Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 36, 2011, 331–340

PRESCRIBING THE PRESCHWARZIAN IN SEVERAL COMPLEX VARIABLES

Rodrigo Hernández

Universidad Adolfo Ibáñez, Facultad de Ciencias y Tecnología
Av. Balmaceda 1625 Recreo, Viña del Mar, Chile; rodrigo.hernandez@uai.cl

Abstract. We solve the several complex variables preSchwarzian operator equation $[Df(z)]^{-1} D^2 f(z) = A(z)$, $z \in \mathbf{C}^n$, where $A(z)$ is a bilinear operator and f is a $\mathbf{C}^n$ valued locally biholomorphic function on a domain in $\mathbf{C}^n$. Then one can define a several variables $f \to f_\alpha$ transform via the operator equation $[Df_\alpha(z)]^{-1} D^2 f_\alpha(z) = \alpha[Df(z)]^{-1} D^2 f(z)$, and thereby, study properties of f_α. This is a natural generalization of the one variable operator $f_\alpha(z)$ in [6] and the study of its univalence properties, e.g., the work of Royster [23] and many others. Möbius invariance and the multivariables Schwarzian derivative operator of Oda [17] play a central role in this work.

1. Introduction

Consider the class $\mathscr{S}$ of functions f holomorphic and univalent in the disk $\mathbf{D} = \{z \colon |z| < 1\}$ with the normalization $f(0) = 0$ and $f'(0) = 1$. Let $\alpha \in \mathbf{C}$, $f \in \mathscr{S}$ and define the integral transform

$$f_\alpha(z) = \int_0^z [f'(w)]^\alpha \, dw, \tag{1.1}$$

where the power is defined by the branch of the logarithm for which $\log f'(0) = 0$, [6]. A question considered in [6] is to determinate the values of α for which $f_\alpha \in \mathscr{S}$. In [23] Royster exhibited non-univalent mappings f_α for each complex $\alpha \neq 1$ with $|\alpha| > 1/3$. In fact, consider functions of the form

$$f(z) = \exp(\mu \log(1 - z)), \tag{1.2}$$

which are univalent if and only if μ lies in ones of the closed disks

$$|\mu + 1| \leq 1, \quad |\mu - 1| \leq 1.$$

Royster showed that for any such value of μ, the function in (1.1) is not univalent for each α with$|\alpha| > 1/3$ and $\alpha \neq 1$. Moreover, Pfaltzgraff using the Ahlfors univalence criterion [1] proved that for any $f \in \mathscr{S}$, if $|\alpha| \leq 1/4$, then f_α is univalent in $\mathbf{D}$, see [19].

Let f be a locally univalent mapping in $\mathbf{D}$ and f_α defined by equation (1.1). Then $f'_\alpha(z) = [f'(z)]^\alpha$, which implies that

$$\frac{f''_\alpha}{f'_\alpha}(z) = \alpha \frac{f''}{f'}(z).$$

doi:10.5186/aasfm.2011.3621
2010 Mathematics Subject Classification: Primary 32H02, 32W99; Secondary 32A10.
Key words: PreSchwarzian derivative, holomorphic mapping, univalence.
The author was partially supported by Fondecyt Grants # 11070055.

If f and g satisfy that $g''/g'(z) = f''/f'(z)$, then $\log(g'(z)) = \log(f'(z))$ when $f'(0) = g'(0)$. Therefore $g = f$ if $f(0) = g(0)$. Thus

$$f_\alpha(z) = \int_0^z [f'(w)]^\alpha \, dw \;\Leftrightarrow\; \frac{f''_\alpha}{f'_\alpha}(z) = \alpha \frac{f''}{f'}(z). \tag{1.3}$$

This equivalence in one variable suggests our idea to define the several variables generalization of f_α via the operator equation

$$[Df_\alpha(z)]^{-1} D^2 f_\alpha(z)(\cdot,\cdot) = \alpha [Df(z)]^{-1} D^2 f(z)(\cdot,\cdot). \tag{1.4}$$

Yoshida [25] developed a complete description of prescribing Oda's Schwarzian derivatives [17] in terms of a completely integrable system of differential equations. The description involves operators $S^k_{ij}f$ and $S^0_{ij}f$ of orders two and three respectively, coefficients of the system and Möbius invariants. In fact, the $S^k_{ij}f$ operators are the operator of least order that vanish for Möbius mappings. This is a strong difference with one complex variable where the third order Schwarzian operator is the lowest order operator annihilated by all Möbius mappings. For $n = 1$, the Möbius group has dimension 3, which allows to set $f(z_0), f'(z_0)$ and $f''(z_0)$ of a holomorphic mapping f at a given point z_0 arbitrarily. It would therefore be pointless to seek a Möbius invariant differential operator of order 2. But for $n > 1$ the number of parameters involved in the value and all derivatives of order 1 and 2 of a locally biholomorphic mapping is $n^2(n+1)/2 + n^2 + n$, and exceeds the dimension of the corresponding Möbius group in $\mathbf{C}^n$, which is n^2+2n. By the definition of the Schwarzian derivatives, we have that $S^k_{ij}F = S^k_{ji}F$ for all k and $\sum_{j=1}^n S^j_{ij}F = 0$ and we see there are exactly $n(n-1)(n+2)/2$ independent terms $S^k_{ij}F$, which is equal to the excess mentioned above.

A different approach to obtain the invariant operators S^k_{ij}, S^0_{ij} has been developed by Molzon and Tamanoi [14]. In addition, Molzon and Pinney had earlier developed equivalent invariant operators in the context of complex manifolds [13].

The operator

$$P_f(z) = [Df(z)]^{-1} D^2 f(z)(\cdot,\cdot)$$

introduced by Pfaltzgraff in [18] is the "natural" way to extend the classical one variable operator preSchwarzian f''/f'. Furthermore, the author in [18] extended the classical univalence criterion of Becker [2] to several variables. The question now is how to extend the equation (1.1) to $\mathbf{C}^n$. It is necessary to understand when one can recover the function f from a given P_f. We shall show a strong connection between this operator and the Schwarzian derivatives operator $SF(z)(\cdot,\cdot)$, introduced in [11]. Indeed, the problem of prescribing P_f can be reduced to understanding how to prescribe $S^k_{ij}f$ in terms of P_f. This is achieved via completely integrable system generated by $S^k_{ij}f$ and corresponding "new differential conditions" on the elements of P_f. We then use this theory to extend the classical single variable problem about the univalence of f_α by using equation (1.4) to define f_α in several complex variables.

2. Oda Schwarzian and Möbius invariants

Let $f\colon \Omega \subset \mathbf{C}^n \to \mathbf{C}^n$ be a locally biholomorphic mapping defined on some domain Ω. Oda in [17] defined the Schwarzian derivatives of $f = (f_1, \dots, f_n)$ as

$$S_{ij}^k f = \sum_{l=1}^{n} \frac{\partial^2 f_l}{\partial z_i \partial z_j} \frac{\partial z_k}{\partial f_l} - \frac{1}{n+1}\left(\delta_i^k \frac{\partial}{\partial z_j} + \delta_j^k \frac{\partial}{\partial z_i}\right) \log J_f, \tag{2.1}$$

where $i, j, k = 1, 2, \ldots, n$, J_f is the jacobian determinant of the differential Df and δ_i^k are the Kronecker symbols. For $n > 1$ the Schwarzian derivatives have the following properties:

$$S_{ij}^k f = 0 \text{ for all } i, j, k = 1, 2, \ldots, n \text{ iff } f(z) = M(z) \tag{2.2}$$

for some Möbius transformation

$$M(z) = \left(\frac{l_1(z)}{l_0(z)}, \ldots, \frac{l_n(z)}{l_0(z)}\right),$$

where $l_i(z) = a_{i0} + a_{i1} z_1 + \cdots + a_{in} z_n$ with $\det(a_{ij}) \neq 0$. Furthermore, for a composition

$$S_{ij}^k (g \circ f)(z) = S_{ij}^k f(z) + \sum_{l,m,r=1}^{n} S_{lm}^r g(w) \frac{\partial w_l}{\partial z_i} \frac{\partial w_m}{\partial z_j} \frac{\partial z_k}{\partial w_r}, \; w = f(z). \tag{2.3}$$

From this chain rule it can be shown that $S_{ij}^k f = S_{ij}^k g$ for all $i, j, k = 1, \ldots, n$ if and only if $g = T \circ f$ for some Möbius transformation. The $S_{ij}^0 f$ coefficients are given by

$$S_{ij}^0 f(z) = J_f^{1/(n+1)} \left(\frac{\partial^2}{\partial z_i \partial z_j} J_f^{-1/(n+1)} - \sum_{k=1}^{n} \frac{\partial}{\partial z_k} J_f^{-1/(n+1)} S_{ij}^k f(z) \right).$$

In his work, Oda gives a description of the functions with prescribed Schwarzian derivatives $S_{ij}^k f$ ([17]). Consider the following overdetermined system of partial differential equations,

$$\frac{\partial^2 u}{\partial z_i \partial z_j} = \sum_{k=1}^{n} P_{ij}^k(z) \frac{\partial u}{\partial z_k} + P_{ij}^0(z) u, \quad i, j = 1, 2, \ldots, n, \tag{2.4}$$

where $z = (z_1, z_2, \ldots, z_n) \in \Omega \subset \mathbf{C}^n$ and $P_{ij}^k(z)$ are holomorphic functions for $i, j, k = 0, \ldots, n$. The system (2.4) is called *completely integrable* if there are at most $n+1$ linearly independent solutions, and is said to be in *canonical form* (see [24]) if the coefficients satisfy

$$\sum_{j=1}^{n} P_{ij}^j(z) = 0, \quad i = 1, 2, \ldots, n.$$

Oda proved that (2.4) is a completely integrable system in canonical form if and only if $P_{ij}^k = S_{ij}^k f$ for a locally biholomorphic mapping $f = (f_1, \ldots, f_n)$, where $f_i = u_i/u_0$ for $1 \leq i \leq n$ and $u_0, u_1, \ldots, u_n$ is a set of linearly independent solutions of the system. For a given mapping f, $u = (J_f)^{-\frac{1}{n+1}}$ is always a solution of (2.4) with $P_{ij}^k = S_{ij}^k f$.

Definition 2.1. We define the *Schwarzian derivative operator* as the operator $S_f(z) \colon T_z\Omega \to T_{f(z)}\Omega$ given by

$$S_f(z)(\vec{v}, \vec{w}) = \left(\vec{v}^{\,t} S^1 f(z) \vec{w}, \ldots, \vec{v}^{\,t} S^n f(z) \vec{w}\right),$$

where $S^k f$ is the $n \times n$ matrix defined by $(S_{ij}^k f)_{ij}$ and $\vec{v} \in T_z\Omega$.

The Schwarzian derivative operator [12] can be rewritten as

$$(2.5)\qquad \begin{aligned} S_f(z)(\vec{v},\vec{w}) = [Df(z)]^{-1}D^2f(z)(\vec{v},\vec{w}) - \frac{1}{n+1}\left(\nabla\log J_f(z)\cdot\vec{v}\right)\vec{w} \\ - \frac{1}{n+1}\left(\nabla\log J_f(z)\cdot\vec{w}\right)\vec{v}, \end{aligned}$$

and the system (2.4) as

$$(2.6)\qquad \operatorname{Hess} u(z)(\cdot,\cdot) = \nabla u(z)\cdot S_f(z)(\cdot,\cdot) + S_f^0(z)(\cdot,\cdot)u(z),$$

where S_f^0 is a $n\times n$ matrix defined by $(S_{ij}^0 f)_{ij}$. We include in this section two lemmas that complement the work of Oda.

Lemma 2.2. *Let $f\colon \Omega\subset \mathbf{C}^n\to\mathbf{C}^n$ be a locally biholomorphic mapping and $u_0 = J_f^{-1/n+1}$. Then*

$$f = \frac{\vec{u}}{u_0} = \left(\frac{u_1}{u_0},\dots,\frac{u_n}{u_0}\right),$$

where $u_0, u_1,\dots,u_n$ are linearly independent solutions of (2.4)

Proof. We will prove that $\vec{u} = fu_0$ is solution of the equation (2.6). It follows that $Dfu_0 + f\nabla u_0 = Du$, from where

$$D^2f\cdot u_0 + 2Df\cdot\nabla u_0 + f\cdot\operatorname{Hess} u_0 = D^2u.$$

Using the system we have that

$$D^2f\cdot u_0 + 2Df\cdot\nabla u_0 - Df\cdot u_0\cdot S_f + Du\cdot S_f + S_f^0\cdot u = D^2u.$$

Considering the equation (2.5) with $u_0 = J_f^{-1/n+1}$ we have that

$$D^2f\cdot u_0 + 2Df\cdot\nabla u_0 - Df\cdot Sf\cdot u_0 = 0,$$

and $D^2u(\cdot,\cdot) = Du(S_f(\cdot,\cdot)) + S_f^0(\cdot,\cdot)u$, hence u_i with $i = 1,\dots,n$ and u_0 are independent solutions of the system (2.4). □

Lemma 2.3. *Let u_0 be a solution of the system (2.4). Then there exists a function $f = \vec{u}/u_0$ where $\vec{u} = (u_1,\dots,u_n)$ and u_i with $i = 0,1,\dots,n$ are independent solutions of the system (2.4) where $u_0 = J_f^{-1/n+1}$. The function f will be holomorphic away from the zero set of u_0.*

Proof. According to the previous lemma we can find $F = \vec{v}/v_0$ where $\{v_0, v_1,\dots,v_n\}$ are a linearly independent solutions of the system (2.4) with $P_{ij}^k = S_{ij}^k$ and $v_0 = J_F^{-1/n+1}$. As u_0 is solution of the system we have that $u_0 = \alpha_0v_0 + \cdots + \alpha_nv_n$. We need to find a Möbius mapping T such that

$$T\circ F = \left(\frac{u_1}{u_0},\dots,\frac{u_n}{u_0}\right) = f,$$

and $J_{T\circ F}^{-1/n+1} = u_0$. We have

$$\begin{aligned} J_{T\circ F}^{-1/n+1}(z) &= J_T^{-1/n+1}(F(z))J_F^{-1/n+1}(z) \\ &= (\lambda_0 + \lambda_1F_1(z) + \cdots + \lambda_nF_n(z))J_F^{-1/n+1}(z) \\ &= \lambda_0v_0 + \lambda_1v_1 + \cdots + \lambda_nv_n, \end{aligned}$$

which will be equal to u_0 if we choose $\lambda_i = \alpha_i$ for all $i = 0,1,\dots,n$. □

3. Results

Let $\Omega \subset \mathbf{C}^n$ be domain.

Theorem 3.1. *Let $f\colon \Omega \to \mathbf{C}^n$ be a locally biholomorphic mapping. The following statements are equivalent:*

(i) $S^0_{ij} f(z) \equiv 0$.
(ii) *There exists a locally biholomorphic mapping $g\colon \Omega \to \mathbf{C}^n$ with $S_g = S_f$ and J_g constant.*
(iii) *There exists a locally biholomorphic mapping $h\colon \Omega \to \mathbf{C}^n$ such that $S_h = S_f$ and $J_h^{-1/n+1} = 1/L(h)$, where $L(w) = \alpha_0 + \alpha_1 w_1 + \cdots + \alpha_n w_n$.*
(iv) *Locally there exists a biholomorphic change of variables such that the system (2.4) with $P^k_{ij} = S^k_{ij} f$ reduces to* $\operatorname{Hess}(u) = 0$.

Proof. (i) $\Rightarrow$ (ii). As $S^0_{ij} f \equiv 0$, the system (2.4) reduces to

$$u_{ij} = \sum_{k=1}^{n} S^k_{ij} u_k.$$

Therefore $u \equiv c$ is solution, thus by Lemma (2.3) there exists a function g such that $J_g \equiv C$.

(ii) $\Rightarrow$ (iii). Let $g = T \circ h$ for some Möbius T to be determined. Then $J_g^{-1/n+1}(z) = J_T^{-1/n+1}(h(z)) J_h^{-1/n+1}(z)$. Since $J_g^{-1/n+1} \equiv C$, we have that

$$C = (a_0 + a_1 h_1 + \cdots + a_n h_n) J_h^{-1/n+1}(z),$$

from where the result obtains after scaling h.

(iii) $\Rightarrow$ (iv). Suppose h has $J_h^{-1/n+1} = 1/L(h)$. The previous argument shows that by choosing T appropriately, we can produce $g = T \circ h$ with $J_g \equiv 1$. Hence $S_g(z)(\cdot,\cdot) = (Dg(z))^{-1} D^2 g(z)(\cdot,\cdot)$, and the system (2.4) reduces to

$$\operatorname{Hess} u(z)(\cdot,\cdot) = \nabla u(z) \cdot S_g(z)(\cdot,\cdot).$$

We consider $D(\nabla u(z)(Dg(z))^{-1})(\cdot,\cdot)$:

$$\begin{aligned} D(\nabla u(z)(Dg(z))^{-1}(\cdot,\cdot) &= \operatorname{Hess} u(z)((Dg(z))^{-1}(\cdot),\cdot) \\ &\quad - \nabla u(z) \cdot (Dg(z))^{-1} D^2 g(z)((Dg(z))^{-1}(\cdot),\cdot) \\ &= \nabla u(z) \cdot S_g(z)((Dg(z))^{-1}(\cdot),\cdot) \\ &\quad - \nabla u(z) \cdot (Dg(z))^{-1} D^2 g(z)((Dg(z))^{-1}(\cdot),\cdot) = 0. \end{aligned}$$

Let φ a local inverse of g. Therefore $U(w) = u(\varphi(w))$ satisfies that $\nabla U = \nabla u \cdot D\varphi = \nabla u(z)(Dg(z))^{-1}$, thus $\operatorname{Hess} U(w) \equiv 0$.

(iv) $\Rightarrow$ (i). Since $\operatorname{Hess} u(s) \equiv 0$, then $u \equiv c$ is a solution of this system (2.4), therefore $S^0_{ij} f \equiv 0$. □

Theorem 3.2. *Let $f\colon \Omega \to \mathbf{C}^n$ be a locally biholomorphic mapping. There exists a function $g\colon \Omega \to \mathbf{C}^n$ locally biholomorphic such that*

$$Dg(z) = Df(z) J_f^{-\frac{2}{n+1}} \tag{3.1}$$

if and only if $S^0_{ij} f \equiv 0$ for all i and j. The function g will have $S_g = S_f$.

Proof. Suppose (3.1) holds. A straightforward calculation shows that

$$(Dg(z))^{-1}D^2g(z)(v,v) = S_f(z)(v,v).$$

The coordinate functions g^i of function g satisfy

$$dg^i = J_f^{-2/n+1} df^i\,.$$

Since $0 = d^2g^i = d^2f^i$ we conclude that J_f must be a constant. By Theorem 3.1 we conclude that $S^0_{ij}f \equiv 0$ for all i and j. Reciprocally, if $S^0_{ij}f \equiv 0$, then there exists a constant solution of the system (2.4), and by Lemma 2.2 there exists a mapping g with $S_g = S_f$ and $J_g^{-1/n+1} \equiv C$. By (2.5), $S_g = P_g = S_f$. □

Remark 3.3. Considering $S^0_{ij}f \equiv 0$ then $cDf = Dg$ for some constant c. When $c = J_f^{-2/n+1}$, we have that

$$P_g(z) = S_f(z) = P_f(z).$$

Goldberg in [7] showed that, in terms of our operator,

$$\mathrm{tr}\{Df(z)^{-1}D^2f(z)(\vec{v}_i,\cdot)\} = \frac{\partial}{\partial z_i}\log J_f(z), \tag{3.2}$$

where $\vec{v}_i = (0,\ldots,1,\ldots,0)$ with 1 in position i. We use this result to prove the next theorem of uniqueness.

Theorem 3.4. *Let f,g be locally biholomorphic mappings defined in Ω. Then $P_f(z)(\cdot,\cdot) = P_g(z)(\cdot,\cdot)$ if and only if $f = T\circ g$, where $T(z) = Az+b$ with A is a $n\times n$ constant matrix and $b\in \mathbf{C}^n$.*

Proof. Let f and g be locally biholomorphic mappings in Ω. As $P_f(z)(\vec{v}_i,\cdot) = P_g(z)(\vec{v}_i,\cdot)$ for all $i = 1,\ldots,n$, then by equation (3.2) we have that

$$\nabla \log J_f(z) = \nabla \log J_g(z)\,. \tag{3.3}$$

Using equation (2.5) we can conclude that $S_f(z) = S_g(z)$ for all z. Hence $g = T\circ f$ for some Mobius mapping T. But $\log J_g(z) = \log J_T(f(z)) + \log J_f(z)$ and equation (3.3) we have that $\log J_T(z)$ is a constant, therefore $T(z) = Az+b$ for some $n\times n$ matrix A and $b\in\mathbf{C}^n$. Reciprocally, if $f = T\circ g$ with $T(z) = Az+b$ for some $n\times n$ matrix A and $b\in\mathbf{C}^n$, it is easy to see that $Df(z) = DT(f(z))Df(z) = ADf(z)$, which implies that $P_f(z) = P_g(z)$. □

Theorem 3.5. *Let $A(z)$ be a bilinear operator defined in Ω by*

$$A(z)(\vec{v},\cdot) = \begin{pmatrix} a^1_{11}v_1+\cdots+a^1_{1n}v_n & \cdot\;\cdot\;\cdot\;\cdot & a^1_{n1}v_1+\cdots+a^1_{nn}v_n \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ \cdot & \cdot\;\cdot\;\cdot\;\cdot & \cdot \\ a^n_{11}v_1+\cdots+a^n_{1n}v_n & \cdot\;\cdot\;\cdot\;\cdot & a^n_{n1}v_1+\cdots+a^n_{nn}v_n \end{pmatrix}$$

where $a^k_{ij} = a^k_{ij}(z)$ and $\vec{v} = (v_1\ldots,v_n)$. Then there exists a function $f\colon \Omega\to\mathbf{C}^n$ locally biholomorphic such that $P_f(z) = A(z)$ if and only if the following statements hold:

(i) $a^k_{ij}(z) = a^k_{ji}(z)$ *for all $i,j,k = 1,\ldots,n$;*

(ii) *there exists a holomorphic function* $\varphi\colon \Omega \to \mathbf{C}$ *such that*

$$a^1_{1j}(z) + a^2_{2j}(z) + \cdots + a^n_{nj}(z) = \frac{\partial \varphi}{\partial z_j}(z) \ \ \forall j = 1, \ldots, n;$$

(iii) $\exp(-\frac{\varphi}{n+1})$ *is a solution of the system (2.4) with* $P^k_{ij}(z)$ *given by*

$$P^k_{ij}(z) = a^k_{ij}(z) - \frac{1}{n+1}\left(\delta^k_i\, tr\,\{A(z)(\vec{v}_j, \cdot) + \delta^k_j\, tr\,\{A(z)(\vec{v}_i, \cdot)\right),$$

$i, j, k = 1, \ldots, n$, and $P^0_{ij}(z)$ are defined in terms of $P^k_{ij}(z)$ such that the integrable condition of the system [25, pages 129–130] *holds.*

Proof. Using (i) and (ii) we have that

$$\mathrm{tr}\{A(z)(\lambda, \cdot)\} = \nabla\varphi(z)\cdot\lambda.$$

For given $A(z)$ we can construct a bilinear mapping $\Lambda(z)(\lambda, \mu)$ as

$$\Lambda(z)(\lambda, \mu) = A(z)(\lambda, \mu) - \frac{1}{n+1}\,\mathrm{tr}\{A(z)(\lambda, \cdot)\}\mu - \frac{1}{n+1}\,\mathrm{tr}\{A(z)(\mu, \cdot)\}\lambda.$$

Each component of $\Lambda(z)$ is P^k_{ij} defined by

$$a^k_{ij}(z) - \frac{1}{n+1}\left(\delta^k_i\,\mathrm{tr}\{A(z)(\vec{v}_j, \cdot) + \delta^k_j\,\mathrm{tr}\{A(z)(\vec{v}_i, \cdot)\right).$$

These coefficients satisfy $\sum_i P^k_{ik} = 0$ for all $k = 1, \ldots, n$. Now we define coefficients P^0_{ij} in terms of P^k_{ij} with $k = 1, \ldots, n$ such that the integrability conditions in [25, pages 129–130] hold. Thus, the system (2.4) is completely integrable and in canonical form. Hence we can construct a function f such that $S_f(z) = \Lambda(z)$. By (iii) we have that

$$J_f^{-1/n+1} = \exp(-\frac{\varphi}{n+1}).$$

As S_f is defined by equation (2.5) we conclude that

$$\mathrm{tr}\{A(z)(\lambda, \cdot)\} = \frac{1}{n+1}\nabla J_f(z)\cdot\lambda,$$

which implies that

$$P_f(z) = (Df(z))^{-1}D^2 f(z)(\cdot, \cdot) = A(z)(\cdot, \cdot).$$

Reciprocally, it is easy to see that $P_f(z)$ satisfies (i), (ii) and (iii). □

Observe that $\alpha[Df(z)]^{-1}D^2 f(z)(\vec{v}, \cdot)$ for a locally biholomorphic function f satisfies (i), (ii) and (iii) of Theorem 3.4.

Definition 3.6. Let f be a locally biholomorphic mapping in Ω such that $f(0) = 0$ and $Df(0) = \mathrm{Id}$. We define f_α in Ω as the locally biholomorphic mapping for which

$$[Df_\alpha(z)]^{-1}D^2 f_\alpha(z)(\cdot, \cdot) = \alpha[Df(z)]^{-1}D^2 f(z)(\cdot, \cdot), \tag{3.4}$$

and $f_\alpha(0) = 0$, $Df_\alpha(0) = \mathrm{Id}$.

As a generalization of the problem raised in [6], one can ask the question of determining the values of α for which the mapping f_α is univalent when f is univalent or even just locally univalent. A partial answer is given below when f is convex in the unit ball $\mathbf{B}^n$. Theorem 3.5 shows another partial result for compact linear invariant families. Since the class of univalent mappings in $\mathbf{B}^n$ fails to be compact ($n > 1$), we think it is unlikely that there exists an $\alpha_0 > 0$ small enough so that f_α is univalent for any $|\alpha| \le \alpha_0$ and f univalent in $\mathbf{B}^n$. An interesting compact family of univalent mappings to consider would be the class S_0 of univalent mappings in $\mathbf{B}^n$ that have a parametric representation.

Example 3.7. Let $f(z_1, z_2) = (\phi_\alpha(z_1), \psi_\alpha(z_2))$ be a locally univalent mapping defined in $\mathbf{B}^2$ such that $\phi_\alpha(z_1)$ and $\psi_\alpha(z_2)$ are defined by the equation (1.1) where ϕ and ψ are locally univalent analytic mappings defined in the unit disc such that $\phi(0) = \psi(0) = 0$, $\phi'(0) = \psi'(0) = 1$ and suppose that $z = (z_1, z_2) \in \mathbf{B}^2$. Its Schwarzian derivatives satisfy

$$S^1_{11}f(z_1,z_2) = \frac{\phi''_\alpha}{\phi'_\alpha}(z_1) = \alpha\frac{\phi''}{\phi'}(z_1), \quad S^2_{22}f(z_1,z_2) = \frac{\psi''_\alpha}{\psi'_\alpha}(z_2) = \alpha\frac{\psi''}{\psi'}(z_2),$$
$$S^1_{22}f(z_1,z_2) = S^2_{11}f(z_1,z_2) = 0.$$

Now, let $f(z) = (\psi(z_1), \phi(z_2))$. Then the corresponding mapping f_α has the property that its Schwarzian derivatives are

$$S^1_{11}f_\alpha(z_1,z_2) = \alpha S^1_{11}f(z_1,z_2) = \alpha\frac{\phi''}{\phi'}(z_1),$$
$$S^2_{22}f_\alpha(z_1,z_2) = \alpha S^2_{22}f(z_1,z_2) = \alpha\frac{\psi''}{\psi'}(z_2),$$
$$S^1_{22}f_\alpha(z_1,z_2) = S^2_{11}f_\alpha(z_1,z_2) = 0.$$

Therefore $S^k_{ij}f = S^k_{ij}f_\alpha$ which implies that there exists a Möbius mapping M such that $M \circ f = f_\alpha$. But $f(0) = 0 = f_\alpha(0)$, $DF(0) = Id = Df_\alpha(0)$ and $\nabla \log J_f = \nabla \log J_{f_\alpha} = \alpha \nabla \log J_f$, then $f = f_\alpha$. Thus

$$f(z) = (\phi(z_1), \psi(z_2)) \Longrightarrow f_\alpha(z) = (\phi_\alpha(z_1), \psi_\alpha(z_2)),$$

where ϕ_α and ψ_α are defined by (1.1). By the way, in this example, if $|\alpha| < 1/4$, then f_α will be univalent in $\mathbf{B}^2$. Moreover, if $\phi(z_1)$ is a univalent mapping defined by (1.2) and $\psi(z_2) = z_2$, then the mapping $f(z) = (\phi(z_1), \psi(z_2))$ is univalent and the corresponding mapping f_α is not univalent if $|\alpha| > 1/3$ and $\alpha \ne 1$.

In [9] the author proved that a locally biholomorphic mapping $f\colon \mathbf{B}^n \to \mathbf{C}^n$ is convex if and only if $1 - \mathrm{Re}\langle [Df(z)]^{-1}D^2f(z)(u,u), z\rangle > 0$ for all $z \in \mathbf{B}^n$ and $u \in \mathbf{C}^n$ with $\|u\| = 1$. Thus, if $0 \le \alpha \le 1$, then f_α is a convex mapping when f is a convex mapping since

$$1 - \mathrm{Re}\langle [Df_\alpha(z)]^{-1}D^2f_\alpha(z)(u,u), z\rangle = 1 - \alpha\mathrm{Re}\langle [Df(z)]^{-1}D^2f(z)(u,u), z\rangle > 0. \tag{3.5}$$

Example 3.8. Let f be a univalent function in $\mathbf{D}$. We consider the Roper–Suffridge extension (see [21]) to $\mathbf{B}^2$ of f to the function

$$\Phi_f(z) = \left(f(z_1), \sqrt{f'(z_1)}z_2\right).$$

Thus,

$$[D\Phi_f(z)]^{-1}[D^2\Phi_f(z)](\vec{v}, \cdot) = \begin{pmatrix} \frac{f''}{f'}(z_1)v_1 & 0 \\ \frac{1}{2}z_2Sf(z_1)v_1 + \frac{1}{2}\frac{f''}{f'}(z_1)v_2 & \frac{1}{2}\frac{f''}{f'}(z_1)v_1 \end{pmatrix}.$$

A straightforward calculation shows that

$$(\Phi_f)_\alpha(z) = \left(f_\alpha(z_1), z_2\sqrt{f'_\alpha(z_1)} + y(z_1)\right),$$

where f_α is defined by equation (1.1) and y satisfies that

$$y'' - \alpha\frac{f''}{f'}y' = \frac{\alpha(\alpha-1)}{4}\left(\frac{f''}{f'}\right)^2 (f')^{\alpha/2}.$$

Moreover, Φ_f is univalent when f is univalent, in fact, if f is convex, then Φ_f is convex. On the other hand, $(\Phi_f)_\alpha$ is univalent if f_α is univalent which holds for $|\alpha| \le 1/4$ for all univalent mappings f.

Theorem 3.9. *Let $f\colon \mathbf{B}^n \to \mathbf{C}^n$ be a locally biholomorphic mapping such that the norm order of the linear invariant family generated by f is $\beta < \infty$. Then f_α is univalent if $|\alpha| \le \dfrac{1}{2\beta+1}$.*

Proof. Let ϕ be a automorphism of $\mathbf{B}^n$ such that $\phi(0) = \zeta$. The mapping $g(z) = D\phi(0)^{-1}Df(\phi(0))^{-1}(f(\phi(z)) - f(\phi(0)))$ belongs to the family generated by f, therefore $\|D^2g(0)\| \le \beta$. But

$$\begin{aligned} D^2g(0)(\cdot,\cdot) &= D\phi(0)^{-1}Df(\zeta)^{-1}Df(w)D^2\phi(0)(\cdot,\cdot) \\ &\quad + D\phi(0)^{-1}Df(\zeta)^{-1}D^2f(\zeta)(D\phi(0)(\cdot), D\phi(0)(\cdot)). \end{aligned}$$

Evaluating in $D\phi(0)^{-1}(\zeta) = \zeta/(1-\|\zeta\|^2)$, multiplication by α and using (3.4) we have that

$$\begin{aligned} \alpha D^2g(0)(\zeta,\cdot) &= \alpha D\phi(0)^{-1}Df(\zeta)^{-1}Df(\zeta)D^2\phi(0)(\zeta,\cdot) \\ &\quad + (1-\|\zeta\|^2)D\phi(0)^{-1}Df_\alpha(\zeta)^{-1}D^2f_\alpha(\zeta)(\zeta, D\phi(0)(\cdot)), \end{aligned}$$

where $D\phi(0)^{-1}D^2\phi(\zeta,\cdot) = -\|\zeta\|^2(\cdot) - \zeta\zeta^*(\cdot)$. Thus, for all vectors $v = D\phi(0)^{-1}(u)$ it follows that

$$(1-\|\zeta\|^2)Df_\alpha(\zeta)^{-1}D^2f_\alpha(\zeta)(\zeta,u) = \alpha D\phi(0)D^2g(0)(\zeta,v) - \alpha\|\zeta\|^2u - \alpha(1-\|\zeta\|^2)\zeta\zeta^*v.$$

Then taking the supremum over all vectors u with norm $\|u\| = 1$, we have that

$$\|(1-\|\zeta\|^2)Df_\alpha(\zeta)^{-1}D^2f_\alpha(\zeta)(\zeta,\cdot) + \alpha\|\zeta\|^2I\| \le |\alpha|(2\beta+1).$$

Hence by the generalization of the Ahlfors and Becker result (see [10, page 350]) we conclude that f_α satisfies the hypothesis of this theorem, so it is univalent in $\mathbf{B}^n$. $\square$

The last corollary is an immediate consequence.

Corollary 3.10. *Let $\mathscr{F}$ be a linearly invariant family of locally biholomorphic mappings defined in $\mathbf{B}^n$ of finite order β. Then f_α is univalent in $\mathbf{B}^n$ for every $f \in \mathscr{F}$ when $|\alpha| \le \dfrac{1}{2\beta+1}$.*

Acknowledgements. I would like to thank Martin Chuaqui and John Pfaltzgraff for their useful suggestions and valuable discussions.

References

[1] Ahlfors, L. V.: Sufficient conditions for quasi-conformal extension. - In: Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), Ann. of Math. Studies 79, 1974, 23–29.

[2] Becker, J.: Löwnersche Differentialgleichung und quasi-konform forttsetzbare schlichte Funktionen. - J. Reine Angew. Math. 225, 1972, 23–43.

[3] Chuaqui, M., and B. Osgood: Sharp distortion theorems associated with the Schwarzian derivative. - J. London Math. Soc. (2) 48, 1993, 289–298.

[4] Chuaqui, M., and B. Osgood: Ahlfors–Weill extensions of conformal mappings and critical points of the Poincaré metric. - Comment. Math. Helv. 69, 1994, 659–668.

[5] Duren, P.: Univalent functions. - Springer-Verlag, 1983.

[6] Duren, P., H. S. Shapiro, and A. L. Shields: Singular measures and domains not of Smirnov type. - Duke Math. J. 33:2, 1966, 247–254.

[7] Goldberg, M. A.: The derivative of a determinant. - Amer. Math. Monthly, 79, 1972, 1124–1126.

[8] Kim, Y. J., and E. P. Merkes: On an integral of poers of spirallike function. - Kyungpook Math. J. 12, 1972, 249–253.

[9] Kikuchi, K.: Starlike and convex mappings in several complex variables. - Pacif. J. Math. 44, 1973, 569–580.

[10] Graham, I., and G. Kohr: Geometric function theory in one and higher dimensions. - Monogr. Textbooks Pure Appl. Math. 255, Marcel Dekker, 2003.

[11] Hernández, R.: Schwarzian derivatives and a linearly invariant family in $\mathbf{C}^n$. - Pacific J. Math. 228:2, 2006, 201–218.

[12] Hernández, R.: Schwarzian derivatives and some criteria for univalence in $\mathbf{C}^n$. - Complex Var. Elliptic Equ. 52:5, 2007, 397–410.

[13] Molzon, R., and K. P. Mortensen: Univalence of holomorphic mappings. - Pacific J. Math. 180:1, 1997, 125–133.

[14] Molzon, R., and H. Tamanoi: Generalized Schwarzians in several variables and Möbius invariant differential operators. - Forum Math. 14, 2002, 165–188.

[15] Nehari, Z.: The Schwarzian derivative and schlicht functions. - Bull. Amer. Math. Soc. 55, 1949, 545–551.

[16] Nehari, Z.: Some criteria of univalence. - Proc. Amer. Math. Soc. 5, 1954, 700–704.

[17] Oda, T.: On Schwarzian derivatives in several variables. - Kokyuroku of R.I.M. 226, Kioto Univ., 1974 (in Japanese).

[18] Pfaltzgraff, J. A.: Subordination chains and univalence of holomorphic mappings in $\mathbf{C}^n$. - Math. Ann. 210, 1974, 55-68.

[19] Pfaltzgraff, J. A.: Univalence of the integral of $f'(z)^\lambda$. - Bull. London Math. Soc. 7, 1975, 254–256.

[20] Pommerenke, Ch.: Linear-invarinte Familien analytischer Funktionen I. - Math. Ann. 155, 1964, 108–154.

[21] Roper, K., and T. Suffridge: Convex mappings on the unit ball in $\mathbf{C}^n$. - J. Anal. Math. 65, 1995, 333–347.

[22] Suffridge, T.: Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions. - In: Complex analysis (Proc. Conf., Univ. Kentucky, Lexington, Ky., 1976), Lecture Notes in Math. 599, Springer-Verlag, New York, 1976, 146–159.

[23] Royster, W. C.: On the univalence of a certain integral. - Michigan Math. J. 12, 1965, 385–387.

[24] Yoshida, M.: Canonical forms of some system of linear partial differential equations. - Proc. Japan Acad. 52, 1976, 473–476.

[25] Yoshida, M.: Orbifold-uniformizing differential equations. - Math. Ann. 267, 1984, 125–142.

Received 5 August 2010