in [0, 8\epsilon]$. So, $\psi(\sigma(t,u)) = 1$, $\forall t \in [0, 8\epsilon]$.

We obtain from (2.10), (2.11) and (2.13) that

$$\begin{aligned}
8\epsilon > \varphi(\sigma(0,u)) - \varphi(\sigma(8\epsilon,u)) &= -\int_0^{8\epsilon} \frac{d\varphi(\sigma(t,u))}{dt}\, dt \\
&= -\int_0^{8\epsilon} \langle \varphi'(\sigma(t,u)), f(\sigma(t,u))\rangle\, dt \\
&= -\int_0^{8\epsilon} \langle \varphi'(\sigma(t,u)), -\frac{\psi(\sigma(t,u))}{\|\varphi'(\sigma(t,u))\|^2}\, g(\sigma(t,u))\rangle\, dt \\
&= \frac{1}{\|\varphi'(\sigma(t,u))\|^2} \int_0^{8\epsilon} \langle \varphi'(\sigma(t,u)), g(\sigma(t,u))\rangle\, dt \geq 8\epsilon.
\end{aligned}$$

So (ii) is also true. □

The following proposition gives a general minimax principle under the $(C)_c$ condition which generalizes Theorem 2.8 of [18] and its proof is similar to Theorem 2.8 of [18].

Proposition 2.8. *Let X be a Banach space and M a metric space. Let M_0 be a closed subspace of M and $\Gamma_0 \subset C(M_0, X)$. Define*

$$\Gamma := \{\gamma \in C(M,X) \colon \gamma|_{M_0} \in \Gamma_0\}.$$

If $\varphi \in C^1(X, \mathbf{R})$ satisfies

$$\infty > c := \inf_{\gamma\in\Gamma} \sup_{u\in M} \varphi(\gamma(u)) > a := \sup_{\gamma_0\in\Gamma_0} \sup_{u\in M_0} \varphi(\gamma_0(u)), \tag{2.14}$$

then, for every $\epsilon \in (0, \frac{c-a}{2})$, $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_M \varphi \circ \gamma \leq c + \epsilon, \tag{2.15}$$

there exists $u \in X$ such that

a) $c - 2\epsilon \leq \varphi(u) \leq c + 2\epsilon$,
b) $\mathrm{dist}(u, \gamma(M)) \leq 2\delta$,
c) $(1 + \|u\|_X)\|\varphi'(u)\|_{X'} < \frac{8\epsilon}{\delta}$.

Proof. Suppose that $\exists \epsilon \in (0, \frac{c-a}{2})$, $\forall \delta > 0$, $\forall \gamma \in \Gamma$ and $\sup\limits_M \varphi\circ\gamma \leq c+\epsilon$, for any $u \in X$, $c - 2\epsilon \leq \varphi(u) \leq c + 2\epsilon$, $\mathrm{dist}(u, \gamma(M)) \leq 2\delta$ but $(1 + \|u\|_X)\|\varphi'(u)\|_{X'} \geq \frac{8\epsilon}{\delta}$. We apply Lemma 2.6 with $S := \gamma(M)$. We assume that

$$c - 2\epsilon > a. \tag{2.16}$$

We see that there is a $\eta \in C([0,1] \times X, X)$, we define $\beta(u) = \eta(1, \gamma(u))$. For every $u \in M_0$, then $\gamma \in \Gamma_0$. By (2.16) we obtain $\varphi(\gamma_0(u)) \leq a < c - 2\epsilon$. Hence, $\gamma_0(u) \notin \varphi^{-1}([c-2\epsilon, c+2\epsilon]) \cap S_{2\delta}$. Then by (ii) of the Lemma 2.6, we get

$$\beta(u) = \eta(1, \gamma_0(u)) = \gamma_0(u),$$

so that $\beta \in \Gamma$. We obtain, from (2.15), $\gamma(u) \in \gamma(M) \cap \varphi^{c+\epsilon} = S \cap \varphi^{c+\epsilon}$. Then by (ii) of the Lemma 2.6, we get that

$$c \leq \sup_{u\in M} \varphi(\beta(u)) = \sup_{u\in M} \varphi(\eta(1, \gamma(u))) \leq c - \epsilon.$$

This is impossible. □

Corollary 2.9. *Under the assumptions of Proposition 2.8, there exists a sequence $\{u_n\} \subset X$ satisfying*

$$\varphi(u_n) \to c, \quad (1 + ||u_n||_X)\|\varphi'(u_n)\|_{X'} \to 0.$$

In particular, if φ satisfies the $(C)_c$ condition, then c is a critical value of φ.

As an application of Proposition 2.8, we have the following result:

Proposition 2.10. (Linking Theorem under the $(C)_c$ condition) *Let $X = Y \oplus Z$ be a Banach space with $\dim Y < \infty$. Let $\rho > r > 0$ and let $z \in Z$ be a fixed element such that $\|z\| = r$. Define*

$$M := \{u = y + \lambda z \colon \|u\| \le \rho,\ \lambda \ge 0,\ y \in Y\},$$
$$M_0 := \{u = y + \lambda z \colon y \in Y,\ \|u\| = \rho,\ \lambda \ge 0 \text{ or } \|u\| \le \rho,\ \lambda = 0\},$$
$$N_r := \{u \in Z \colon \|u\| = r\}.$$

Let $\varphi \in C^1(X, \mathbf{R})$ be such that

$$b := \inf_{N_r} \varphi > a := \max_{M_0} \varphi.$$

Then $c \ge b$ and there exists a $(C)_c$-sequence of φ where

$$c := \inf_{\gamma \in \Gamma} \max_{u \in M} \varphi(\gamma(u)), \quad \Gamma := \{\gamma \in C(M, X) \colon \gamma|_{M_0} = I_d\}.$$

In particular, if φ satisfies the $(C)_c$ condition, then c is a critical value of φ.

Remark 2.11. Proposition 2.10 extends Theorem 2.12 of [18], where the conclusion was that there was a $(PS)_c$-sequence for φ and some $c \ge b$.

Proof. The proof is similar to that of Theorem 2.12 of [18]. In order to apply Proposition 2.8, we first show that: $c \ge b$.

Let us prove that, for every $\gamma \in \Gamma$, $\gamma(M) \cap N_r \ne \emptyset$. Denote by P the projection onto Y such that $PZ = \{0\}$ and by R a retraction from $Y \oplus Rz \backslash \{z\}$ to M_0. If $\gamma(M) \cap N_r = \emptyset$, then the map

$$u \mapsto R(P\gamma(u) + \|(1 - P)\gamma(u)\| r^{-1} z)$$

is a retraction from M to M_0. This is impossible since M is homeomorphic to a finite dimensional ball. In fact, just assume, by contradiction, that $R\colon M \to M_0$ is a retraction and let U be the interior of M_0. For each $t \in [0, 1]$, we introduce the homotopy

$$H(t, u) = (1 - t)u + tR(P\gamma(u) + \|(1 - P)\gamma(u)\| r^{-1} z).$$

It is easy to check that

$$\forall u \in M_0,\ \forall t \in [0, 1],\ H(t, u) = u \ne 0.$$

Hence, the topological degree $\deg(H(t, \cdot), U, 0)$ is well defined for every $t \in [0, 1]$.

By the well-known properties of the topological degree, we deduce

$$\deg(R, U, 0) = \deg(H(1, \cdot), U, 0) = \deg(I_d|_{M_0}, U, 0) = 1.$$

We obtain, by existence of the topological degree, that

$$0 \in R(P\gamma(u) + \|(1 - P)\gamma(u)\| r^{-1} z) \subset M_0.$$

A contradiction. Hence we obtain, for every $\gamma \in \Gamma$, that

$$\max_{u \in M} \varphi(\gamma(u)) = \max_{u \in \gamma(M)} \varphi(u) \ge \inf_{u \in N_r} \varphi(u) = b.$$

Therefore,

$$c = \inf_{\gamma \in \Gamma} \max_{u \in M} \varphi(\gamma(u)) \geq \inf_{u \in N_r} \varphi(u) = b,$$

i.e., $c \geq b$.

By Proposition 2.8, if we take $\epsilon = \frac{1}{n}$ and let $n \to \infty$, then we know that there exists a $(C)_c$-sequence of φ. □

Proposition 2.12. (Lemma 2.14 of [18]) *If Ω is a bounded domain in $\mathbf{R}^N$, $N \geq 3$, and $a(x) \in L^{\frac{N}{2}}(\Omega)$, then*

$$\lambda_1 := \inf_{u \in H_0^1(\Omega), |u|_2 = 1} \int_\Omega [|\nabla u|^2 - a(x)u^2]\, dx > -\infty.$$

The following result is well-known, for the reader's convenience we will give the proof.

Lemma 2.13. *Let Ω be a bounded domain in $\mathbf{R}^N$ and $a(x) \in L^{\frac{N}{2}}(\Omega)$. Then the sequence of all eigenvalues $\{\lambda_j\}_{j=1}^{+\infty}$ of the problem*

$$\begin{cases} -\Delta u - a(x)u = \lambda u, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

satisfies

$$-\infty < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots,$$

and $\lim_{j \to \infty} \lambda_j = +\infty$.

Proof. By Proposition 2.12, it follows that $\lambda_1 > -\infty$. Therefore, there is a λ_0 large enough such that

$$\int_\Omega [|\nabla u|^2 - a(x)u^2]\, dx + \int_\Omega \lambda_0 u^2\, dx > 0,$$

for any $u \in H_0^1(\Omega)$. So we can define an equivalent inner product on $H_0^1(\Omega)$ by

$$(u, v)_{\lambda_0} = \int_\Omega [\nabla u \cdot \nabla v - a(x)uv]\, dx + \int_\Omega \lambda_0 uv\, dx, \ \forall u, v \in H_0^1(\Omega).$$

By the Poincaré inequality and the Riesz representation theorem, we know that for any $u \in L^2(\Omega)$, there exists a unique $w \in H_0^1(\Omega)$ such that

$$\int_\Omega uv\, dx = (w, v)_{\lambda_0}, \ \forall v \in H_0^1(\Omega).$$

For $u \in H_0^1(\Omega)$, define $K_{\lambda_0} \colon L^2(\Omega) \longrightarrow H_0^1(\Omega)$ by $w = K_{\lambda_0} u$, then K_{λ_0} is a bounded linear operator. If $i \colon H_0^1(\Omega) \longrightarrow L^2(\Omega)$ is the natural embedding operator, then the Sobelev embedding theorem shows that i is a compact operator and for any $u, v \in H_0^1(\Omega)$ we have

$$(K_{\lambda_0} \circ i(u), v)_{\lambda_0} = \int_\Omega uv\, dx.$$

Since $K_{\lambda_0} \circ i$ is a compact operator from $H_0^1(\Omega)$ to $H_0^1(\Omega)$ and $(K_{\lambda_0} \circ i(u), u)_{\lambda_0} > 0$ for $u \neq 0$, we see that by Hilbert–Schmidt theory (see e.g. Section 4 of Chapter 4 of [3]), it follows that the sequence of all eigenvalues $\{\mu_j\}_{j=1}^{+\infty}$ of $K_{\lambda_0} \circ i$ satisfies $\mu_1 > \mu_2 > \mu_3 > \ldots > \mu_n > \ldots > 0,\ \mu_j \to 0$ (as $j \to +\infty$), and

$$\lambda_j = \frac{1}{\mu_j} - \lambda_0,\ \ (j = 1, 2, 3, \ldots)$$

is the sequence of all eigenvalues of (1.2) and the corresponding eigenfunctions satisfy

$$\int_\Omega e_i e_j \, dx = \delta_{ij}. \qquad \square$$

3. The proof of the main result

In this section, we prove our main result Theorem 1.1. According to Lemma 2.13, let

$$-\infty < \lambda_1 < \lambda_2 \leqslant \lambda_3 < \cdots \lambda_n < \lambda_{n+1} \leq \lambda_{n+2} < \cdots$$

be the sequence of all eigenvalues of the problem:

$$\begin{cases} -\Delta u - a(x)u = \lambda u, & x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

with $\lim\limits_{j\to\infty} \lambda_j = +\infty$, and let $e_1, e_2, \ldots, e_n, \ldots$ be all the corresponding eigenvectors such that

$$\int_\Omega e_i e_j \, dx = \delta_{ij}.$$

Following the notation in the proof of Lemma 2.13, we denote an equivalent inner product in $H_0^1(\Omega)$ as

$$(u, v)_{\lambda_0} = \int_\Omega [\nabla u \cdot \nabla v - a(x)uv] \, dx + \int_\Omega \lambda_0 uv \, dx,\ \ \forall u, v \in H_0^1(\Omega),$$

where $\lambda_0 + \lambda_1 > 0$, and

$$\lambda_1 = \inf_{u \in H_0^1(\Omega),\, |u|_2 = 1} \int_\Omega [|\nabla u|^2 - a(x)u^2] \, dx.$$

If

$$Y := \operatorname{span}\{e_1, e_2, \ldots, e_n\}$$

and

$$Z := \{u \in H_0^1(\Omega) \colon \int_\Omega uv \, dx = 0,\ \ v \in Y\},$$

then we know that $\dim Y < +\infty,\ H_0^1(\Omega) = Y \oplus Z$. From the definition of Y, Z and Lemma 2.13, we have the following lemma.

Lemma 3.1. (Lemma 2.15 of [18])

$$\delta := \inf_{u \in Z,\, |\nabla u|_2 = 1} \int_\Omega [|\nabla u|^2 - a(x)u^2] \, dx > 0. \tag{3.1}$$

Proof. For every $u \in Z$, we have $\int_\Omega u e_i \, dx = 0$ $(1 \leq i \leq n)$. Let

$$u = \sum_{i=n+1}^{+\infty} c_i e_i,$$

where $c_i = \int_\Omega u e_i \, dx$ $(i = n+1, n+2, \ldots)$, hence

$$\begin{aligned}(u,u)_{\lambda_0} &= \Big(\sum_{i=n+1}^{+\infty} c_i e_i, \sum_{j=n+1}^{+\infty} c_j e_j \Big)_{\lambda_0} = \sum_{i=n+1}^{+\infty} \sum_{j=n+1}^{+\infty} c_i c_j (e_i, e_j)_{\lambda_0} \\ &= \sum_{i=n+1}^{+\infty} \sum_{j=n+1}^{+\infty} c_i c_j \Big[\int_\Omega (\nabla e_i \cdot \nabla e_j - a(x) e_i e_j) \, dx + \lambda_0 \int_\Omega e_i e_j \, dx \Big] \\ &= \sum_{i=n+1}^{+\infty} \sum_{j=n+1}^{+\infty} c_i c_j (\lambda_i + \lambda_0) \int_\Omega e_i e_j \, dx = \sum_{i=n+1}^{+\infty} c_i^2 (\lambda_i + \lambda_0) \\ &\geq (\lambda_{n+1} + \lambda_0) \sum_{i=n+1}^{+\infty} c_i^2 = (\lambda_{n+1} + \lambda_0) \int_\Omega u^2 \, dx.\end{aligned}$$

So for every $u \in Z$, we have

$$\int_\Omega [|\nabla u|^2 - a(x) u^2] \, dx \geq \lambda_{n+1} \int_\Omega u^2 \, dx.$$

Take a minimizing sequences $\{u_n\}_{n=1}^{+\infty} \subset Z$ such that

$$\|u_n\| = |\nabla u_n|_2 = 1, \quad 1 - \int_\Omega a(x) u_n^2 \, dx \to \delta.$$

Without loss of generality, let

$$u_n \rightharpoonup u \text{ in } H_0^1(\Omega).$$

By the Sobelev's embedding theorem, we may assume that

$$u_n \to u \text{ in } L^2(\Omega).$$

So we get

$$\delta = 1 - \int_\Omega a(x) u^2 \, dx \geq \int_\Omega [|\nabla u|^2 - a(x) u^2] \, dx \geq \lambda_{n+1} \int_\Omega u^2 \, dx.$$

If $u = 0$, then $\delta = 1$. If $u \neq 0$, then $\delta \geq \lambda_{n+1} \int_\Omega u^2 \, dx > 0$. □

Lemma 3.2. *For every $u \in Y$, we have*

$$\int_\Omega [|\nabla u|^2 - a(x) u^2] \, dx \leq \lambda_n \int_\Omega u^2 \, dx. \tag{3.2}$$

Proof. If $u \in Y$, then

$$u = \sum_{i=1}^{n} c_i e_i,$$

where $c_i = \int_\Omega ue_i\,dx (i = 1, 2, \ldots, n)$. Hence

$$\begin{aligned}(u,u)_{\lambda_0} &= \Big(\sum_{i=1}^n c_ie_i, \sum_{j=1}^n c_je_j\Big)_{\lambda_0} = \sum_{i=1}^n\sum_{j=1}^n c_ic_j(e_i,e_j)_{\lambda_0}\\ &= \sum_{i=1}^n\sum_{j=1}^n c_ic_j\Big[\int_\Omega(\nabla e_i\cdot\nabla e_j - a(x)e_ie_j)\,dx + \lambda_0\int_\Omega e_ie_j\,dx\Big]\\ &= \sum_{i=1}^n\sum_{j=1}^n c_ic_j(\lambda_i+\lambda_0)\int_\Omega e_ie_j\,dx = \sum_{i=1}^n c_i^2(\lambda_i+\lambda_0)\\ &\le (\lambda_n+\lambda_0)\sum_{i=1}^n c_i^2 = (\lambda_n+\lambda_0)\int_\Omega u^2\,dx.\end{aligned}$$

Thus for $u \in Y$, by the definition of $(\cdot,\cdot)_{\lambda_0}$, it follows that

$$\int_\Omega [|\nabla u|^2 - a(x)u^2]\,dx \le \lambda_n \int_\Omega u^2\,dx. \qquad \Box$$

Lemma 3.3. *Under (f_1), (f_2), (f_3) and (f_5) for when $p = 2$, the functional I defined by (2.1) possesses the linking geometric structure, i.e. for $\rho > r > 0$, let $z = \frac{e_{n+1}}{||e_{n+1}||}r \in Z$ and define*

$$\begin{aligned}M^\rho &:= \{u = y + \mu z\colon \|u\| \le \rho,\ \mu \ge 0,\ y \in Y\},\\ M_0^\rho &:= \{u = y + \mu z\colon y \in Y,\ \|u\| = \rho,\ \mu \ge 0 \ \text{ or }\ \|u\| \le \rho,\ \mu = 0\},\\ N_r &:= \{u \in Z\colon \|u\| = r\},\quad c = \inf_{\gamma\in\Gamma}\max_{u\in M^\rho} I(\gamma(u)),\\ \Gamma &= \{\gamma \in C(M^\rho, H_0^1(\Omega))\colon \gamma|_{M_0^\rho} = I_d\}.\end{aligned}$$

If $I \in C^1(X, \mathbf{R})$, then

$$b = \inf_{N_r} I > a = \max_{M_0^\rho} I.$$

Proof. We hope to find $0 < r < 1 < \rho$ such that

$$b = \inf_{N_r} I > a = \max_{M_0^\rho} I.$$

Using (f_1) and (f_2), we obtain

$$(\forall\epsilon > 0)(\exists c_\epsilon > 0)\colon |F(x,s)| \le \epsilon|s|^2 + C_\epsilon|s|^q, \tag{3.3}$$

for any $x \in \Omega$ and $s \in \mathbf{R}^1$. For every $u \in N_r$, we have that $u \in Z$ and $\|u\| = r$. We deduce from Lemma 3.1, (3.3) and the Sobolev embedding theorem that

$$\begin{aligned}I(u) &= \frac{1}{2}\int_\Omega\Big[|\nabla u|^2 - a(x)u^2\Big]dx - \int_\Omega F(x,u)\,dx\\ &\ge \frac{\delta}{2}\|u\|^2 - \epsilon\int_\Omega |u|^2\,dx - C_\epsilon|u|_q^q \ge \frac{\delta}{2}\|u\|^2 - C\epsilon\|u\|^2 - \widetilde{C_\epsilon}\|u\|^q\\ &\ge \frac{\delta}{2}r^2 - o(r^2) \quad \Big(\frac{o(r^2)}{r^2} \to 0 \text{ as } r \to 0\Big).\end{aligned}$$

Then there exists $r > 0$ such that $b = \inf\limits_{\|u\|=r, u\in Z} I(u) > 0$.

For every $u \in M_0^\rho$, if $u = y+\mu z,\ \|u\| \le \rho,\ \mu = 0$, then $u = y \in Y$. By Lemma 3.2 and hypothesis (f_5), we know that

$$I(u) = \frac{1}{2}\int_\Omega \Big[|\nabla u|^2 - a(x)u^2\Big]dx - \int_\Omega F(x,u)\,dx$$
$$\le \frac{\lambda_n}{2}\int_\Omega |u|^2\,dx - \int_\Omega F(x,u)\,dx \le \int_\Omega \Big[\frac{\lambda_n}{2}u^2 - F(x,u)\Big]\,dx \le 0.$$

It follows from (f_3) that

$$\forall N,\ \exists C_N \text{ such that } F(x,s) \ge Ns^2 - C_N \tag{3.4}$$

for any $x \in \Omega$ and $s \in^1$. For $u \in M_0^\rho, u = y+\mu z, \mu \ge 0$, we have by (3.4) that

$$I(u) = \frac{1}{2}\int_\Omega \Big[|\nabla u|^2 - a(x)u^2\Big]dx - \int_\Omega F(x,u)\,dx$$
$$\le \frac{1}{2}\|u\|^2 + |a|_{\frac{N}{2}}\frac{|u|^2_{2^*}}{2} - \int_\Omega (Nu^2 - C_N)\,dx.$$

Since $M_0^\rho = Y \oplus RZ$, we have $\dim(Y\oplus RZ) < \infty$.

On the finite dimensional space $Y \oplus RZ$, all norms are equivalent, so we have

$$I(u) \le \frac{1}{2}\|u\|^2 + C\|u\|^2 - N\tilde{C}\|u\|^2 + \tilde{C}_N \le (\frac{1}{2} + C - N\tilde{C})\|u\|^2 + \tilde{C}_N.$$

Fixed N with $\frac{1}{2} + C - N\tilde{C} < 0$, then

$$I(u) \to -\infty \text{ as } \|u\| = \rho \to +\infty.$$

Take ρ large enough, r small enough with $\rho > 1 > r > 0$. Then

$$\max_{M_0^\rho} I(u) \le 0 \le \frac{\delta}{4}r^2 \le \frac{\delta}{2}r^2 - o(r^2) \le \inf_{N_r} I(u).$$

Hence,

$$b = \inf_{N_r} I(u) > a = \max_{M_0^\rho} I(u).$$

□

Lemma 3.4. *If (f_2), (f_3) and (f_4) hold, then the functional I defined by (2.1) satisfies the $(C)_c$ condition for $c \in \mathbf{R}^1$.*

Proof. Suppose that $\{u_n\} \subset H_0^1(\Omega)$ is a $(C)_c$ sequence for $I(u)$, that is,

$$I(u_n) \to c,\ \|I'(u_n)\|_*(1+\|u_n\|) \to 0,$$

which shows that

$$c = I(u_n) + o(1),\ \langle I'(u_n), u_n\rangle = o(1), \tag{3.5}$$

where $o(1) \to 0$ as $n \to 0$.

(i) (u_n) is bounded in $H_0^1(\Omega)$. For this purpose, we suppose, by contradiction, that

$$\|u_n\| \to +\infty, \tag{3.6}$$

and let $w_n = \frac{u_n}{\|u_n\|}$. Then $w_n \in H_0^1(\Omega)$ with

$$\|w_n\| = 1.$$

Passing to a subsequence, there exists a $w \in H_0^1(\Omega)$ such that

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega).$$

Since Ω is bounded, by the Sobolev's embedding theorem we may assume that

$$\begin{cases} w_n(x) \to w(x) & \text{a.e. in } \Omega, \\ w_n \to w & \text{in } L^q(\Omega),\ 2 \le q < 2^*. \end{cases} \tag{3.7}$$

Let $\Omega_{\neq} = \{x \in \Omega \colon w(x) \neq 0\}$, then

$$\lim_{n\to+\infty} w_n(x) = \lim_{n\to+\infty} \frac{u_n(x)}{\|u_n\|} = w(x) \neq 0 \text{ in } \Omega_{\neq}$$

and (3.6) implies that

$$|u_n| \to +\infty \text{ a.e. in } \Omega_{\neq}. \tag{3.8}$$

By (f_3), we see that

$$\lim_{n\to+\infty} \frac{F(x,u_n(x))}{|u_n(x)|^2} = +\infty \text{ a.e. in } \Omega_{\neq}.$$

This means that

$$\lim_{n\to+\infty} \frac{F(x,u_n(x))}{|u_n(x)|^2} |w_n(x)|^2 = +\infty \text{ a.e. in } \Omega_{\neq}. \tag{3.9}$$

By (f_3), there is an $N_0 > 0$ such that

$$\frac{F(x,s)}{|s|^2} > 1, \tag{3.10}$$

for any $x \in \Omega$ and $s \in \mathbf{R}^1$ with $|s| \ge N_0$. Since $F(x,s)$ is continuous on $\bar\Omega \times [-N_0, N_0]$, there is an $M > 0$ such that

$$|F(x,s)| \le M, \tag{3.11}$$

for $(x,t) \in \bar\Omega \times [-N_0, N_0]$. From (3.10) and (3.11), we see that there is a constant C, such that for any $(x,s) \in \bar\Omega \times \mathbf{R}^1$, we have

$$F(x,s) \ge C,$$

which shows that

$$\frac{F(x,u_n(x)) - C}{\|u_n\|^2} \ge 0.$$

This means that

$$\frac{F(x,u_n(x))}{|u_n(x)|^2} |w_n(x)|^2 - \frac{C}{\|u_n\|^2} \ge 0. \tag{3.12}$$

Since by (3.5) we have that

$$c = I(u_n) + o(1) = \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\int_\Omega a(x)u_n^2\,dx - \int_\Omega F(x,u_n)\,dx + o(1),$$

which shows that

$$\|u_n\|^2 - \int_\Omega a(x)u_n^2\,dx = 2c + 2\int_\Omega F(x,u_n)\,dx + o(1). \tag{3.13}$$

Since $\|w_n\|^2 = 1$ and $\frac{2c}{\|u_n\|^2} = o(1),\ n \to \infty$, we have

$$\frac{1}{2} - \frac{1}{2}\int_\Omega a(x)w_n^2\,dx = \int_\Omega \frac{F(x,u_n)}{u_n^2} w_n^2\,dx + o(1). \tag{3.14}$$

We claim that $|\Omega_{\neq}| = 0$.

If $|\Omega_{\neq}| \neq 0$, then by the Fatou's Lemma, (f_3) and the Hölder's inequality, we get

$$\begin{aligned}
+\infty = (+\infty)|\Omega_{\neq}| &= \Big[\int_{\Omega_{\neq}} \liminf_{n\to+\infty} \frac{F(x,u_n(x))}{|u_n(x)|^2}|w_n(x)|^2\,dx - \int_{\Omega_{\neq}} \limsup_{n\to+\infty} \frac{C}{\|u_n\|^2}\,dx\Big] \\
&= \int_{\Omega_{\neq}} \liminf_{n\to+\infty} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^2}|w_n(x)|^2 - \frac{C}{\|u_n\|^2}\Big)\,dx \\
&\le \liminf_{n\to+\infty} \int_{\Omega_{\neq}} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^2}|w_n(x)|^2 - \frac{C}{\|u_n\|^2}\Big)\,dx \\
&\le \liminf_{n\to+\infty} \int_{\Omega} \Big(\frac{F(x,u_n(x))}{|u_n(x)|^2}|w_n(x)|^2 - \frac{C}{\|u_n\|^2}\Big)\,dx \\
&= \liminf_{n\to+\infty} \int_{\Omega} \frac{F(x,u_n(x))}{\|u_n\|^2}\,dx - \limsup_{n\to+\infty} \int_{\Omega} \frac{C}{\|u_n\|^2}\,dx \\
&= \liminf_{n\to+\infty} \int_{\Omega} \frac{F(x,u_n(x))}{\|u_n\|^2}\,dx \le \frac{1}{2} - \frac{1}{2}\int_{\Omega} a(x)w_n^2\,dx + o(1) \\
&\le \frac{1}{2} + C|a(x)|_{\frac{N}{2}} + o(1) < +\infty,
\end{aligned}$$

which is a contradiction. This shows that

$$|\Omega_{\neq}| = 0.$$

Hence $w(x) = 0$ a.e. in Ω.

Since $I(tu_n)$ is continuous in $t \in [0,1]$, there exists $t_n \in [0,1]$, $n = 1, 2, \ldots$, such that

$$I(t_n u_n) = \max_{0\le t\le 1} I(tu_n).$$

As $\langle I'(u_n), u_n\rangle = o(1)$, we see that

$$\langle I'(t_n u_n), t_n u_n\rangle = o(1).$$

By (f_4), we then get for $t \in [0,1]$ that

$$\begin{aligned}
2I(tu_n) \le 2I(t_n u_n) &= 2I(t_n u_n) - \langle I'(t_n u_n), t_n u_n\rangle + o(1) \\
&= \int_{\Omega} [t_n u_n f(x, t_n u_n) - 2F(x, t_n u_n)]\,dx + o(1) \\
&\le \int_{\Omega} [u_n f(x, u_n) - 2F(x, u_n) + C_*]\,dx + o(1). \\
&\le (\|u_n\|^2 + 2c - \|u_n\|^2 + o(1)) + C_*|\Omega| + o(1) \\
&\le 2c + C_*|\Omega| + o(1)
\end{aligned}$$

where we use (3.5) and (3.13). On the other hand, since the functional $\chi\colon \mathscr{D}_0^{1,2}(\Omega) \to \mathbf{R}\colon u \mapsto \int_{\Omega} a(x)u^2\,dx$ is weakly continuous when $u \in L^{\frac{N}{2}}(\Omega)$, by (f_2) and $w_n \to 0$ in $L^q(\Omega)$, we get for any $R > 0$, that

$$2I(Rw_n) = \|Rw_n\|^2 - R^2\int_{\Omega} a(x)w_n^2\,dx - 2\int_{\Omega} F(x, Rw_n)\,dx = R^2 + o(1).$$

So we have

$$R^2 + o(1) = 2I(Rw_n) \le 2c + C_*|\Omega| + o(1).$$

Letting $n \to \infty$ we get

$$R^2 \le C_*|\Omega| + 2c.$$

Letting $R \to \infty$ we get a contradiction. This proves that $\|u_n\| \le C < +\infty$ for some constant C.

(ii) $\{u_n\}$ has a convergent subsequence in $H_0^1(\Omega)$. Since $\|u_n\| \le C$, passing to a subsequence, we may assume that there exists $u_0 \in H_0^1(\Omega)$ such that

$$u_n \rightharpoonup u_0 \text{ in } H_0^1(\Omega).$$

By $|\Omega| < +\infty$ and the Sobelev's embedding theorem, we may assume that

$$\begin{cases} u_n \to u_0 & \text{in } L^q(\Omega),\ 2 \le q < 2^*, \\ u_n \to u_0 & \text{a.e. in } \Omega. \end{cases} \tag{3.15}$$

By (f_2)(3.15) and the Lebesgue's dominated convergent theorem, we have that

$$\begin{cases} \int_\Omega f(x,u_n)u_n\,dx \to \int_\Omega f(x,u_0)u_0\,dx, \\ \int_\Omega f(x,u_n)u_0\,dx \to \int_\Omega f(x,u_0)u_0\,dx. \end{cases} \tag{3.16}$$

On the other hand,

$$\begin{aligned} \|u_n - u_0\|^2 = \langle I'(u_n) - I'(u_0), u_n - u_0\rangle + \int_\Omega a(x)(u_n - u_0)^2\,dx \\ + \int_\Omega [f(x,u_n) - f(x,u_0)](u_n - u_0)\,dx. \end{aligned}$$

By $I'(u_n) \to 0$ and $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$, we know that

$$\langle I'(u_n) - I'(u_0), u_n - u_0\rangle \to 0.$$

By (3.16), we obtain

$$\Big|\int_\Omega [f(x,u_n) - f(x,u_0)](u_n - u_0)\,dx\Big| \to 0.$$

Since the functional $\chi\colon \mathscr{D}_0^{1,2}(\Omega) \to \mathbf{R}\colon u \mapsto \int_\Omega a(x)u^2\,dx$ is weakly continuous when $u \in L^{\frac{N}{2}}(\Omega)$, we obtain

$$\Big|\int_\Omega a(x)(u_n - u_0)^2\,dx\Big| \to 0.$$

Hence,

$$u_n \to u_0 \text{ in } H_0^1(\Omega).$$

Therefore, for any $c \in \mathbf{R}$, $I(u)$ satisfies the $(C)_c$ condition. □

The Proof of Theorem 1.1. Combing the results of Lemma 3.3 and Lemma 3.4, we will complete the proof by applying Proposition 2.10.

Remark 3.5. If $\lambda_1 > 0$, then it suffices to use the mountain-pass theorem instead of the linking theorem to prove the existence of nontrivial solutions of (1.1).

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