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SCHWARZIAN DERIVATIVES OF CONVEX MAPPINGS

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Abstract. A simple proof is given for Nehari's theorem that an analytic function f which maps the unit disk onto a convex region has Schwarzian norm $\|\mathscr{S}f\| \le 2$. The inequality in sharper form leads to the conclusion that no convex mapping with $\|\mathscr{S}f\| = 2$ can map onto a quasidisk. In particular, every bounded convex mapping has Schwarzian norm $\|\mathscr{S}f\| < 2$. The analysis involves a structural formula for the pre-Schwarzian of a convex mapping, which is studied in further detail.

1. Introduction

Let f be a function analytic and locally univalent in the unit disk $\mathbf{D}$, and let

$$\mathscr{S}f = (f''/f')' - \tfrac{1}{2}(f''/f')^2$$

denote its Schwarzian derivative. Nehari [12] proved that if

$$|\mathscr{S}f(z)| \le \frac{2}{(1-|z|^2)^2}, \quad z \in \mathbf{D}, \tag{1}$$

then f is univalent in $\mathbf{D}$. In the converse direction a result of Kraus [11], rediscovered by Nehari [12], says that univalence of f implies $|\mathscr{S}f(z)| \le 6(1-|z|^2)^{-2}$. Both of the constants 2 and 6 are best possible. However, Nehari [13] later showed that the inequality (1) holds whenever f maps the disk conformally onto a convex region. For a proof he approximated a general convex mapping by a mapping onto a convex polygon, then invoked the Schwarz–Christoffel formula and used some delicate algebraic manipulations to arrive at the desired conclusion. In view of the technical difficulty of Nehari's proof, it may be worthwhile to observe that a direct analytic argument, based only on the Schwarz lemma, leads to the same result.

Recall first that if f maps the disk conformally onto a convex region, then the function

$$g(z) = 1 + \frac{zf''(z)}{f'(z)} \tag{2}$$

has positive real part in $\mathbf{D}$. (See for instance [6].) Since $g(0) = 1$, this says that g is subordinate to the half-plane mapping $\ell(z) = (1+z)/(1-z)$, so that $g(z) = \ell(\omega(z))$

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for some Schwarz function ω. In other words,

$$\frac{zf''(z)}{f'(z)} = \frac{1+\omega(z)}{1-\omega(z)} - 1 = \frac{2\omega(z)}{1-\omega(z)},$$

where ω is analytic and has the property $|\omega(z)| \le |z|$ in $\mathbf{D}$. With the notation $\varphi(z) = \omega(z)/z$, this gives the representation

$$\frac{f''(z)}{f'(z)} = \frac{2\varphi(z)}{1-z\varphi(z)} \tag{3}$$

for the pre-Schwarzian, where φ is analytic and satisfies $|\varphi(z)| \le 1$ in $\mathbf{D}$. Straightforward calculation now gives the Schwarzian of f in the form

$$\mathscr{S}f(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{2\varphi'(z)}{(1-z\varphi(z))^2}. \tag{4}$$

But $|\varphi'(z)| \le (1-|\varphi(z)|^2)/(1-|z|^2)$ by the invariant form of the Schwarz lemma, so we conclude that

$$|\mathscr{S}f(z)| \le 2\frac{1-|\varphi(z)|^2}{(1-|z|^2)(1-|z\varphi(z)|)^2} \le \frac{2}{(1-|z|^2)^2}, \tag{5}$$

which is the inequality (1).

In other language, the inequality (5) says that the Schwarzian norm

$$\|\mathscr{S}f\| = \sup_{z\in\mathbf{D}}(1-|z|^2)^2|\mathscr{S}f(z)|$$

of a convex mapping is no larger than 2. The bound is best possible since the parallel strip mapping

$$L(z) = \frac{1}{2}\log\frac{1+z}{1-z} \tag{6}$$

has Schwarzian $\mathscr{S}L(z) = 2(1-z^2)^{-2}$.

Nehari [13] also stated that $\|\mathscr{S}f\| < 2$ if the convex mapping f is bounded. We will show that the statement is correct, although Nehari's proof appears to be erroneous (more about this later). Ahlfors and Weill [1] showed that any analytic function with $\|\mathscr{S}f\| < 2$ is not only univalent, but maps the disk onto a Jordan domain and actually has a quasiconformal extension to the whole plane. As a consequence, every bounded convex domain is a quasidisk.

However, the last statement follows easily from a standard geometric characterization of quasidisks. On the other hand, using a known property of John domains, we will show in Section 3 of this paper that the image $f(\mathbf{D})$ of a convex function with Schwarzian norm $\|\mathscr{S}f\| = 2$ can not be a quasidisk on the Riemann sphere. This will allow us to conclude indirectly that every bounded convex mapping has Schwarzian norm less than 2. The results are illustrated by some examples in Section 4.

The structural formula (3) plays an important role in our analysis, and this is studied in some detail. In Section 2 we develop a sharper form of Nehari's inequality $\|\mathscr{S}f\| \le 2$ for convex mappings f, and we find that certain geometric properties of f correspond to analytic properties of the function φ that generates its pre-Schwarzian.

After an earlier version of this paper was completed we became aware of a paper by Koepf [10], which contains our theorem that $f(\mathbf{D})$ is not a quasidisk when f is convex and $\|\mathscr{S}f\| = 2$. However, Koepf's proof appeals to Nehari's theorem that

$\|\mathscr{S}f\| < 2$ for every bounded convex mapping, and the error in Nehari's proof was not observed and corrected until now.

2. A closer look at convex mappings

We now take a closer look at the expression (3) for the pre-Schwarzian of a convex mapping. Observe first that the formula gives also a sufficient condition for convexity. In other words, if f is analytic and locally univalent in $\mathbf{D}$ and if f''/f' has the form (3) for some analytic function φ with $|\varphi(z)| \le 1$, then f is univalent and it maps the disk conformally onto a convex region. Indeed, the assumption (3) implies that the function (2) has positive real part, and a familiar argument (cf. [6, p. 43]) completes the proof. Note that f maps the disk onto a half-plane precisely when $\varphi(z) \equiv e^{i\theta}$, a unimodular constant.

The representation (3) says that the function

$$\varphi(z) = \frac{f''(z)/f'(z)}{2 + zf''(z)/f'(z)}$$

satisfies $|\varphi(z)|^2 \le 1$, which gives by simple calculation the stronger inequality

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \ge \frac{1}{4}(1 - |z|^2)\left|\frac{f''(z)}{f'(z)}\right|^2 \tag{7}$$

for every convex mapping f. Strict inequality holds for all $z \in \mathbf{D}$ unless $\varphi(z) \equiv e^{i\theta}$, which means that f is a half-plane mapping.

We now return to Schwarzian derivatives of convex mappings and the formula (4), with $|\varphi(z)| \le 1$ in $\mathbf{D}$. Recall that $(1 - |z|^2)|\varphi'(z)| \le 1 - |\varphi(z)|^2$, with strict inequality for all $z \in \mathbf{D}$ unless $\varphi(z) \equiv e^{i\theta}$ or φ is a Möbius automorphism of the disk. Applying this inequality, we find after short calculation that

$$(1 - |z|^2)^2|\mathscr{S}f(z)| + 2\left|\frac{\varphi(z) - \overline{z}}{1 - z\varphi(z)}\right|^2 \le 2. \tag{8}$$

Strict inequality holds in (8) for all $z \in \mathbf{D}$ unless $\varphi(z) \equiv e^{i\theta}$ or φ is a Möbius self-mapping of $\mathbf{D}$. In the first case, f is a half-plane mapping and $\mathscr{S}f(z) \equiv 0$. In either case, equality holds for all $z \in \mathbf{D}$. In view of the relation (3), the inequality (8) reduces to

$$(1 - |z|^2)^2|\mathscr{S}f(z)| + 2\left|\overline{z} - \frac{1}{2}(1 - |z|^2)\frac{f''(z)}{f'(z)}\right|^2 \le 2. \tag{9}$$

In this form the inequality appears in a paper of Kim and Minda [9], with a more geometric proof based on estimates of the hyperbolic metric. For mappings onto convex polygons, Nehari's proof also led him to an equivalent form of the inequality (9) (cf. [13, formula (7)]).

For $z = 0$, the inequality (9) says that convex univalent functions $f(z) = z + a_2z^2 + \dots$ satisfy the coefficient inequality

$$\left|a_3 - a_2^2\right| \le \frac{1}{3}\left(1 - |a_2|^2\right),$$

a result due to Hummel [8] and given a shorter proof by Trimble [15]. Conversely, a Koebe transform

$$F(z) = \frac{f\left(\frac{z+\zeta}{1+\overline{\zeta}z}\right) - f(\zeta)}{(1-|\zeta|^2)f'(\zeta)} = z + A_2(\zeta)z^2 + A_3(\zeta)z^3 + \dots$$

shows that the coefficient inequality implies the inequality (9).

Clearly, the inequality (9) gives a stronger form of the result that $\|\mathscr{S}f\| \le 2$ for all convex mappings. It also implies that

$$\left|\overline{z} - \frac{1}{2}(1-|z|^2)\frac{f''(z)}{f'(z)}\right|^2 \le 1,$$

which reduces to the inequality (7) and is a sufficient condition for convexity. Thus the inequality (9) provides a necessary and sufficient condition for convexity.

On the other hand, the inequality $\|\mathscr{S}f\| \le 2$ is far from sufficient for convexity. In fact, for no $\varepsilon > 0$ does the condition $\|\mathscr{S}f\| \le \varepsilon$ imply that f is a convex mapping. This can be seen by an example constructed in the paper [3]. There it is found that for $0 < t < 1$ the function

$$f(z) = \frac{(1+z)^a - (1-z)^a}{(1+z)^a + (1-z)^a}, \quad \text{where } a = \sqrt{1+t},$$

has Schwarzian $\mathscr{S}f(z) = -2t(1-z^2)^{-2}$ and maps the disk onto a nonconvex region bounded by two circular arcs.

Nehari's proof that $\|\mathscr{S}f\| \le 2$ for all convex mappings has a significant implication for the Schwarz–Christoffel construction. If f is a mapping of the unit disk onto the interior of an n-gon, the Schwarz–Christoffel formula states that

$$f'(z) = \frac{C}{(z-z_1)^{2\beta_1} \cdots (z-z_n)^{2\beta_n}}, \tag{10}$$

where C is a complex constant, $z_k \in \partial\mathbf{D}$ are the preimages of the vertices, and $2\beta_k\pi$ are the exterior angles at the vertices of the polygon, with $-1 < \beta_k < 1$ and $\beta_1 + \cdots + \beta_n = 1$. The polygon is convex if and only if all $\beta_k > 0$. The convex polygon is bounded if and only if $0 < \beta_k < \frac{1}{2}$ for all k. According to Nehari's calculations in [13], if a function f has a derivative of the form (10) with arbitrary parameters $z_k \in \partial\mathbf{D}$ and $0 < \beta_k < 1$, then $\|\mathscr{S}f\| \le 2$. It then follows from the earlier theorem of Nehari [12] that f provides a *univalent* mapping onto the interior of a convex n-gon. A more direct proof can be given as follows. The formula (10) leads to the expression

$$\frac{zf''(z)}{f'(z)} = -2\sum_{k=1}^{n} \frac{\beta_k z}{z - z_k},$$

which implies that

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} = 1 - 2\sum_{k=1}^{n} \beta_k \,\mathrm{Re}\left\{\frac{z}{z-z_k}\right\} > 1 - 2\sum_{k=1}^{n} \frac{1}{2}\beta_k = 0, \quad z \in \mathbf{D},$$

because all β_k are positive and their sum is 1. Because f is locally univalent, we conclude that f is univalent and convex in $\mathbf{D}$.

We have shown that convex mappings of the disk have the representation (3) for some analytic function φ with $|\varphi(z)| \le 1$. It is interesting to ask how the geometric properties of the mapping are encoded in behavior of φ. When φ is a unimodular constant, one obtains a half-plane mapping, whereas an automorphism of the disk generates a mapping onto a parallel strip or an infinite sector. Our next result describes the situation when φ is a finite Blaschke product of higher degree.

Theorem 1. *Let f be a convex mapping satisfying* (3) *for some analytic function φ with $|\varphi(z)| \leq 1$. Then φ is a finite Blaschke product of degree $n \geq 2$ if and only if f maps $\mathbf{D}$ onto the interior of a (bounded or unbounded) convex $(n+1)$-gon.*

Proof. Suppose first that φ is a finite Blaschke product of degree n,

$$\varphi(z) = e^{i\theta} \prod_{k=1}^{n} \frac{z - \alpha_k}{1 - \overline{\alpha_k} z}, \quad |\alpha_k| < 1. \tag{11}$$

After rotating f we may assume that $e^{i\theta} = 1$. The right-hand side of (3) is a rational function with poles at the roots of $z\varphi(z) = 1$; that is, at $n+1$ distinct points $z_1, \ldots, z_{n+1}$ on the unit circle. A partial fraction expansion gives

$$\frac{\varphi(z)}{1 - z\varphi(z)} = -\sum_{k=1}^{n+1} \frac{\beta_k}{z - z_k} \tag{12}$$

for some complex constants $\beta_k \neq 0$. We claim that $\beta_1 + \cdots + \beta_{n+1} = 1$. To see this, combine (11) and (12) to write

$$\sum_{k=1}^{n+1} \frac{\beta_k}{z - z_k} = \frac{\prod_{k=1}^{n} (z - \alpha_k)}{z \prod_{k=1}^{n} (z - \alpha_k) - \prod_{k=1}^{n} (1 - \overline{\alpha_k} z)}. \tag{13}$$

On the right-hand side of (13) is a quotient of two monic polynomials, the numerator of degree n and the denominator of degree $n+1$. But the left-hand side has the form

$$\sum_{k=1}^{n+1} \frac{\beta_k}{z - z_k} = \frac{(\beta_1 + \cdots + \beta_{n+1}) z^n + \ldots}{\prod_{k=1}^{n+1} (z - z_k)},$$

and so $\beta_1 + \cdots + \beta_{n+1} = 1$.

Next we show that all β_k are real. Write $\beta_k = b_k + i c_k$, so that

$$\frac{z\varphi(z)}{z\varphi(z) - 1} = \sum_{k=1}^{n+1} \frac{\beta_k z}{z - z_k} = \sum_{k=1}^{n+1} \frac{b_k z}{z - z_k} + i \sum_{k=1}^{n+1} \frac{c_k z}{z - z_k}.$$

For $|z| = 1$ and $z \neq z_1, \ldots, z_{n+1}$, we infer that

$$\begin{aligned} \frac{1}{2} = \operatorname{Re}\left\{ \frac{z\varphi(z)}{z\varphi(z) - 1} \right\} &= \sum_{k=1}^{n+1} b_k \operatorname{Re}\left\{ \frac{z}{z - z_k} \right\} - \sum_{k=1}^{n+1} c_k \operatorname{Im}\left\{ \frac{z}{z - z_k} \right\} \\ &= \frac{1}{2} - \sum_{k=1}^{n+1} c_k \operatorname{Im}\left\{ \frac{z}{z - z_k} \right\}, \end{aligned}$$

since $b_1 + \cdots + b_{n+1} = 1$. But $\left| \operatorname{Im}\left\{ \frac{z}{z - z_k} \right\} \right| \to \infty$ as z tends to z_k along the unit circle, so we deduce that $c_1 = c_2 = \cdots = c_{n+1} = 0$. Hence all of the constants β_k are real. In view of (3), we conclude that the convex mapping f has a derivative of the form (10), with n replaced by $n+1$. It follows geometrically that f maps the disk locally onto a polygonal region with exterior angles $2\beta_k \pi$ at the vertices. Since f is convex, all of the angles are positive, and so f maps the disk onto the interior of a convex $(n+1)$-gon.

Conversely, suppose that f maps the disk onto the interior of a convex $(n+1)$-gon. Then by (3) and the Schwarz–Christoffel formula, we have

$$\frac{z\varphi(z)}{1-z\varphi(z)} = -\sum_{k=1}^{n+1} \frac{\beta_k z}{z-z_k} = r(z), \tag{14}$$

say, where z_k are distinct points on the unit circle and β_k are positive numbers whose sum is 1. In particular, φ is a rational function, analytic in the unit disk, given by

$$z\varphi(z) = \frac{r(z)}{1+r(z)}. \tag{15}$$

But $\mathrm{Re}\{r(z)\} = -\frac{1}{2}$ for all $z \in \partial\mathbf{D}$ with $z \neq z_1, \ldots, z_{n+1}$, which implies that $|\varphi(z)| = 1$. The exceptional points z_k are poles of $r(z)$, and so $|\varphi(z_k)| = 1$ as well, by (15). Thus $|\varphi(z)| \equiv 1$ on $\partial\mathbf{D}$ and φ is a finite Blaschke product. Finally, because the relation (14) shows that $z\varphi(z) = 1$ at precisely $n+1$ points $z_1, \ldots, z_{n+1}$ on the unit circle, it follows that $z\varphi(z)$ winds about the origin $n+1$ times, and so the Blaschke product φ has degree n. □

The next theorem describes the functions φ that generate bounded convex mappings.

Theorem 2. *Let f be a convex mapping of the unit disk, and let φ be the analytic function associated with f by the formula* (3). *Then the image $f(\mathbf{D})$ is bounded if and only if*

$$\limsup_{|z|\to 1} \frac{1-|z|}{|1-z\varphi(z)|} < \frac{1}{2}. \tag{16}$$

Proof. If the relation (16) holds, then it follows from the representation (3) that

$$|f'(z)| \leq \frac{C}{(1-|z|)^a}, \quad R < |z| < 1,$$

for some constants $C > 0$, $a < 1$, and $R > 0$. Hence $f(z)$ is bounded in $\mathbf{D}$.

Conversely, suppose that f is a convex mapping with bounded image $\Omega = f(\mathbf{D})$. Then as proved in Section 3 of this paper, f has Schwarzian norm $\|\mathscr{S}f\| \leq 2t$ for some $t < 1$. Consider the function

$$g(z) = \frac{f(z)}{1+a_2 f(z)}, \quad \text{where } a_2 = \tfrac{1}{2} f''(0).$$

Then $\|\mathscr{S}g\| = \|\mathscr{S}f\| \leq 2t$, and since $g''(0) = 0$ it follows from [4] that

$$\left|\frac{g''(z)}{g'(z)}\right| \leq \frac{2t|z|}{1-|z|^2}, \quad z \in \mathbf{D}.$$

Therefore,

$$\limsup_{|z|\to 1} (1-|z|^2) \left|\frac{g''(z)}{g'(z)}\right| < 2.$$

On the other hand, since $\|\mathscr{S}f\| < 2$, we also know from [4] that $-1/a_2 \notin \overline{\Omega}$, and consequently $|1 + a_2 f(z)| \geq \delta > 0$ for all $z \in \mathbf{D}$. But

$$\frac{g''(z)}{g'(z)} = \frac{f''(z)}{f'(z)} - \frac{2a_2 f'(z)}{1+a_2 f(z)},$$

and the boundedness of Ω implies that $(1-|z|^2)|f'(z)| \to 0$ as $|z| \to 1$, so we infer that

$$\limsup_{|z|\to 1}\,(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| < 2.$$

In view of (3), we conclude that

$$\limsup_{|z|\to 1}\frac{(1-|z|)|\varphi(z)|}{|1-z\varphi(z)|} < \frac{1}{2}.$$

Since $|\varphi(z) \le 1$, the relation (16) follows. Indeed, if L denotes the "lim sup" in (16) and $\{z_n\}$ is an extremal sequence, then $L = 0$ unless $\limsup|\varphi(z_n)| = 1$. $\square$

3. Convex mappings with $\|\mathscr{S}f\| = 2$

We have seen that $\|\mathscr{S}f\| \le 2$ for every convex mapping of the disk. We now show that $\|\mathscr{S}f\| < 2$ when the convex mapping is bounded. We will prove this indirectly as a consequence of the stronger statement that the image of a convex mapping with $\|\mathscr{S}f\| = 2$ cannot be a quasidisk on the Riemann sphere. Because a bounded convex domain is a quasidisk, it will then follow that bounded convex mappings cannot have Schwarzian norm $\|\mathscr{S}f\| = 2$, and so $\|\mathscr{S}f\| < 2$.

For a study of convex mappings f with $\|\mathscr{S}f\| = 2$, our analysis will be based on the formula (3) that expresses the pre-Schwarzian of a convex mapping in terms of an analytic function φ with $|\varphi(z)| \le 1$. We may confine attention to the case where $|\varphi(z)| < 1$ in $\mathbf{D}$, since the functions $\varphi(z) \equiv e^{i\theta}$ correspond to half-plane mappings f, for which $\mathscr{S}f(z) \equiv 0$. It will be useful to know how φ changes when f is precomposed with a Möbius automorphism of the disk. We state the result as a lemma.

Lemma 1. *Let f be a convex mapping with*

$$\frac{f''(z)}{f'(z)} = \frac{2\varphi(z)}{1-z\varphi(z)}$$

for some function analytic φ with $|\varphi(z)| < 1$ in $\mathbf{D}$. For fixed $z_0 \in \mathbf{D}$, define the Möbius automorphism $\sigma(z) = (z+z_0)/(1+\overline{z_0}z)$ and let $g = f\circ\sigma$. Then g is a convex mapping of $\mathbf{D}$ and

$$\frac{g''(z)}{g'(z)} = \frac{2\lambda(z)}{1-z\lambda(z)}, \quad \textit{where } \lambda(z) = \frac{\varphi(\sigma(z))-\overline{z_0}}{1-z_0\varphi(\sigma(z))}.$$

Proof. Calculations give

$$\begin{aligned}\frac{g''(z)}{g'(z)} &= \frac{f''(\sigma(z))}{f'(\sigma(z))}\,\sigma'(z) + \frac{\sigma''(z)}{\sigma'(z)}\\ &= \frac{2}{1+\overline{z_0}z}\left[\frac{(1-|z_0|^2)\varphi(\sigma(z))}{(1+\overline{z_0}z)-(z+z_0)\varphi(\sigma(z))} - \overline{z_0}\right]\\ &= 2\frac{\varphi(\sigma(z))-\overline{z_0}}{(1-z_0\varphi(\sigma(z)))-z(\varphi(\sigma(z))-\overline{z_0})} = \frac{2\lambda(z)}{1-z\lambda(z)}.\end{aligned}$$ $\square$

Theorem 3. *Let f be a convex mapping with Schwarzian norm $\|\mathscr{S}f\| = 2$, and suppose that $(1-|z_0|^2)^2|\mathscr{S}f(z_0)| = 2$ for some point $z_0 \in \mathbf{D}$. Then f maps the disk onto a parallel strip.*

Proof. According to (3), the pre-Schwarzian of f has the form

$$\frac{f''(z)}{f'(z)} = \frac{2\varphi(z)}{1 - z\varphi(z)}$$

for some analytic function φ with $|\varphi(z)| \le 1$ in $\mathbf{D}$. In fact, $|\varphi(z)| < 1$ for all $z \in \mathbf{D}$, since $\varphi(z) \equiv e^{i\theta}$ would imply that f is a half-plane mapping with $\|\mathscr{S}f\| = 0$. The hypothesis implies that equality occurs in the inequality (8), so that φ is a Möbius self-mapping of $\mathbf{D}$. Since $(1 - |z_0|^2)^2|\mathscr{S}f(z_0)| = 2$, it also follows from (8) that $\varphi(z_0) = \overline{z_0}$. Now let $\sigma(z) = (z + z_0)/(1 + \overline{z_0}z)$ and define $g = f \circ \sigma$. Then by Lemma 1,

$$\frac{g''(z)}{g'(z)} = \frac{2\lambda(z)}{1 - z\lambda(z)}, \quad \text{where } \lambda(z) = \frac{\varphi(\sigma(z)) - \overline{z_0}}{1 - z_0\varphi(\sigma(z))}.$$

In particular, λ is a Möbius self-mapping of $\mathbf{D}$ with $\lambda(0) = 0$, since $\sigma(0) = z_0$ and $\varphi(z_0) = \overline{z_0}$. Hence λ is a rotation, and so $\lambda(z) = e^{i\theta}z$ for some unimodular constant $e^{i\theta}$. This shows that

$$\frac{g''(z)}{g'(z)} = \frac{2e^{i\theta}z}{1 - e^{i\theta}z^2} = \frac{L_\theta''(z)}{L_\theta'(z)},$$

where $L_\theta(z) = e^{-i\theta/2}L(e^{i\theta/2}z)$ is a rotation of the parallel strip mapping L defined by (6). Integration now yields $g(z) = \alpha L_\theta(z) + \beta$ for some constants $\alpha \ne 0$ and β, so that

$$f(z) = g(\sigma^{-1}(z)) = \alpha L_\theta(\sigma^{-1}(z)) + \beta,$$

and f maps the disk onto a parallel strip. $\square$

In order to treat the case where $(1 - |z|^2)^2|\mathscr{S}f(z)| < 2$ in $\mathbf{D}$, we will appeal to a known result about John domains. A bounded simply connected domain $\Omega \subset \mathbf{C}$ is a *John domain* if there is a constant $A > 0$ such that for every crosscut C of Ω the inequality $\operatorname{diam} H \le A \operatorname{diam} C$ holds for one component H of $\Omega \setminus C$. (See for instance Pommerenke [14].) Every bounded quasidisk is a John domain. It was found in [5] that if $\|\mathscr{S}f\| \le 2$ and $f(\mathbf{D})$ is a John domain, then $f(\mathbf{D})$ is a quasidisk. The following lemma is implicit in the proof of Theorem 6 in [5].

Lemma 2. *Let f be analytic and univalent in $\mathbf{D}$ with Schwarzian norm $\|\mathscr{S}f\| \le 2$, and suppose that its image $f(\mathbf{D})$ is a John domain. Then no sequence of normalized Koebe transforms of f can converge locally uniformly to a parallel strip mapping.*

Here a *normalized Koebe transform* is understood to mean a Koebe transform followed by an appropriate Möbius transformation to produce a function F with $F(0) = 0$, $F'(0) = 1$, and $F''(0) = 0$.

Now for the remaining case where the supremum that defines the Schwarzian norm is not attained in $\mathbf{D}$.

Theorem 4. *Let f be a convex mapping with $\|\mathscr{S}f\| = 2$ but $(1-|z|^2)^2|\mathscr{S}f(z)| < 2$ for all points $z \in \mathbf{D}$. Then the image $f(\mathbf{D})$ is not a quasidisk on the Riemann sphere.*

Taken together, Theorems 3 and 4 have the following consequence.

Corollary. *If f is a convex mapping with $\|\mathscr{S}f\| = 2$, then $f(\mathbf{D})$ is not a quasidisk on the Riemann sphere. In particular, every bounded convex mapping of $\mathbf{D}$ has Schwarzian norm $\|\mathscr{S}f\| < 2$.*

Proof of theorem. Because of the hypothesis, there exists a sequence of points z_n in $\mathbf{D}$ with $|z_n| \to 1$ such that
$$(1-|z_n|^2)^2|\mathscr{S}f(z_n)| \to 2 \quad \text{as } n \to \infty.$$
By the inequality (8), this implies that
$$\frac{\varphi(z_n)-\overline{z_n}}{1-z_n\varphi(z_n)} \to 0 \quad \text{as } n \to \infty. \tag{17}$$
In other words, the hyperbolic distance $d(\varphi(z_n), \overline{z_n})$ tends to 0. It follows from (17) that
$$\frac{1-|z_n|^2}{1-z_n\varphi(z_n)} \to 1 \quad \text{as } n \to \infty, \tag{18}$$
since
$$\frac{1-|z_n|^2}{1-z_n\varphi(z_n)} - 1 = z_n\frac{\varphi(z_n)-\overline{z_n}}{1-z_n\varphi(z_n)}.$$
But the representation (4) of $\mathscr{S}f$ shows that
$$2|\varphi'(z_n)|\left|\frac{1-|z_n|^2}{1-z_n\varphi(z_n)}\right|^2 = (1-|z_n|^2)^2|\mathscr{S}f(z_n)| \to 2,$$
and so the relation (18) implies that
$$|\varphi'(z_n)| \to 1 \quad \text{as } n \to \infty. \tag{19}$$

There is no loss of generality in assuming that $f(0)=0$ and $f'(0)=1$, since the Schwarzian is invariant under postcomposition with affine transformations, which preserve convexity. Now make a further normalization by defining
$$f_0(z) = \frac{f(z)}{1+a_2f(z)}, \quad \text{where } a_2 = \frac{1}{2}f''(0).$$
By a result of Chuaqui and Osgood [3], a normalized function f with $\|\mathscr{S}f\| \le 2$ can not take the value $-1/a_2$, so f_0 is analytic and univalent, with $f_0(0)=0$, $f_0'(0)=1$, and $f_0''(0)=0$. Also, $\mathscr{S}f_0 = \mathscr{S}f$, since the Schwarzian is preserved under Möbius transformation. Therefore, $\|\mathscr{S}f_0\| = 2$ and $(1-|z|^2)^2|\mathscr{S}f_0(z)| < 2$ for all $z \in \mathbf{D}$. But by a theorem of Gehring and Pommerenke ([7], Theorem 2), the properties $f_0(0)=0$, $f_0'(0)=1$, $f_0''(0)=0$, and $\|\mathscr{S}f_0\| \le 2$ imply that f_0 is either a rotation L_θ of the strip mapping L given by (6), or else f_0 has a homeomorphic extension to $\overline{\mathbf{D}}$. But the first alternative is impossible because $(1-|z|^2)^2|\mathscr{S}L_\theta(z)| = 2$ for all points z on some diameter of the disk, whereas $(1-|z|^2)^2|\mathscr{S}f_0(z)| < 2$ for all $z \in \mathbf{D}$. Thus we infer in particular that f_0 is bounded.

Next we define the Koebe transform
$$g_n(z) = \frac{f_0(\sigma_n(z)) - f_0(z_n)}{(1-|z_n|^2)f_0'(z_n)}, \quad \text{where } \sigma_n(z) = \frac{z+z_n}{1+\overline{z_n}z}.$$
Then g_n is univalent with $g_n(0)=0$, $g_n'(0)=1$, and $\|\mathscr{S}g_n\| = 2$, by the Möbius invariance of the Schwarzian norm. We normalize further by defining
$$h_n(z) = \frac{g_n(z)}{1+b_ng_n(z)}, \quad \text{where } b_n = \tfrac{1}{2}g_n''(0),$$
so that h_n has the additional property $h_n''(0)=0$, and again $\|\mathscr{S}h_n\| = 2$. Note that
$$\mathscr{S}h_n = \mathscr{S}g_n = \mathscr{S}(f_0 \circ \sigma_n) = \mathscr{S}(f \circ \sigma_n).$$

Thus by Lemma 1 and the relation (4), we see that

$$\mathscr{S}h_n(z) = \mathscr{S}(f \circ \sigma_n)(z) = \frac{2\lambda_n'(z)}{(1 - z\lambda_n(z))^2}, \tag{20}$$

where

$$\lambda_n(z) = \frac{\varphi(\sigma_n(z)) - \overline{z_n}}{1 - z_n\varphi(\sigma_n(z))}.$$

Then

$$\lambda_n(0) = \frac{\varphi(z_n) - \overline{z_n}}{1 - z_n\varphi(z_n)} \to 0$$

by (17), and direct calculation gives

$$|\lambda_n'(0)| = \frac{(1 - |z_n|^2)^2}{|1 - z_n\varphi(z_n)|^2}|\varphi'(z_n)|,$$

so that $|\lambda_n'(0)| \to 1$, in view of (18) and (19). Hence by a normal family argument we may assume after passing to a subsequence that $\lambda_n(z)$ converges to $e^{i\theta}z$ locally uniformly in $\mathbf{D}$, for some angle θ. It then follows from (20) that

$$\mathscr{S}h_n(z) \to \frac{2e^{i\theta}}{(1 - e^{i\theta}z^2)^2} = \mathscr{S}L_\theta(z), \tag{21}$$

locally uniformly in $\mathbf{D}$, where again $L_\theta(z) = e^{-i\theta/2}L(e^{i\theta/2}z)$, a rotation of the parallel strip mapping L.

In view of the normalizations, we can conclude from (21) that $h_n(z) \to L_\theta(z)$ locally uniformly in $\mathbf{D}$. Indeed, the function $u_n = (h_n')^{-1/2}$ satisfies the differential equation $u'' + \frac{1}{2}(\mathscr{S}h_n)u = 0$ and the initial conditions $u_n(0) = 1$ and $u_n'(0) = 0$, since $h_n'(0) = 1$ and $h_n''(0) = 0$. Similarly, the function $v_\theta = (L_\theta')^{-1/2}$ satisfies the differential equation $u'' + \frac{1}{2}(\mathscr{S}L_\theta)u = 0$ and the initial conditions $v_\theta(0) = 1$ and $v_\theta'(0) = 0$, since $L_\theta'(0) = 1$ and $L_\theta''(0) = 0$. Thus we conclude from (21) that $u_n(z) \to v_\theta(z)$, so that $h_n'(z) \to L_\theta'(z)$ locally uniformly in $\mathbf{D}$. But $h_n(0) = L_\theta(0) = 0$, so this implies that $h_n(z)$ converges locally uniformly to $L_\theta(z)$. In particular, the function f_0 has Schwarzian norm $\|\mathscr{S}f_0\| \le 2$ and a sequence of its Koebe transforms converges locally uniformly to a parallel strip mapping. Thus by Lemma 2 the image $f_0(\mathbf{D})$ cannot be a John domain, so it is not a quasidisk. Hence $f(\mathbf{D})$ is not a quasidisk on the Riemann sphere. □

Finally, we comment on Nehari's argument in [13] that a mapping onto a bounded convex domain must have $\|\mathscr{S}f\| < 2$. He approximates the given bounded convex domain by a convex polygon, and by geometric considerations he shows that all the inner angles are bounded below by a positive constant when the polygon is sufficiently close to the domain. He then appeals to Carathéodory's theorem on convex polytopes to reduce the analysis to quadrilaterals with one of the angles bounded below as before (a crucial point) and he works with the corresponding Schwarz–Christoffel mappings. Via this argument, one is led to determining the supremum of the Schwarzian norms of mappings onto quadrilaterals where one exterior angle, $2\pi\alpha_1$ in Nehari's notation, is uniformly bounded above away from π. As he states, the corresponding estimate would then apply to the norm of all bounded convex mappings. He shows correctly that if the norm of the Schwarzian of the mapping onto a bounded quadrilateral *is* *equal* to 2 then the quadrilateral must degenerate to an *unbounded* polygon with only two vertices, for which he deduces

$$\|Sf\| \le 2(1-(1-2\alpha_1)^2) < 2.$$

The proper conclusion is that the norm can never be equal to 2 for bounded quadrilaterals with the restriction on α_1. But the norm *can* be arbitrarily close to 2, for example by taking mappings onto long, thin rectangles (for which $\alpha_1 = 1/4$). This degeneracy of four vertices coalescing to just two vertices points exactly to the problem of the supremum not being a maximum. We have not been able to rescue Nehari's approach, ingenious as it is.

4. Examples

As illustrations of the preceding results, it will be instructive to consider some specific examples of convex mappings and their Schwarzian derivatives.

Example 1. *Sectors.* Consider first the class of conformal mappings

$$w = f(z) = \left(\frac{1+z}{1-z}\right)^a, \quad 0 < a \le 2,$$

of the unit disk onto an infinite sector with angle $a\pi$ at the origin. Simple calculations give the Schwarzian derivative

$$\mathscr{S}f(z) = \frac{2(1-a^2)}{(1-z^2)^2}.$$

Thus f has Schwarzian norm $\|\mathscr{S}f\| = 2|1-a^2| \le 6$, and $\|\mathscr{S}f\| \le 2$ for $a \le \sqrt{2}$. The sector is convex for $a \le 1$, and $\|\mathscr{S}f\|$ decreases to 0 as the sector widens to a half-plane. In particular, for $a \le 1$ these functions provide examples of unbounded convex mappings with Schwarzian norm less than 2.

Example 2. *Half-strip.* The function

$$w = f(z) = \sin^{-1}\left(i\frac{1+z}{1-z}\right)$$

maps the unit disk onto the half-strip $\{w = u+iv\colon -\frac{\pi}{2} < u < \frac{\pi}{2},\ v > 0\}$. Its Schwarzian derivative is found to be

$$\mathscr{S}f(z) = \frac{3z - \frac{1}{2}(1+z^2)}{(1-z)^2(1+z^2)^2}.$$

Hence $(1-x^2)^2\mathscr{S}f(x) \to 2$ as $x \to 1-$ along the real axis. But $\|\mathscr{S}f\| \le 2$ because the half-strip is convex, so we conclude that $\|\mathscr{S}f\| = 2$. On the other hand, the half-strip is not a quasidisk since its boundary has a cusp at infinity, so the result is compatible with Theorems 3 and 4.

Example 3. *Parabolic region.* The function

$$w = f(z) = \frac{4}{\pi^2}\left[\cosh^{-1}\left(\frac{1+z}{1-z}\right)\right]^2$$

maps the unit disk conformally onto the region Ω inside the parabola $v^2 = 4(u+1)$, where $w = u+iv$. (See Bieberbach [2, p. 111] for the construction.) Note that the function $[\cosh^{-1}(\zeta)]^2$ is single-valued although $\cosh^{-1}(\zeta)$ is not. Laborious calculations show that the Schwarzian derivative is

$$\mathscr{S}f(z) = \frac{3 - 2z + 3z^2}{8z^2(1-z)^2} - \frac{3/2}{z(1-z)^2\left[\cosh^{-1}\left(\frac{1+z}{1-z}\right)\right]^2}.$$

The apparent singularity at $z = 0$ is removed by cancellation of the two terms, since

$$\left[\cosh^{-1}\left(\frac{1+z}{1-z}\right)\right]^2 = 4z + \dots$$

for z near zero. We know that $\|\mathscr{S}f\| \le 2$ because f is a convex mapping, and it is not difficult to see that $(1-x^2)^2\mathscr{S}f(x) \to 2$ as $x \to 1-$ along the real axis. Hence $\|\mathscr{S}f\| = 2$ and we conclude from Theorems 3 and 4 that Ω is not a quasidisk. But the last fact is again apparent geometrically, since the boundary of Ω has a cusp at infinity.

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