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A UNIT DISC ANALOGUE OF THE BANK–LAINE CONJECTURE DOES NOT HOLD

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Abstract. The 1982 conjecture due to Bank and Laine claims the following: If $A(z)$ is a transcendental entire function of order of growth $\rho(A) \in [0, \infty) \setminus \mathbf{N}$, then $\max\{\lambda(f_1), \lambda(f_2)\} = \infty$, where f_1, f_2 are linearly independent solutions of $f'' + A(z)f = 0$ and $\lambda(g)$ stands for the exponent of convergence of the zeros of g. This conjecture has been verified in the case $\rho(A) \leq 1/2$, while counterexamples have been found in the cases $\rho(A) \in \mathbf{N} \cup \{\infty\}$. The aim of this paper is to illustrate that no growth condition on $A(z)$ alone yields a unit disc analogue of the Bank–Laine conjecture. The main discussion yields solutions to two open problems recently stated by Cao and Yi.

1. Introduction

The celebrated 1982 paper by Bank and Laine [3] opened up a new chapter in the oscillation theory of solutions of

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is entire [17]. Finding a growth condition on $A(z)$ such that every fundamental solution base $\{f_1, f_2\}$ of (1.1) satisfies

$$\max\{\lambda(f_1), \lambda(f_2)\} = \infty \tag{1.2}$$

has aroused wide interest during the last three decades. Here and in what follows,

$$\lambda(g) = \inf\left\{\alpha > 0\colon \sum_{n=1}^{\infty} |z_n|^{-\alpha} < \infty\right\}$$

stands for the exponent of convergence of the zeros $\{z_n\}$ of an entire function g, while

$$\rho(g) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

is the order of growth of g. If $A(z)$ is a polynomial, then it is well-known [3, 17] that all solutions f of (1.1) are entire and satisfy $\lambda(f) \leq \rho(f) < \infty$. Hence, in all attempts to obtain (1.2), we need to assume that $A(z)$ is transcendental.

Bank and Laine [3] introduced a method for constructing equations of the form (1.1) with zero-free solution bases. This construction depends on a certain entire

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parameter function φ. If φ is a polynomial, then $\rho(A)$ is a positive integer, while if φ is transcendental, then $\rho(A) = \infty$. Hence it seems plausible that (1.2) holds whenever $A(z)$ is transcendental and satisfies $\rho(A) \in [0,\infty) \setminus \mathbf{N}$. This is widely known as the Bank–Laine conjecture, which we abbreviate as the BL-conjecture.

To get an intuitive idea on the theory behind the BL-conjecture, we outline the general approach from [3]: Let $\{f_1, f_2\}$ be a fundamental solution base of (1.1) with Wronskian $W(f_1, f_2) = c \neq 0$. Denote $E = f_1 f_2$. Then

$$4A(z) = \left(\frac{E'}{E}\right)^2 - \left(\frac{c}{E}\right)^2 - 2\frac{E''}{E}. \tag{1.3}$$

A standard Nevanlinna theory reasoning applied to (1.3) results in

$$2T(r,E) = 2\overline{N}\left(r, \frac{1}{E}\right) + T(r,A) + S(r,E). \tag{1.4}$$

Now, suppose that some growth condition on a finite-order entire $A(z)$ forces $\rho(E) = \infty$. Then, by (1.4), it follows that $\lambda(E) = \infty$, which clearly yields (1.2).

The BL-conjecture was verified in [3] in the case $\rho(A) < 1/2$ by means of Wiman–Valiron theory and the $\cos\pi\rho$-theorem. The case $\rho(A) = 1/2$ was proved independently by Rossi [24] and Shen [27]. The method in [24] is based on the Beurling–Tsuji estimate for harmonic measure, while the method in [27] relies on the Carleman integral inequality. We note that the BL-conjecture still remains unsolved [18].

Proceeding to the case of the unit disc $\mathbf{D}$, we need the following definitions. Let g be an analytic function in $\mathbf{D}$. The exponent of convergence of the zeros $\{z_n\}$ of g is given by

$$\mu(g) = \inf\left\{\beta > 0\colon \sum_{n=1}^{\infty}(1-|z_n|)^{\beta+1} < \infty\right\},$$

while the order of growth of g is

$$\sigma(g) = \limsup_{r\to 1^-} \frac{\log^+ T(r,g)}{-\log(1-r)}.$$

Recall that the inequality $\sigma(g) \geq \mu(g)$ always holds. If g has the growth rate

$$\limsup_{r\to 1^-} \frac{T(r,g)}{-\log(1-r)} = \infty,$$

then g is called admissible. The Nevanlinna error term $S(r,g)$ is of growth $o(T(r,g))$, provided that g is admissible. We say that g belongs to the Korenblum space $A^{-\infty}$ [15] if there exists a constant $q \in [0,\infty)$ such that

$$\sup_{z\in\mathbf{D}}(1-|z|^2)^q|g(z)| < \infty. \tag{1.5}$$

Further, g belongs to the Hardy space H^p [9], $p \in (0,\infty]$, if

$$\sup_{0\leq r<1}\int_0^{2\pi}|g(re^{i\theta}|^p\,d\theta < \infty, \quad p<\infty,$$

$$\sup_{z\in\mathbf{D}}|g(z)| < \infty, \quad p=\infty.$$

The harmonic counterpart of the H^p space is denoted by h^p [9].

The space $A^{-\infty}$ includes the classical H^p-spaces, for if $g \in H^p$, $0 < p \leq \infty$, then (1.5) holds for $q = 1/p$ [9, p. 36]. Moreover, if $g \in A^{-\infty}$, then g is non-admissible, and so $\sigma(g) = 0$. While discussing differential equations in the unit disc, the functions in the Korenblum space have sometimes been called $\mathscr{H}$-functions since the appearance of [12].

Next, let $A(z)$ be analytic in $\mathbf{D}$, and let $\{f_1, f_2\}$ be a fundamental solution base of (1.1). We wish to find a growth condition on $A(z)$ such that

$$\max\{\mu(f_1), \mu(f_2)\} = \infty. \tag{1.6}$$

This would be a unit disc analogue of the BL-conjecture. If $A(z)$ is an $\mathscr{H}$-function, then it is known that all solutions f of (1.1) satisfy $\mu(f) \le \sigma(f) < \infty$ [12]. Hence, for (1.6) to hold, $A(z)$ cannot be an $\mathscr{H}$-function.

The aim of this paper is, however, to illustrate that no growth condition on $A(z)$ alone implies (1.6). This discussion is carried out in Section 2. The main result is Theorem 1 below, which will be proved in Section 4. The proof requires that we compare the growth of characteristics $T(r, \varphi)$ and $T(r, e^{\varphi})$ for a function φ analytic in $\mathbf{D}$. This comparison, to be presented in Section 3, is closely related to the well known paper by Clunie [8] in the case of the complex plane. The main discussion yields solutions to two open problems recently stated by Cao and Yi in [4], see Section 2. Concluding remarks on zero-free solution bases are given in Section 5.

2. Construction of zero-free solution bases

Following the reasoning in [3, p. 356], let φ be an analytic function in $\mathbf{D}$, and let h denote a primitive function of e^{φ}, that is, $h' = e^{\varphi}$. Set $g = -(\varphi + h)/2$. Then a simple computation shows that the functions $f_1 = e^g$ and $f_2 = e^{g+h}$ are linearly independent solutions of (1.1), where

$$A = -\frac{1}{4}(e^{2\varphi} + (\varphi')^2 - 2\varphi'') \tag{2.1}$$

is analytic in $\mathbf{D}$. Note that the choice $\varphi(z) = -\log(1-z)$ implies $A(z) \equiv 0$ and gives raise to linearly independent solutions $f_1(z) = 1 - z$ and $f_2(z) = 1$ having no zeros in $\mathbf{D}$. Suppose more generally that φ belongs to the Bloch space $\mathscr{B}$ [1, 22], that is,

$$||\varphi||_{\mathscr{B}} = \sup_{z\in\mathbf{D}}(1 - |z|^2)|\varphi'(z)| < \infty.$$

Then

$$|\varphi(z)| \le ||\varphi||_{\mathscr{B}} \left(1 + \frac{1}{2}\log\left(\frac{1+|z|}{1-|z|}\right)\right) \quad \text{and} \quad |\varphi^{(j)}(z)| \le \frac{2^{4(j-1)}||\varphi||_{\mathscr{B}}}{(1-|z|^2)^j}, \tag{2.2}$$

where $z \in \mathbf{D}$ is arbitrary and $j = 1, 2$. A substitution to (2.1) now shows that $A(z)$ is an $\mathscr{H}$-function, and hence every solution of (1.1) is of finite order of growth [12]. In the following example we consider cases where $A(z)$ is not an $\mathscr{H}$-function.

Example 1. (1) The choice $\varphi(z) = \frac{1+z}{1-z}$ in (2.1) leads to a coefficient $A(z)$ growing exponentially, yet $A(z)$ is of bounded characteristic. Indeed,

$$T(r, e^{2\varphi}) = 2m(r, e^{\varphi}) = \frac{1}{\pi}\int_0^{2\pi} \frac{1-|z|^2}{|1-z|^2}\, d\arg(z) = 2, \quad 0 < r = |z| < 1,$$

and therefore $e^{2\varphi}$ is of bounded characteristic. Moreover, $(\varphi')^2 \in H^p$ for all $p < 1/4$ and $\varphi'' \in H^p$ for all $p < 1/3$ [9], and hence both $(\varphi')^2$ and φ'' are of bounded characteristic.

(2) For $k \in \mathbf{N} \setminus \{1\}$ and $c \in (-1, 0)$, let

$$\psi(z) = \sum_{n=1}^{\infty} k^{cn} z^{k^n}$$

be the function studied by Hayman [10] and Littlewood [20]. Since ψ is bounded in $\overline{\mathbf{D}}$, it is analytic and of bounded characteristic in $\mathbf{D}$. If k is large enough, then ψ' is of unbounded characteristic. However, a calculation based on Cauchy's integral formula shows that

$$\sup_{z \in \mathbf{D}} (1 - |z|^2)^j |\psi^{(j)}(z)| < \infty \tag{2.3}$$

for all $j \in \mathbf{N}$. Choose $\varphi = \psi'$, and suppose on the contrary that e^{φ} is of bounded characteristic. By [11, p. 174] we have $\Re\varphi \in h^1$. Kolmogorov's theorem [9, p. 57] now gives $\Im\varphi \in h^p$ for all $p \in (0, 1)$. Hence $\varphi \in H^p$ for all $p \in (0, 1)$. This is a contradiction, since all functions in the Hardy spaces have bounded characteristic. Moreover, φ clearly maps $[0, 1)$ onto $[0, \infty)$, and hence, by (2.3), the function $A(z)$ in (2.1) grows exponentially. In particular, $A(z)$ is of unbounded characteristic.

(3) The choice $\varphi(z) = \frac{1+z}{1-z}\left(\log \frac{1}{1-z} + 1\right)^p$, $p \geq 0$, in (2.1) leads to a coefficient $A(z)$ of growth $\sigma_{\log}(A) = p$, where

$$\sigma_{\log}(A) = \limsup_{r \to 1^-} \frac{\log^+ T(r, A)}{\log(-\log(1 - r))}$$

is the logarithmic order of $A(z)$ [7]. This calculation requires a fair amount of work, but the details are essentially worked out in [7, pp. 172–174].

(4) The choice $\varphi(z) = \left(\frac{1+z}{1-z}\right)^{p+1}$, $p \geq 0$, in (2.1) leads to a coefficient $A(z)$ of growth $\sigma(A) = p$. The details can be worked out easily, see [6, p. 753].

By Example 1 it seems plausible that no growth condition on a finite-order $A(z)$ alone implies (1.6). The main result below shows that it is possible to construct a function $A(z)$ analytic in $\mathbf{D}$ and of arbitrarily rapid growth such that (1.1) possesses two linearly independent solutions each having no zeros. This is a unit disc analogue of the corresponding reasoning in [3, p. 356].

Theorem 1. *Let $\Lambda(r)$ be an increasing and continuous function defined on the interval $[0, 1)$ such that $\Lambda''(r) > 0$ and*

$$\lim_{r \to 1^-} \frac{\Lambda(r)}{-\log(1 - r)} = \infty. \tag{2.4}$$

Then it is possible to construct a function $A(z)$ analytic in $\mathbf{D}$ of growth

$$\lim_{r \to 1^-} \frac{T(r, A)}{\Lambda(r)} = \infty \tag{2.5}$$

such that (1.1) possesses linearly independent solutions f_1, f_2 each having no zeros. Moreover, the product function $E = f_1 f_2$ satisfies

$$\lim_{r \to 1^-} \frac{T(r, E)}{\Lambda(r)} = \infty. \tag{2.6}$$

In light of cases (3) and (4) in Example 1, the assumption (2.4) does not seem too restrictive. When proving Theorem 1 in Section 4, we rely on a Linden–Shea construction [19, 25] on an analytic function of prescribed asymptotic growth. This construction depends on (2.4). It may be possible that (2.4) can be weakened to "$\Lambda(r)$ is unbounded", but this requires new results on asymptotic growth.

Remark 1. Let $b \in (0, e^{-6})$ be an arbitrary constant. The proof of Theorem 1 shows that, in addition to (2.6), the product function $E = f_1 f_2$ also satisfies

$$\limsup_{r\to 1^-} \frac{T\left(r, \frac{E'}{E}\right)}{\Lambda(1-b(1-r))} \leq 1. \tag{2.7}$$

We note that the constant e^{-6} may not be the best possible.

We conclude that no growth condition on $A(z)$ alone yields a unit disc analogue of the BL-conjecture. This settles one open problem stated in a recent paper by Cao and Yi [4]. Another open problem in [4] is stated as follows:

> *Let $A(z)$ be a non-admissible analytic function in $\mathbf{D}$. Suppose that f_1, f_2 are two linearly independent solutions of (1.1), and set $E = f_1 f_2$. It is known that in this case $\sigma(E) \leq \sigma(f_1) = \sigma(f_2)$. Can we obtain an equality instead of an inequality here?*

We will demonstrate that a strict inequality $\sigma(E) < \sigma(f_1) = \sigma(f_2)$ typically holds. To begin with, recall that $f_1 = e^g$ and $f_2 = e^{g+h}$ are linearly independent solutions of (1.1), where the coefficient $A(z)$ is given by (2.1). Since h denotes a primitive function of e^φ, and since

$$E = f_1 f_2 = e^{2g+h} = e^{-\varphi}, \tag{2.8}$$

our claim is intuitively clear. For example, suppose that φ is an unbounded analytic function in $\mathbf{D}$ with its range in the right-half plane, say $\varphi(z) = \frac{1+z}{1-z}$. Then E is bounded in $\mathbf{D}$ and $A(z)$, as defined in (2.1), is of bounded characteristic (and hence non-admissible). Meanwhile, $\sigma(f_1) = \sigma(f_2) = \infty$. Hence it is possible that the solutions f_1, f_2 are of infinite order of growth, while their product E belongs to H^∞. A strict inequality may hold even in the finite-order case by [13, p. 1052]. Indeed, for $\alpha \geq 0$ and for a fixed branch, the functions

$$f_j(z) = (1-z)^{\frac{\alpha+2}{2}} \exp\left((-1)^j \left(\frac{1}{1-z}\right)^{\alpha+1}\right), \quad j = 1, 2, \tag{2.9}$$

are linearly independent solutions of (1.1), where

$$A(z) = -\frac{\alpha(\alpha+2)}{4(1-z)^2} - \frac{(\alpha+1)^2}{(1-z)^{2\alpha+4}}. \tag{2.10}$$

It is clear that $\sigma(f_1) = \sigma(f_2) = \alpha$, while the product function E is again in H^∞.

We note that the example involving (2.9) and (2.10) illustrates the sharpness of the main result in [16] and is more elementary than the examples in [16].

3. Comparison of $T(r, \varphi)$ and $T(r, e^\varphi)$

The function $A(z)$ in Theorem 1 will be constructed by means of (2.1). To prove Theorem 1, we need to know how $T(r, \varphi)$ and $T(r, e^\varphi)$ are related to each other. Referring to the paper by Clunie [8] for a parallel discussion in the case of complex plane, we believe that the discussion below is also of independent interest.

We remind the reader that φ may be unbounded, with its range in the left-half plane, while $e^\varphi \in H^\infty$. The Möbius transformation $\varphi(z) = -\frac{1+z}{1-z}$ is a typical example of such a case. For $T(r, e^{\varphi})$ to be unbounded, the range of φ should have an unbounded intersection with the right half-plane. If $p > 0$, then the functions φ in cases (3) and (4) of Example 1 satisfy

$$\sup_{z\in\mathbf{D}}(1-|z|^2)|\varphi(z)| = \infty. \tag{3.1}$$

The growth rate (3.1) is, however, not necessary for $T(r, e^{\varphi})$ to be unbounded. Indeed, let φ be the lacunary series

$$\varphi(z) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} z^{2^n}$$

studied by Pommerenke [22]. The reasoning in [22, p. 694] shows that $\varphi \in \mathscr{B}$, and that φ has radial limits almost nowhere on $\partial\mathbf{D}$. This means that φ is of unbounded characteristic [9, p. 17], and the reasoning in Example 1(2) shows that e^{φ} is of unbounded characteristic as well. An alternative proof for the latter claim can be achieved as follows: The function e^{φ} has the radial limit 0 almost nowhere on $\partial\mathbf{D}$ by means of Privalov's uniqueness theorem [23, p. 325]. All other radial limits of e^{φ} on a set of positive Lebesgue measure are prevented by the properties of φ. Hence e^{φ} is of unbounded characteristic.

Lemma 1. *Let φ be analytic and admissible in $\mathbf{D}$. Then e^{φ} is admissible, and*

$$\limsup_{r\to 1^-} \frac{T(r, e^{\varphi})}{T(r, \varphi)} = \infty. \tag{3.2}$$

In what follows, we will be dealing with exceptional sets. In the literature they are typically handled with [2, Lemma C] due to Bank. However, in Section 4 we will make full use of the following slight generalization of Bank's lemma. The proof is an easy modification of that in [2] and is therefore omitted.

Lemma 2. *Let $r_0 \in [0,1)$, and suppose that $g(r)$ and $h(r)$ are nondecreasing functions on $[r_0,1)$ such that $g(r) \le h(r)$ for all $r \notin F$, where the set $F \subset [r_0,1)$ has a finite logarithmic measure $\operatorname{lm}(F) = \int_F dr/(1-r)$. Set $s_b(r) = 1 - b(1-r)$ for any $b \in (0, e^{-\operatorname{lm}(F)})$. Then $g(r) \le h(s_b(r))$ for all $r \in [r_0, 1)$.*

Proof of Lemma 1. Let $k \in \mathbf{N}$. Using the first and the second fundamental theorems, we conclude that

$$\begin{aligned}(2k-1)T(r,\varphi) &\le \sum_{n=-k}^{k} \overline{N}\left(r, \frac{1}{\varphi - 2\pi n i}\right) + S(r,\varphi) \\ &\le N\left(r, \frac{1}{e^{\varphi}-1}\right) + S(r,\varphi) \le T(r, e^{\varphi}) + S(r,\varphi)\end{aligned} \tag{3.3}$$

outside of a possible exceptional set $F \subset [0,1)$ for which $\operatorname{lm}(F) < \infty$. Hence

$$\limsup_{r\to 1^-} \frac{T(r, e^{\varphi})}{T(r, \varphi)} \ge \limsup_{\substack{r\to 1^- \\ r\notin F}} \frac{T(r, e^{\varphi})}{T(r, \varphi)} \ge 2k-1.$$

Since $k \in \mathbf{N}$ can be chosen arbitrarily large, we deduce that (3.2) holds. To prove that e^{φ} is admissible, let $b \in (0, e^{-\operatorname{lm}(F)})$. By Lemma 2 and (3.3), we have

$$\left(2k - \frac{3}{2}\right) T(r,\varphi) \le T(s_b(r), e^{\varphi}), \quad r \in [0,1). \tag{3.4}$$

Since $\log(1 - s_b(r)) = (1 + o(1)) \log(1 - r)$ and φ is admissible, it follows that e^{φ} is admissible as well. □

By letting $k \to \infty$ in (3.4), we deduce that

$$\lim_{r\to 1^-} \frac{T(s_b(r), e^{\varphi})}{T(r,\varphi)} = \infty. \tag{3.5}$$

This is closely related to (3.2). We proceed to a more complicated approach that allows us to replace "lim sup" in (3.2) by "lim", or alternatively, to take $b = 1$ in (3.5). We state this result as the following unit disc analogue of a well known plane result [8, Theorem 2] due to Clunie.

Lemma 3. *Suppose that φ is analytic in $\mathbf{D}$ and of unbounded characteristic. Then e^{φ} is of unbounded characteristic, and*

$$\lim_{r\to 1^-} \frac{T(r, e^{\varphi})}{T(r,\varphi)} = \infty. \tag{3.6}$$

In particular, if φ is admissible, then e^{φ} is admissible.

Proof. By the discussion in Example 1(2), e^{φ} is of unbounded characteristic. Hence it remains to prove (3.6).

We require the following statement [21, p. 276]: If f is meromorphic in $\mathbf{D}$ and of unbounded characteristic, then for all $a \in \mathbf{C}$ outside a set of zero capacity, depending on f, we have

$$\lim_{r\to 1^-} \frac{N\left(r, \frac{1}{f-a}\right)}{T(r,f)} = 1.$$

It is known that a point set in $\mathbf{C}$ of zero capacity cannot have a continuum as its subset, and that the set of all complex rational points in $\mathbf{C}$ forms a set of zero capacity. Further, the union of two sets of zero capacity is also of zero capacity.

Let $F \subset \mathbf{C}$ be the exceptional set of zero capacity related to the functions φ and e^{φ} of unbounded characteristic. Then a point $a \in \mathbf{C} \setminus \{0\}$ can be found such that the point a itself and the points

$$w_n = \log|a| + i(\arg(a) + 2n\pi), \quad n \in \mathbf{Z},$$

all avoid the set F, for otherwise F would include a continuum. For such a the points w_n form an infinite zero sequence of $e^{\varphi} - a$. Moreover,

$$\lim_{r\to 1^-} \frac{N\left(r, \frac{1}{e^{\varphi}-a}\right)}{T(r, e^{\varphi})} = 1 \quad \text{and} \quad \lim_{r\to 1^-} \frac{N\left(r, \frac{1}{\varphi-w_n}\right)}{T(r,\varphi)} = 1 \tag{3.7}$$

for all $n \in \mathbf{Z}$. Let $k \in \mathbf{N}$ be any constant. Then

$$N\left(r, \frac{1}{e^{\varphi}-a}\right) \geq \sum_{n=-k}^{k} N\left(r, \frac{1}{\varphi - w_n}\right). \tag{3.8}$$

Combining (3.7) and (3.8), we conclude that

$$\liminf_{r\to 1^-} \frac{T(r, e^{\varphi})}{T(r,\varphi)} \geq 2k + 1.$$

Since k can be chosen arbitrarily large, the assertion (3.6) follows. □

4. Proof of Theorem 1

We require the following result on constructing an analytic function in $\mathbf{D}$ with a prescribed asymptotic growth.

Theorem 2 ([19, 25])**.** *Let $\Lambda(r)$ be an increasing, convex and continuous function defined on the interval $[0,1)$ such that* (2.4) *holds. Then there exists a function f analytic in $\mathbf{D}$ such that $T(r,f)\sim\Lambda(r)$ as $r\to 1^-$.*

Remark 2. An exceptional set $F\subset[0,1)$ typically appears after using either the lemma on the logarithmic derivative (LLD) or the second main theorem (SMT). In both cases $\operatorname{lm}(F)$ is uniformly bounded from above. This can be seen by choosing $\phi(r)=1-r$ and $\psi(r)=\log^2 r$ in [5, Theorems 3.4.1 and 4.2.1]. More precisely, let f be meromorphic in $\mathbf{D}$ and of unbounded characteristic, and let $r_0\in[0,1)$ be such that $T(r_0,f)\ge e$. Then the inequalities in LLD and SMT hold for $r\notin[0,r_0]\cup F$, where $F\subset[r_0,1)$ satisfies $\operatorname{lm}(F)\le 2$. Hence Lemma 2 would be applicable for any choice of $b\in(0,e^{-2})$.

We will make use of this reasoning. Due to (2.4), there exists a constant $r_0\in[0,1)$ such that $\Lambda(r_0)\ge e$. In what follows, the constant r_0 may not be the same each time it occurs. Set $s_b(r)=1-b(1-r)$ for $b\in(0,e^{-4})$. Define $\Lambda_0(r)=\Lambda(s_b(r))$. Clearly $\Lambda_0(r)$ is increasing and continuous on $[0,1)$, and satisfies

$$\lim_{r\to 1^-}\frac{\Lambda_0(r)}{-\log(1-r)}\ge\lim_{r\to 1^-}\frac{\Lambda(s_b(r))}{-\log(1-s_b(r))}\left(1+\frac{4}{-\log(1-r)}\right)=\infty.$$

Moreover, $\Lambda_0''(r)=\Lambda''(s_b(r))b^2>0$, so that $\Lambda_0(r)$ is convex. By Theorem 2 there exists a function φ analytic in $\mathbf{D}$ such that $T(r,\varphi)\sim\Lambda_0(r)$ as $r\to 1^-$. For this particular φ, define $A(z)$ as in (2.1). Clearly φ is admissible, and hence by [5, Theorem 3.4.1] and Remark 2, we have

$$T(r,\varphi^{(j)})=m(r,\varphi^{(j)})\le m(r,\varphi)+m\left(r,\frac{\varphi^{(j)}}{\varphi}\right)=(1+o(1))T(r,\varphi)\tag{4.1}$$

for all $r\in[r_0,1)$ outside of respective exceptional sets $F_j\subset[r_0,1)$ satisfying $\operatorname{lm}(F_j)\le 2$ for $j=1,2$. Denote $F=F_1\cup F_2$, so that $\operatorname{lm}(F)\le 4$. Combining (2.1) and (4.1) with Lemma 3, we conclude that

$$T(r,A)\ge\frac{3}{2}T(r,e^{\varphi}),\quad r\notin[0,r_0)\cup F.$$

By Lemma 2, we have

$$\frac{T(s_b(r),A)}{\Lambda(s_b(r))}\ge\frac{3}{2}\cdot\frac{T(r,e^{\varphi})}{T(r,\varphi)}\cdot\frac{T(r,\varphi)}{\Lambda_0(r)}\ge\frac{T(r,e^{\varphi})}{T(r,\varphi)},\quad r\in[r_0,1).\tag{4.2}$$

The assertion (2.5) now follows by means of (4.2) and Lemma 3. Using (2.8), we deduce that

$$\frac{T(r,E)}{\Lambda(r)}=\frac{T(r,e^{\varphi})}{T(r,\varphi)}\cdot\frac{T(r,\varphi)}{\Lambda_0(r)}\cdot\frac{\Lambda_0(r)}{\Lambda(r)}\ge\frac{T(r,e^{\varphi})}{T(r,\varphi)}\cdot\frac{T(r,\varphi)}{\Lambda_0(r)}.$$

The assertion (2.6) now follows by Lemma 3.

Proof of (2.7). We note that the formulas (2.8) and (4.1) yield

$$T\left(r,\frac{E'}{E}\right)=T(r,\varphi')=(1+o(1))T(r,\varphi)\sim(1+o(1))\Lambda_0(r).$$

To see that (2.7) holds, it remains to use Lemma 2 in the case $\operatorname{lm}(F) \le 2$. □

5. Concluding remarks on zero-free solution bases

The solutions f_1, f_2 in (2.9) are of the form

$$f_j(z) = \exp\left(g(z) + (-1)^j h(z)\right), \quad j = 1, 2, \tag{5.1}$$

where g and h are analytic in $\mathbf{D}$. Indeed, by choosing $g(z) = \frac{\alpha+2}{2}\log(1-z)$ and $h(z) = (1-z)^{-\alpha-1}$, we see that (5.1) reduces to (2.9). In the general case, for f_1, f_2 in (5.1) to be linearly independent solutions of (1.1), the function h must be non-constant. Moreover, by substituting f_1, f_2 in (1.1), we get, after a simplification, that the functions g, h depend on each other by the equation

$$\frac{h''}{h'} = -2g'. \tag{5.2}$$

Hence $h' = Ce^{-2g}$ for some $C \in \mathbf{C} \setminus \{0\}$. The product function $E = f_1 f_2 = e^{2g}$ and the ratio $G = f_2/f_1 = e^{2h}$ are now easy to deal with. Recall that $S_G = 2A$, where

$$S_G = \left(\frac{G''}{G'}\right)' - \frac{1}{2}\left(\frac{G''}{G'}\right)^2$$

is the Schwarzian derivative of G. Since $G = e^{2h}$, an easy computation yields

$$2A = S_h - 2(h')^2. \tag{5.3}$$

In particular, if h is a (constant times) Möbius transformation, then $S_h = 0$, and so $A = -(h')^2$. This is the theory behind [13, Example 5.3], for example. In the general case, using (5.2), we may write (5.3) alternatively as

$$A = -g'' - (g')^2 - (h')^2. \tag{5.4}$$

Hence a zero-free solution base as well as the coefficient $A(z)$ of (1.1) can be written in terms of two parameter functions g, h by means of (5.1) and (5.4).

We note that the solutions f_1, f_2 in (5.1) can also be written in terms of the BL-method described in Section 2, and hence this approach is not new. However, sometimes this approach is easier to use in constructing examples of zero-free solution bases. Suitable computer software is also useful.

Corresponding to the reasoning in the beginning of Section 2, we have the following claim: If $g \in \mathscr{B}$, then the solutions f_1, f_2 in (5.1) are of finite order. To prove this, we first observe that $h' = Ce^{-2g}$ is an $\mathscr{H}$-function by (2.2). Since

$$|h(z)| \le |h(0)| + \int_0^{|z|} |h'(\zeta)||d\zeta|,$$

it follows that h is also an $\mathscr{H}$-function. Let $\sigma_M(f)$ be the maximum modulus order of a function f analytic in $\mathbf{D}$. The sharp estimates $\sigma(f) \le \sigma_M(f) \le \sigma(f) + 1$ are well-known. We now conclude that $\sigma(f_j) \le \sigma_M(f_j) < \infty$ for $j = 1, 2$ [16].

In a private communication, Gröhn (University of Eastern Finland) pointed out that if

$$(1 - |z|^2)|g'(z)| \le 1/2, \quad z \in \mathbf{D},$$

then h is univalent in $\mathbf{D}$. This claim follows by applying [23, p. 172] on (5.2).

The discussion above applies in the case of $\mathbf{C}$ as well. Indeed, if g is a polynomial, then $\rho(h) = \deg(g) = \rho(A)$ by $h' = Ce^{-2g}$ and (5.4). If h in turn is a polynomial, then so is h', and hence g must be a constant function. In this case $\deg(h) = 1$ and $A(z)$ is a constant function. The latter case is the only possibility for constructing entire solutions of finite order. If g is transcendental entire, then so is h, and $A(z)$ in (5.4) is of infinite order of growth by [8, Theorem 2].

We note that if the starting point is a non-vanishing analytic function E (hence a BL-function), then the method due to Shen in [26] can also be used in constructing zero-free solution bases. Indeed, it turns out that E is then a product of two linearly independent zero-free solutions. This method is originally written in the case of $\mathbf{C}$, but it is clearly valid in the case of $\mathbf{D}$ as well. For example, the choice $E(z) = \frac{(1-z)^{\alpha+2}}{2(\alpha+1)}$ for a fixed $\alpha \geq 0$ in [26] yields

$$A(z) = -\frac{1}{4}\left(\frac{1}{E(z)^2} + 2\frac{E''(z)}{E(z)} - \left(\frac{E'(z)}{E(z)}\right)\right) = -\frac{\alpha(\alpha+2)}{4(1-z)^2} - \frac{(\alpha+1)^2}{(1-z)^{2\alpha+4}}$$

as in (2.10). The functions in the corresponding zero-free solution base

$$g_j(z) = E(z)^{\frac{1}{2}} \exp\left(\frac{(-1)^j}{2}\int_0^z \frac{d\zeta}{E(\zeta)}\right), \quad j = 1, 2,$$

then reduce to constant multiples of the functions f_1, f_2 in (2.9), respectively.

We have seen that it is possible to construct zero-free solutions bases $\{f_1, f_2\}$ for (1.1) in the cases of $\mathbf{C}$ and $\mathbf{D}$. It has recently been proved [14] in both cases that arbitrary linear combinations $f = C_1 f_1 + C_2 f_2$ typically have the maximal quantity of zeros when compared to the growth of f_1, f_2.

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