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SPLITTING-TYPE VARIATIONAL PROBLEMS WITH x-DEPENDENT EXPONENTS

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Abstract. In this article we prove regularity results for locally bounded minimizers $u\colon \mathbf{R}^n \supset \Omega \to \mathbf{R}^N$ of functionals of the type

$$\int_\Omega \left[(1+|\nabla_1 u|^2)^{\frac{p(x)}{2}} + (1+|\nabla_2 u|^2)^{\frac{q(x)}{2}}\right] dx,$$

where p and q are Lipschitz-functions and $\nabla u = (\nabla_1 u, \nabla_2 u)$ is an arbitrary decompositon of the gradient of u. Related functionals are the topic of the paper [Br3], but the situation here is not covered.

1. Introduction

The study of regularity properties for minimizers $u\colon \Omega \to \mathbf{R}^N$ of energies

$$I[u,\Omega] := \int_\Omega F(\nabla u)\,dx, \tag{1.1}$$

where Ω denotes an open set in $\mathbf{R}^n$ and where $F\colon \mathbf{R}^{nN} \to [0,\infty)$ satisfies an anisotropic growth condition, i.e.,

$$C_1|Z|^{\overline{p}} - c_1 \le F(Z) \le C_2|Z|^{\overline{q}} + c_2, \quad Z \in \mathbf{R}^{nN}$$

with constants $C_1, C_2 > 0$, $c_1, c_2 \ge 0$ and exponents $1 < \overline{p} \le \overline{q} < \infty$, was introduced by Marcellini (see [Ma1] and [Ma2]) and was widely investigated by many authors in the last years, see the references at the end of the paper. Starting from the research of Esposito, Leonetti and Mingione [ELM1] it is known that in general minimizers of (1.1) stay not regular if one allows an additional x-dependence and considers minimizers of functionals

$$J[u,\Omega] := \int_\Omega F(\cdot,\nabla u)\,dx \tag{1.2}$$

for $F\colon \Omega\times\mathbf{R}^{nN} \to [0,\infty)$. Already in the autonomous situation it is well-known that we have no hope for regularity for minimizers of (1.1) if $\overline{p}$ and $\overline{q}$ are too far apart (compare the counterexamples of [Gi2] and [Ho]). The best known bound is

$$\overline{q} < \overline{p} + 2,$$

proven in [BF1] and [ELM2]. To get better results additional assumptions are necessary. Thus Fuchs and Bildhauer consider decomposable integrands which means we have

$$F(Z) = f(\widetilde{Z}) + g(Z_n)$$

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for $Z = (Z_1, \dots, Z_n)$ with $Z_i \in \mathbf{R}^N$ and $\widetilde{Z} = (Z_1, \dots, Z_{n-1})$ (note that this condition is only an example, we could consider every other decomposition of ∇u into two parts). Bildhauer, Fuchs and Zhong assume power growth conditions for the C^2-functions f and g with exponents $\overline{p} \le \overline{q}$ and get a very general regularity theory in case $\overline{p} \ge 2$ (see [BF3], [BF4] and [BFZ]). In [Br2] we generalize these statements under the assumption

$$f(\widetilde{Z}) = a(|\widetilde{Z}|) \quad \text{and} \quad g(Z_n) = b(|Z_n|)$$

where a and b are N-functions. Here the main assumptions are (h stands for a or b)

$$\frac{h'(t)}{t} \approx h''(t)$$

and superquadratic growth of h. In [Br3] we extend the results for an x-dependence without severe restrictions. In this paper we focus our attention on the regularity properties for minimizers of functionals of the following type

$$\mathscr{F}[u] := \int_\Omega \left[(1 + |\widetilde{\nabla} u|^2)^{\frac{p(x)}{2}} + (1 + |\partial_n u|^2)^{\frac{q(x)}{2}} \right] dx. \tag{1.3}$$

Now the functions

$$a(x,t) := \big(1 + t^2\big)^{\frac{p(x)}{2}} - 1 \quad \text{and} \quad b(x,t) := \big(1 + t^2\big)^{\frac{q(x)}{2}} - 1$$

satisfy all conditions assumed in [Br3] (if $p, q \ge 2$) except

$$|\partial_\gamma h'(x,t)| \le c h'(x,t) \text{ for all } (x,t) \in \overline{\Omega} \times \mathbf{R}_0^+ \tag{1.4}$$

and all $\gamma \in \{1, \dots, n\}$ for a constant $c \ge 0$. Note that (1.4) is the main hypothesis to handle the terms involving derivatives with respect to x in [Br3]. The functions a and b in the functional above do *not* fulfill (1.4), here the best estimation is

$$|\partial_\gamma h'(x,t)| \le c(\epsilon)(1 + t^2)^{\frac{\epsilon}{2}} h'(x,t) \text{ for all } (x,t) \in \overline{\Omega} \times \mathbf{R}_0^+ \tag{1.5}$$

for every $\epsilon > 0$ with a constant $c(\epsilon) > 0$.

Let us state our new result.

Theorem 1.1. *Let $u \in L^\infty_{\mathrm{loc}}(\Omega, \mathbf{R}^N)$ be a local minimizer of (1.3) in the class $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{R}^N)$ and $p, q \in W^{1,\infty}_{\mathrm{loc}}(\Omega, [2, \infty))$. Then we have*

(a) *partial $C^{1,\alpha}$-regularity if $p \le q < p + 2$ on Ω (for $n \ge 5$ we additionally need $p > \|q - p\|_\infty (n-2)/2$);*
(b) *full $C^{1,\alpha}$-regularity for $n = 2$;*
(c) *full $C^{1,\alpha}$-regularity for $N = 1$ if $\|p - q\|_\infty < 2$.*

Remark 1.1. Results due to minimizers of functionals like in (1.3) are not found in literature. A similar problem is minimizing

$$\int_\Omega (1 + |\nabla w|^2)^{\frac{p(x)}{2}} \, dx.$$

Regularity results are stated in [CM].

Our result is not restricted to the special integrand in (1.3). We can also consider functions $a, b \colon \Omega \times [0, \infty) \to [0, \infty)$ which satisfy all assumptions from [Br3] except

(A5) together with (1.5). This means the theorem above covers functionals of the type

$$\mathscr{F}[u] := \int_\Omega \Big[a(\cdot, |\widetilde{\nabla} u|) + b(\cdot, |\partial_n u|) \Big] \, dx$$

provided we have for $h \in \{a, b\}$ and all $x \in \Omega$:

(A1) $h(x, \cdot)$ is a N-function (in the sense of [Ad]);

(A2) $\frac{h'(x,t)}{t} \approx h''(x,t)$ for all $t \geq 0$;

(A3) $a(x,t) \leq cb(x,t)$ for large t (can be weakened if $n = 2$ or $N = 1$);

(A4) $\frac{h'(x,t)}{t} \geq h_0 > 0$ for all $t \geq 0$;

(A5)* for every $\epsilon > 0$ we can find $c(\epsilon) > 0$ such that

$$|\partial_\gamma h'(x,t)| \leq c(\epsilon)(1+t^2)^{\frac{\epsilon}{2}} h'(x,t)$$

for large t;

(A6) $b(x,t) \leq ct^\omega a(x,t)$ for large t for an $\omega > 0$ (ω arbitrary if $n = 2$ and $\omega < 2$ if $n \geq 3$);

(A7) $\frac{h'(x,t)}{t} \leq h''(x,t)$ for $t \geq 0$, if $\omega < 1$;

(A8) $a(x,t) \geq \vartheta t^{\frac{\omega+2\epsilon}{2}(n-2)}$ for large t and $\vartheta > 0$;

(A9) $\operatorname{argmin}_{y \in B} a(y,t)$ is independent of t for all $B \Subset \Omega$;

(A10) $a(x,t) \leq \theta_1 \, t^{\theta_2 |x-y|} a(y,t)$ for large t and all $x, y \in B$ ($\theta_1, \theta_2 > 0$).

Note that the assumptions (A7)–(A10) disappear if $n = 2$ or $N = 1$ and (A7) and (A8) are only important for $n \geq 5$. Further functionals which are covered by the theory in this paper but not by [Br3] are given if we define

$$a(x,t) := \int_0^t (1+s^2)^{\frac{p(x)-2}{2}} s \, ds \quad \text{or} \quad a(x,t) := \int_0^t (1+s)^{p(x)-2} \Theta(s) \, ds$$

and $b(x,t)$ replacing p by q. Here $\Theta \in C^1([0,\infty),[0,\infty))$ has to perform $\Theta'(t)t \approx \Theta(t)$, $\Theta(0) = 0$ and $\Theta'(t) \geq \Theta_0 > 0$.

Remark 1.2. Let us compare the statements of Theorem 1.1 with the power growth situation: Fuchs and Bildhauer [BF3] proved full regularity for $n = 2$ in the superquadratic situation which we can exactly reproduce. In [BF4] they analyse the general vector case and get partial regularity under the assumptions $p \leq q \leq p+2$ and $q \leq pn/(n-2)$. The first one is nearly the same as in Theorem 1.1, we can not allow an equality. If we have a look at the second one this corresponds to $p > \|q-p\|_\infty (n-2)/2$ in case of constants p and q but without equality, too. Only the scalar case is a real restriction: In [BFZ] no condition between p and q is needed, but we have to suppose $\|p-q\|_\infty < 2$.

The bound $q < p+2$ for functionals with (p,q)-growth firstly appears in [ELM2] (in the autonomous situation).

Remark 1.3. If $n = 2$ then we do not have to assume local boundedness of the minimizer. The idea to remove this is outlined in [Bi] (section 4). In 2D it is possible to consider subquadratic problems with restriction between p and q. In this case one can follow the approach of [BF6] and [Br4].

From our proof follows that we do not need superquadratic growth if $N = 1$. We only have to suppose $p > 1$ on Ω. Then the regularized problem (compare Lemma 2.2) has a Lipschitz-solution by [BF2] (Thm. 1.2).

If $n \leq 4$, then we can deduce from $p \geq 2$ and $p \leq q < p+2$ the inequality $p > \|q-p\|_\infty (n-2)/2$.

2. Proof of Theorem 1.1

Let

$$a(x,t) := \left(1+t^2\right)^{\frac{p(x)}{2}} - 1 \quad \text{and} \quad b(x,t) := \left(1+t^2\right)^{\frac{q(x)}{2}} - 1.$$

It is easy to prove that these functions satisfy the assumptions (A1)–(A4) from [Br3] as well as (A9) and (A10) (compare the list in the introduction). Hence we define the regularization as there (originally it was introduced in [BF6]): for $h \in \{a,b\}$ let

$$h_M(x,t) := \int_0^t s g_M(x,s)\, ds$$

where $M \gg 1$ and

$$g_M(x,t) := g(x,0) + \int_0^t \eta(s) g'(x,s)\, ds, \quad g(x,t) := \frac{h'(x,t)}{t}.$$

Here $\eta \in C^1([0,\infty))$ denotes a cut-off function with the properties $0 \leq \eta \leq 1$, $\eta' \leq 0$, $|\eta'| \leq c/M$, $\eta \equiv 1$ on $[0, 3M/2]$ and $\eta \equiv 0$ on $[2M,\infty)$. From [Br3] we quote the following properties of h_M.

Lemma 2.1. *For the sequence (h_M) we have:*

(i) *$h_M \in C^2(\Omega \times [0,\infty))$ is a N-function, $\frac{h'_M(x,t)}{t} \geq h_0 > 0$ for all $x \in \Omega$, all $t \geq 0$ and uniformly in M;*

(ii) *$h_M \leq h$ and $h''_M \leq c(M)$ on $\Omega \times \mathbf{R}_0^+$;*

(iii) *we have for positive constants $\overline{\epsilon}, \overline{h}$*

$$\overline{\epsilon}\, \frac{h'_M(x,t)}{t} \leq h''_M(x,t) \leq \overline{h}\, \frac{h'_M(x,t)}{t}$$

uniformly in M;

(iv) *if we have $p \leq q$, then*

$$a_M(x,t) \leq \overline{c} b_M(x,t) \text{ for all } x \in \Omega \text{ and all } t \geq 0;$$

(v) *(1.5) extends to h_M uniformly in M:*

$$|\partial_\gamma h'_M(x,t)| \leq c(\epsilon)(1+t^2)^{\frac{\epsilon}{2}} h'_M(x,t) \text{ for all } (x,t) \in \Omega \times \mathbf{R}_0^+$$

and all $\gamma \in \{1,\dots,n\}$;

(vi) *from $q - p \leq \omega$ for a positive number ω follows*

$$b_M(x,t) \leq c t^\omega a_M(x,t) \text{ uniformly in } M;$$

(vii) *h_M and h_M^{-1} satisfy uniform Δ_2-conditions, which follows from (iii);*

(viii) *we get from (iii) and monotonicity of h'_M*

$$\lambda h'_M(x,t) t \leq h_M(x,t) \leq h'_M(x,t) t \ \text{ uniformly in } M.$$

Only (v) is not the same as in [Br3] and need a slight comment: the estimation follows from

$$\frac{h'_M(x,t)}{t} = \eta(t)\frac{h'(x,t)}{t} + \int_0^t \left\{-\frac{\eta'(s)}{s}\right\} h'(x,s)\, ds$$

and (1.5) using $\eta'(s) \leq 0$.

Letting $F_M(x, Z) := a_M(x, |\widetilde{Z}|) + b_M(x, |Z_n|)$ we define for $B \Subset \Omega$

$$\mathscr{F}_M[w] := \int_B F_M(\cdot, \nabla w)\, dx$$

and u_M as the unique minimizer of $\mathscr{F}_M$ in $u + W_0^{1,2}(B, \mathbf{R}^N)$. We obtain for u_M the following regularity properties.

Lemma 2.2. (i) *u_M belongs to the space $W^{2,2}_{\mathrm{loc}}(B, \mathbf{R}^N)$;*

(ii) *$a_M(\cdot, |\nabla \widetilde{u}_M|)|\widetilde{\nabla} u_M|^2$ and $b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2$ are elements of $L^1_{\mathrm{loc}}(B)$;*

(iii) *if $n = 2$ or $N = 1$ then we have $u_M \in W^{1,\infty}_{\mathrm{loc}}(B, \mathbf{R}^N)$;*

(iv) *for $\gamma \in \{1, \ldots, n\}$ $\partial_\gamma u_M$ solves*

$$\int_B D_P^2 F_M(\cdot, \nabla u_M)(\nabla w, \nabla \varphi)\, dx + \int_B \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \nabla \varphi\, dx = 0$$

for all $\varphi \in W_0^{1,2}(B, \mathbf{R}^N)$ with $\mathrm{spt}(\varphi) \Subset B$;

(v) *u_M is in $W^{1,2}(B, \mathbf{R}^N)$ uniformly bounded and we have*

$$\sup_M \int_B F_M(\cdot, \nabla u_M)\, dx < \infty;$$

(vi) *if we have $u \in L^\infty_{\mathrm{loc}}(\Omega, \mathbf{R}^N)$, then $\sup_M \|u_M\|_\infty < \infty$.*

Proof. By construction of F_M we obtain the following growth conditions (compare Lemma 2.2)

$$\lambda |X|^2 \le D_P^2 F_M(x, Z)(X, X) \le \Lambda_M (1 + |Z|^2)^{\frac{\epsilon}{2}} |X|^2,$$
$$|\partial_\gamma D_P F_M(x, Z)| \le \Lambda_M (1 + |Z|^2)^{\frac{1+\epsilon}{2}}$$

for all $X, Z \in \mathbf{R}^{nN}$, all $\gamma \in \{1, \ldots, n\}$ and all $x \in \overline{B}$ for positive constants λ, Λ_M. If we follow the approach of [BF2] (Lemma 2.8 with $\alpha = 0$) for $p = 2$ and $q = 2 + \epsilon$, we see $\nabla u_M \in L^4_{\mathrm{loc}}(B, \mathbf{R}^{nN})$. Note that in case $\alpha = 0$ modulus dependence is not necessary. From the same proof we deduce $u_M \in W^{2,2}_{\mathrm{loc}}(B, \mathbf{R}^N)$ and so the first two statements of the Lemma. If we quote [BF2] (Thm. 1.1), then follows $u_M \in W^{1,\infty}_{\mathrm{loc}}(B, \mathbf{R}^N)$ for $n = 2$ or $N = 1$ (we can choose ϵ small enough to reach $q < p(n+1)/n$). By approximation we get (iv), which is of course valid for $\varphi \in C_0^\infty(B, \mathbf{R}^N)$. We can adopt the last two statements from [Br3]. □

Partial regularity. Now we have to prove the higher integrability stated in [Br3] (Theorem 1.1). This means we have to show

$$a_M(\cdot, |\nabla \widetilde{u}_M|)|\widetilde{\nabla} u_M|^2, b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2 \in L^1_{\mathrm{loc}}(B) \text{ uniformly.} \tag{2.1}$$

If we follow the lines of [Br3] (Section 2), we get by Young's inequality and Lemma 2.2 (v) for a suitable cut-off function $\eta \in C_0^\infty(B)$ and $k \in \mathbf{N}$ large enough

$$\begin{aligned}\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2\, dx &\le c(\eta) + c(\eta) \int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|^{2\epsilon}\, dx \\ &\le c(\eta, \tau) + \tau \int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|^2\, dx.\end{aligned} \tag{2.2}$$

This is a consequence of an integration-by-parts argument, a Caccioppoli-type inequality, following by standard calculations from Lemma 2.2 (iv), and finally (1.5) resp. Lemma 2.1 (v) (of course we also need the uniform bounds from Lemma 2.2 (v)).

We remark that (2.2) is the analogy of inequality (2.5) in [Br3]. Whereas (2.7) of [Br3] now reads as

$$\int_B \eta^{2k} a_M(\cdot, |\widetilde{\nabla} u_M|)|\widetilde{\nabla} u_M|^2\,dx \le c(\eta) + c(\eta)\int_B \eta^{2k} b_M(\cdot, |\partial_n u_M|)|\partial_n u_M|^2\,dx. \tag{2.3}$$

If we combine (2.2) and (2.3) and choose τ small enough we get (2.1) and can pass to the limit (details can be found in [Br3]).

As usual the key to the partial regularity is the following lemma.

Lemma 2.3. *Assume the assumptions of Theorem 1.1 and fix $L > 0$. Then there is a $C^*(L)$ such that for every $\tau \in (0, 1/4)$ exists an $\kappa = \kappa(\tau, L) > 0$ with the following property: If*

$$|(\nabla u)_{x,r}| \le L \quad \textit{and} \quad E(x,r) + r^{\gamma^*} \le \kappa \tag{2.4}$$

for a ball $B_r(x) \Subset \Omega$, this implies

$$E(x, \tau r) \le C^* \tau^2 [E(x,r) + r^{\gamma^*}], \tag{2.5}$$

where $\gamma^* \in (0,2)$ is arbitrary and $(f)_{x,r}$ denotes the mean value of a function f over the ball $B_r(x)$.

Here we have

$$E(x,r) := \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2\,dy + \fint_{B_r(x)} \overline{a}(\cdot, |\nabla u - (\nabla u)_{x,r}|)\,dy$$

for a small radius r, where $\overline{a}(x,t) := a(x,t)t^{\omega+2\epsilon}$ and $\omega := \|p-q\|_\infty < 2$. The ϵ-term in the definition of $\overline{a}$ is the modification of the excess function in [Br3] and compensates the additional power ϵ in (1.5). Since $\epsilon > 0$ is arbitrary and $\omega < 2$, we can reach $\omega + 2\epsilon < 2$ and the well-definedness of the excess function follows from (2.1) by the lines of [Br3] (section 3).

Proof of Lemma 2.3. Thanks to the modification of $\overline{a}$ in the excess function we can prove Lemma 2.3 as in [Br3]. In the proof of the strong convergence of the scaled functions we need the convergence (letting $A_K(r) := B_r \cap [\lambda_m|\widetilde{\nabla} u_m| > K]$, $K > 3L$, $B_r \Subset B_1$)

$$\lambda_m^{-2}\int_{\overline{A}_K(r)} \overline{a}(\lambda_m|\nabla u_m|)\,dy \longrightarrow 0, \quad m \to \infty. \tag{2.6}$$

Here u_m as a scaling of u on the unit ball and λ_m converge to zero, details, also for (2.6), can be found in [Br2] and [Br3]. Since (compare [Fu], after (3.25))

$$\lambda_m^{-2}\int_{\overline{A}_K(r)} \overline{a}(\lambda_m|\widetilde{\nabla} u_m|)\,dy \le c(r)\left(\int_{A_K(r)} |\lambda_m \widetilde{\nabla} u_m|^{\frac{\omega+2\epsilon}{2}n}\,dz\right)^{2/n},$$

we conclude (2.6) by

$$a(x,t) \ge \vartheta t^{\frac{\omega+2\epsilon}{2}(n-2)},$$

which follows from $p > \|p-q\|_\infty (n-2)/2$ for a suitable choice of ϵ and the definition of ω. Now we have

$$\lambda_m^{-2}\int_{\overline{A}_K(r)} \overline{a}(\lambda_m|\widetilde{\nabla} u_m|)\,dy \le c(r)\left(\int_{A_K(r)} a(\cdot, |\lambda_m \widetilde{\nabla} u_m|)^{\frac{n}{n-2}}\,dz\right)^{2/n}$$

and the r.h.s. vanishes for $m \to \infty$, for details we refer again to [Br2] and [Br3]. For

$$\lambda_m^{-2} \int_{A_K(r)} b(|\lambda_m \partial_n u_m|) |\lambda_m \partial_n u_m|^\omega \, dy$$

the same arguments are applicable. This finally leads to (2.6), the last missing step in the contradiction of the proof of Lemma 2.3 from which the claim of Theorem 1.1 follows by standard arguments. □

Full regularity for $n = 2$. In [BF6], (2.5), the authors prove an inequality of the form (sum over $\gamma \in \{1, 2\}$)

$$\begin{aligned} &\int_{B_r(z)} D_P^2 F_M(\cdot, \nabla u_M)(\partial_\gamma \nabla u_M, \partial_\gamma \nabla u_M) \, dx \\ &\leq c(\tau)(R-r)^{-\beta} + \tau \int_{B_R(z)} \left(a_M(\cdot, |\partial_1 u_M|)^2 + b_M(\cdot, |\partial_2 u_M|)^2 \right) \, dx. \end{aligned} \tag{2.7}$$

Here is $B_r(z) \Subset B_R(z) \Subset B$, $\tau > 0$ arbitrary and $\beta > 0$ a suitable exponent. On account of the x-dependence we have additionally to the terms in [BF6] the integral

$$-\int_{B_R(z)} \eta^2 \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \partial_\gamma \nabla u_M \, dx,$$

where $\eta \in C_0^\infty(B_R(z))$ is a suitable cut-off function. Using (1.5) and the splitting-structure we estimate this by

$$\begin{aligned} &c \int_{B_R(z)} \eta^2 a'_M(\cdot, |\partial_1 u_M|)(1 + |\partial_1 u_M|^2)^{\frac{\epsilon}{2}} |\partial_\gamma \partial_1 u_M| \, dx \\ &+ c \int_{B_R(z)} \eta^2 b'_M(\cdot, |\partial_2 u_M|)(1 + |\partial_2 u_M|^2)^{\frac{\epsilon}{2}} |\partial_\gamma \partial_2 u_M| \, dx. \end{aligned}$$

As a consequence of Young's inequality we can bound the first term by (compare Lemma 2.1 (viii))

$$\tau' \int_{B_R(z)} \eta^2 \frac{a'_M(\cdot, |\partial_1 u_M|)}{|\partial_1 u_M|} |\partial_\gamma \partial_1 u_M|^2 \, dx + c(\tau') \int_{B_R(z)} \eta^2 a_M(\cdot, |\partial_1 u_M|)(1 + |\partial_1 u_M|^2)^\epsilon \, dx.$$

For $\tau' \ll 1$ one can absorb the first integral in the l.h.s. of (2.7). Here we used the inequality

$$\frac{a'_M(\cdot, |\widetilde{Z}|)}{|\widetilde{Z}|} |\widetilde{P}|^2 \leq c D_P^2 F_M(x, Z)(P, P)$$

for $Z, P \in \mathbf{R}^{nN}$ (compare Lemma 2.1, part (iii)). For the second one we obtain

$$\begin{aligned} &c(\tau') \int_{B_R(z)} \eta^2 a_M(\cdot, |\partial_1 u_M|)(1 + |\partial_1 u_M|^2)^\epsilon \, dx \\ &\leq \tau'' \int_{B_R(z)} a_M(\cdot, |\partial_1 u_M|)^2 \, dx + c(\tau'') \int_{B_R(z)} (1 + |\partial_1 u_M|^2)^{2\epsilon} \, dx. \end{aligned}$$

We can handle the r.h.s. conveniently, since we may assume $\epsilon \leq 1/2$ and receive (compare Lemma 2.2, part (v))

$$\int_{B_R(z)} (1 + |\partial_1 u_M|^2)^{2\epsilon} \, dx \leq c + \int_{B_R(z) \cap [|\partial_1 u_M| > 1]} a_M(\cdot, |\partial_1 u_M|) \, dx \leq c.$$

Analogously we can incorporate the term

$$\int_{B_R(z)} \eta^2 b_M'(\cdot, |\partial_2 u_M|)(1+|\partial_2 u_M|^2)^{\frac{\epsilon}{2}} |\partial_\gamma \partial_2 u_M|\, dx,$$

hence (2.7) follows. In [BF6] we can find the inequality

$$\begin{aligned} &\int_{B_\rho(z)} \left(a_M(\cdot, |\partial_1 u_M|)^2 + b_M(\cdot, |\partial_2 u_M|)^2\right)\, dx \\ &\le c(R-\rho)^{-2} + c\int_{B_r(z)} D_P^2 F_M(\cdot, \nabla u_M)(\partial_\gamma \nabla u_M, \partial_\gamma \nabla u_M)\, dx \end{aligned} \tag{2.8}$$

for $\rho \in (0,R)$, and $r = (\rho + R)/2$. In order to show this the authors of [BF6] use an additional cut-off function and Sobolev's embedding $W^{1,1} \hookrightarrow L^2$, valid for $n = 2$, as well as the uniform growth conditions for a_M and b_M (compare Lemma 2.1). In our approach we obtain on the r.h.s. of this inequality additionally the term (if we estimate $\nabla_x a_M$ and $\nabla_x b_M$ using (1.5))

$$\left[\int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)(1+|\partial_1 u_M|^2)^{\frac{\epsilon}{2}}\, dx\right]^2 + \left[\int_{B_r(z)} b_M(\cdot, |\partial_2 u_M|)(1+|\partial_2 u_M|^2)^{\frac{\epsilon}{2}}\, dx\right]^2.$$

We can handle both terms in a similar way and show the proceeding for the first one. By Hölder's inequality we receive the upper bound

$$\begin{aligned} Y_M := &\left[\int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)^{s\chi}\, dx\right]^{\frac{2}{\chi}} \\ &\cdot \left[\int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)^{\frac{\chi - s\chi}{\chi - 1}} (1+|\partial_1 u_M|^2)^{\frac{\epsilon}{2}\frac{\chi}{\chi-1}}\, dx\right]^{2\frac{\chi-1}{\chi}}. \end{aligned}$$

Here we have $s \in (0,1)$ and $\chi \in (1,2)$ such that $s\chi > 1$. For the second integral Y_M^2 follows by Lemma 2.2 (v)

$$Y_M^2 = \int_{B_r(z)\cap[|\partial_1 u_M|\le 1]} \ldots + \int_{B_r(z)\cap[|\partial_1 u_M|>1]} \ldots \le c + \int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)\, dx \le c.$$

Note that we have for $t \ge 1$

$$a_M(x,t)^{\frac{\chi - s\chi}{\chi-1}} (1+t^2)^{\frac{\epsilon}{2}\frac{\chi}{\chi-1}} \le c a_M(x,t)$$

for ϵ small enough, since $s\chi > 1$ (remember Lemma 2.1 (i)). Now we get, using Jensen's and Young's inequality

$$\begin{aligned} Y_M &\le c\left[\int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)^{s\chi}\, dx\right]^{\frac{2}{\chi}} \le c\int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)^{2s}\, dx \\ &\le \tau''' \int_{B_r(z)} a_M(\cdot, |\partial_1 u_M|)^2\, dx + c(\tau'''). \end{aligned}$$

So we have to add

$$\tau''' \int_{B_r(z)} \left(a_M(\cdot, |\partial_1 u_M|)^2 + b_M(\cdot, |\partial_2 u_M|)^2\right)\, dx \tag{2.9}$$

on the r.h.s. of (2.8). Combining (2.7)–(2.9) we have shown (for a suitable choice of τ and τ''')

$$\int_{B_\rho(z)} \big(a_M(\cdot,|\partial_1 u_M|)^2 + b_M(\cdot,|\partial_2 u_M|)^2\big)\,dx$$
$$\le c(R-r)^{-\beta} + \frac{1}{2}\int_{B_R(z)} \big(a_M(\cdot,|\partial_1 u_M|)^2 + b_M(\cdot,|\partial_2 u_M|)^2\big)\,dx.$$

From [Gi1] (Lemma 5.1, p. 81) we obtain uniform bounds on $a_M(\cdot,|\partial_1 u_M|)$ and $b_M(\cdot,|\partial_2 u_M|)$ in $L^2_{\mathrm{loc}}(B)$. Now we get the uniform boundedness of u_M in $W^{2,2}_{\mathrm{loc}}(B,\mathbf{R}^N)$ (compare Lemma 2.1 (i) and (2.7)) and we can reproduce the proof of [BF6] for the rest, whereby the terms which appear additionally on account of (1.5) are uncritical.

Full regularity for $N = 1$. In case $N = 1$ it is possible to modify the N-function in [Br3], (4.4). Therefore we need the inequalities

$$b_M(x,t) \le ct^{2-2\epsilon} a_M(x,t) \quad \text{and} \quad a_M(x,t) \le ct^{2-2\epsilon} b_M(x,t). \tag{2.10}$$

By Lemma 2.1 (vi) this follows from $\|p-q\|_\infty < 2$ for $\epsilon \ll 1$. So we can separate the mixed integrands of the terms

$$\int_B \eta^{2k} a_M(\cdot,|\widetilde{\nabla} u_M|)|\Gamma_{n,M}^{\frac{\alpha+2\epsilon}{2}}\,dx \quad \text{and} \quad \int_B \eta^{2k} b_M(\cdot,|\partial_n u_M|)\widetilde{\Gamma}_M^{\frac{\alpha+2\epsilon}{2}}\,dx,$$

which occur additionally to the integrals in [Br3]. Finally we get instead of [Br3], (4.6),

$$\int_B \eta^{2k} b_M(\cdot,|\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha+2}{2}}\,dx$$
$$\le c(\eta)\left[\ldots + \int_B \eta^{2k} b_M(\cdot,|\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha+2\epsilon}{2}}\,dx + \int_B \eta^{2k} a_M(\cdot,|\widetilde{\nabla} u_M|)\widetilde{\Gamma}_M^{\frac{\alpha+2\epsilon}{2}}\,dx\right]$$

as well as an analogous inequality for $a_M(\cdot,|\widetilde{\nabla} u_M|)\widetilde{\Gamma}_M^{\frac{\alpha+2}{2}}$ instead of [Br3], (4.7). The key for this is an integrating-by-parts argument as in the vector-valued situation (compare the step: partial regularity), the growth conditions of a_M and b_M and a Caccioppoli-type inequality only valid if $N = 1$. Since we may assume $\epsilon \le 1/2$, the first integral on the r.h.s. is bounded by (using Young's inequality)

$$\tau \int_B \eta^{2k} b_M(\cdot,|\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha+2}{2}}\,dx + c(\tau)\int_B \eta^{2k} b_M(\cdot,|\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha}{2}}\,dx$$

for an arbitrary $\tau > 0$. For the second one we can argue similarly and we obtain by absorbing the τ-terms

$$\int_B \eta^{2k} b_M(\cdot,|\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha+2}{2}}\,dx + \int_B \eta^{2k} a_M(\cdot,|\widetilde{\nabla} u_M|)|\widetilde{\Gamma}_M^{\frac{\alpha+2}{2}}\,dx$$
$$\le c(\eta)\left[\int_{\mathrm{spt}(\eta)} \eta^{2k} b_M(\cdot,|\partial_n u_M|)|\Gamma_{n,M}^{\frac{\alpha}{2}}\,dx + \int_{\mathrm{spt}(\eta)} \eta^{2k} a_M(\cdot,|\widetilde{\nabla} u_M|)\widetilde{\Gamma}_M^{\frac{\alpha}{2}}\,dx\right].$$

Now we can iterate as in [Br3] and obtain arbitrary high integrability of ∇u_M uniform in M (the starting point is $\alpha = 0$, see Lemma 2.2, part (v)). This is enough to end up the proof as mentioned there (full regularity follows by DeGiorgi-type arguments using Stampacchia's Lemma [St], Lemma 5.1, p. 219).

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