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A FAMILY OF SMOOTH CANTOR SETS

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Abstract. We show that the Cantor set C_p associated to the sequence $\{1/n^p\}_n$, $p>1$, is a smooth attractor. Moreover, it is smoothly conjugate to the 2^{-p}-middle Cantor set. We also study the convolution of Hausdorff measures supported on these sets and the structure and size of the sumset C_p+C_q.

1. Introduction and statement of main results

1.1. Introduction. A Cantor set is a compact, perfect and totally disconnected set in some topological space. We deal with Cantor sets in the real line with the usual topology.

There is a way to construct zero Lebesgue measure Cantor sets that consists in successively removing *gaps*, that is, bounded open intervals, from an initial closed interval; the construction is done by steps and the lengths of the removed gaps are prescribed by the values of a positive and summable sequence. The precise definition, which appeared in [BT54], is given in Section 2. For example the 'classical' middle-r Cantor set A_r, $0<r<1/2$, that is defined by

$$A_r=\Big\{(1-r)\sum_{j\geq 0}a_jr^j\colon a_j\in\{0,1\}\Big\},$$

is the one associated to the sequence $\big\{\xi,r\xi,r\xi,r^2\xi,r^2\xi,r^2\xi,r^2\xi,\dots\big\}$, where $\xi=1-2r$. Here, the ratio between the lengths of the gaps of consecutive steps is constant, which reflects the 'linearity' of the set. Note that $A_{1/3}$ is the classical ternary Cantor set.

We will mainly focus on the family of *p-Cantor sets* C_p, $p>1$, which are defined through the above construction using the sequence $\{1/n^p\}_{n\geq 1}$. At any fixed step the removed gaps have strictly decreasing lengths, which reflects the nonlinear nature of this set. Despite its nonlinearity, this is a family of well behaved Cantor sets, since in [CMPS05] and [GMS07] it is shown that $0<\mathscr{H}^{1/p}(C_p)<P_0^{1/p}(C_p)<+\infty$, where $\mathscr{H}^t$ and P_0^t denotes the t-dimensional Hausdorff measure and packing premeasure respectively; in particular,

$$\dim C_p=\overline{\dim}_B C_p=1/p,$$

where dim and $\overline{\dim}_B$ denote the Hausdorff and upper Box dimensions. See the book of Mattila [Mat95] for the definitions of these measures and dimensions. In this article we discover further properties of this family of Cantor sets, showing that it

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is closely related to the family of middle-r Cantor sets and so it can be viewed as a nonlinear version of the classical linear case.

1.2. Statements of main results. Let us recall that an *iterated function system* (IFS) is a finite set $\{f_0, \ldots, f_n\}$ of self maps defined on a nonempty closed subset $X \subset \mathbf{R}$ such that each f_i is a strict contraction, that is, there is a constant $0 < c < 1$ such that

$$|f_i(x) - f_i(y)| \leq c|x - y|, \quad \forall x, y \in X, \quad i = 0, \ldots, n.$$

Hutchinson [Hut81] proved that to each IFS one can associate an unique nonempty compact invariant set, that is, a set K that verifies

$$K = \bigcup_{i=0}^{n} f_i(K).$$

Moreover, given a probability vector $(p_0, \ldots, p_n)$ with $\sum_{i=0}^{n} p_i = 1$ and $p_i \in (0,1)$, there is a unique probability measure μ supported on K, called *invariant measure*, such that

$$\mu(A) = \sum_{i=0}^{n} p_i \mu(f_i^{-1}(A)) \quad \text{for every Borel set } A. \tag{1.1}$$

It is well known, and easy to verify, that the before mentioned sets A_r are also the attractors of the IFS of contracting similitudes $\{g_{r,0}, g_{r,1}\}$ defined on $[0,1]$, where $g_{r,i} = rx + i(1-r)$. These are the simplest examples of *regular* or *dynamically defined Cantor sets*, where in general, the derivatives of the functions of the IFS are assumed be at least ϵ-Hölder continuous for some $0 < \epsilon < 1$ (see Section 2). We write $\mathscr{C}^{1+\epsilon}$-regular to emphasize that the functions of the system are of class $\mathscr{C}^{1+\epsilon}$. An important feature of regular Cantor sets is that their Hausdorff and Box dimensions coincide. In addition, their Hausdorff and packing measures on this dimensional value are finite and positive; see the book of Falconer [Fal97]. This motivates the following theorem which will be proved in Section 3.

Theorem 1. *The set C_p is a $\mathscr{C}^{1+1/p}$-regular Cantor set. Moreover, this is the highest degree of smoothness that can be attained by any other regular system that has this set as attractor.*

In view of the above result, C_p has a $\mathscr{C}^{1+1/p}$-differentiable structure. By a result of Sullivan [Sul88], this structure can be classified by its scaling function (defined in Section 4). More precisely, two regular IFS $\{f_0, f_1\}$ and $\{\tilde{f}_0, \tilde{f}_1\}$ are *equivalent* if they are smoothly conjugate, that is, if there exists a smooth homeomorphism h, termed *conjugacy*, such that

$$h \circ f_i = \tilde{f}_i \circ h, \quad i = 0, 1;$$

here by *smooth* we mean that h and its inverse are at least $\mathscr{C}^1$. Then, the result in [Sul88] says that the scaling function is a complete invariant: two $\mathscr{C}^{1+\epsilon}$-regular systems are equivalent if and only if their scaling functions coincide. Moreover, there is a conjugacy which is $\mathscr{C}^{1+\epsilon}$; for a proof of this see [PT96] and [BF97]. Let us denote by $\{f_{p,0}, f_{p,1}\}$ the IFS which has C_p as attractor given by Theorem 1. In Section 4 we apply the result of Sullivan to obtain

Theorem 2. *The systems $\big(C_p, \{f_{p,0}, f_{p,1}\}\big)$ and $\big(A_{2^{-p}}, \{2^{-p}x, 2^{-p}x + (1-2^{-p})\}\big)$ are $\mathscr{C}^{1+1/p}$-conjugate. In particular, C_p is $\mathscr{C}^{1+1/p}$-diffeomorphic to $A_{2^{-p}}$.*

Also in Section 4 we exhibit an example of a $\mathscr{C}^1$-regular Cantor set which has the same scaling function as $A_{2^{-p}}$ but it is not bi-Lipschitz conjugate to it (see Example 13).

In order to introduce the results of the last section, recall that a measure is of pure type if it is either absolutely continuous or purely singular with respect to $\mathscr{L}$, the Lebesgue measure on $\mathbf{R}$. Henceforth, absolutely continuous or singular will be meant with respect to $\mathscr{L}$. Let $\mathscr{H}_p$ denotes the measure $\mathscr{H}^{1/p}|_{C_p}$. We are interested in determine whether the convolution measure $\mathscr{H}_p * \mathscr{H}_{p'}$ is of pure type; in Section 5 we obtain partial results. Also we investigate the size of the sumset $C_p + C_{p'}$, which is relevant in this setting since it contains the support of $\mathscr{H}_p * \mathscr{H}_{p'}$.

Due to a classical result of Newhouse, the thickness of a Cantor set is a useful tool to determine whether the sum of two of these sets has nonempty interior. Through an estimate of thickness, we provide sufficient conditions on the parameters p and p' so that $C_p + C_{p'}$ has nonempty interior. We show that in order to have analogous conditions to the classical case, it is necessary to consider a local version of thickness.

After the above estimates we concentrate on the convolution of measures and the dimensional behaviour of sumsets, but from a measure theoretical point of view. For any pair of sets $E, F \subset \mathbf{R}$, with $\dim F = \overline{\dim}_B F$, it is well known that

$$\dim(E+F) \leq \dim(E \times F) \leq \dim E + \dim F;$$

see Mattila [Mat95]. Hence it is always true that

$$\dim(C_p + C_{p'}) \leq \min\bigl(\dim C_p + \dim C_{p'}, 1\bigr).$$

Hence $\mathscr{H}_p * \mathscr{H}_{p'}$ is trivially singular if $\dim C_p + \dim C_{p'} < 1$ because $\mathscr{L}(C_p + C_{p'}) = 0$. We prove that the convolution is absolutely continuous when $\dim C_p + \dim C_{p'} > 1$, with the possible exception of a small set in the parameter space. More precisely, let p' be fixed and $\bar{p}$ be such that $\dim C_{p'} + \dim C_{\bar{p}} = 1$. Also, let us denote with $\nu \in L^2$ ($\nu \notin L^2$) the fact that the measure ν has (does not have) a density in $L^2(\mathbf{R})$. Then, for any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ (which decreases to 0 with ϵ) such that

$$\dim\bigl\{p \in (1, \bar{p} - \epsilon) \colon \mathscr{H}_p * \mathscr{H}_{p'} \notin L^2\bigr\} \leq 1 - \delta. \tag{1.2}$$

In particular, $\mathscr{H}_p * \mathscr{H}_{p'} \in L^2$ for $\mathscr{L}$-a.e. p such that $\dim C_p + \dim C_{p'} > 1$.

Observe that (1.2) implies that

$$\dim\bigl\{p \in (1, \bar{p} - \epsilon) \colon \mathscr{L}(C_p + C_{p'}) = 0\bigr\} < 1 - \delta. \tag{1.3}$$

Moreover, we show that

$$\dim\bigl\{p \in (\bar{p} + \epsilon, \infty) \colon \dim(C_p + C_{p'}) < \dim C_p + \dim C_{p'}\bigr\} < 1 - \delta. \tag{1.4}$$

In particular, the formula

$$\dim(C_p + C_{p'}) = \min\bigl(\dim C_p + \dim C_{p'}, 1\bigr)$$

holds for almost every p.

We can replace $C_{p'}$ and $\mathscr{H}_{p'}$ above by any compact $K \subset \mathbf{R}$ and a suitable measure; besides, more general families of Cantor sets can be used instead of $\{C_p\}_p$; see Theorems 19 and 21.

These last results are a consequence of the Peres–Schlag projection theorem; see [PS00]. In that paper, the dimensional bounds of exceptions (1.3) and (1.4) are obtained for families of homogeneous Cantor sets, each of these sets being by definition an attractor of an IFS of similitudes, all of them with the same ratio of contraction.

1.3. Background and related works. In connection with dynamically defined Cantor sets, Bamón et al. in [BMPV97] considered central Cantor sets, which by definition satisfy that on each step the removed gaps have the same length. In that paper those central Cantor sets that are $\mathscr{C}^{k+\epsilon}$ or $\mathscr{C}^{\infty}$-regular are characterized in terms of the decay of the sequence. Moreover, a classification of these sets is provided up to local and global diffeomorphisms.

The structure and dimension of sums of Cantor sets are relevant in different areas such as diophantine approximations in number theory and homoclinic tangencies in smooth dynamics. In the latter context, Palis (see [PT93]) asked whether the sum of two regular Cantor sets has zero Lebesgue measure or contains an open interval. There are particular cases where this is not true, as it was shown by Sannami [San92], but Moreira and Yoccoz [MY01] proved that generically (in the $\mathscr{C}^{1+\epsilon}$ topology on regular Cantor sets) the conjecture is true. Nevertheless, the question for the self-similar case is still open, although for the special case of $A_r + A_r$ it is true, see Cabrelli et al. [CHM97].

Related to the size of sumsets, if C_1 and C_2 are strictly nonlinear $\mathscr{C}^2$-regular Cantor sets, the formula $\dim(C_1 + C_2) = \min\big(\dim C_1 + \dim C_2, 1\big)$ is true under some explicit conditions on the IFS; see Moreira [M98]. For the linear case, given a compact set $K \subset \mathbf{R}$, the equality

$$\dim(K + A_r) = \min\big(\dim K + \dim A_r, 1\big) \text{ for } \mathscr{L}\text{-a.e. } r \tag{1.5}$$

was established by Peres and Solomyak [PS98]. It was improved in [PS00] as we mentioned above. Recently Peres and Shmerkin [PS09] found the exceptional set for $K = A_s$: when $\dim A_r + \dim A_s \le 1$, equality holds if and only if $\log r/\log s$ is irrational. This condition also appears in the study of the topological structure of the sumset when $\dim A_s + \dim A_r > 1$; see Mendes and Oliveira [MO94] and Cabrelli et al. [CHM02].

In order to motivate our study of convolution of measures, let

$$\mu_r(A) = \frac{1}{2}\mu_r\left(g_{r,0}^{-1}(A)\right) + \frac{1}{2}\mu_r\left(g_{r,1}^{-1}(A)\right) \quad \text{for every Borel set } A.$$

In this particular case, $\mu_r = \mathscr{H}^{d_r}|_{A_r}$ ([Hut81]), where $d_r = \dim A_r$ and $\mathscr{H}^{d_r}|_{A_r}$ is the normalized restriction of the Hausdorff measure to A_r. Now let us look at the convolution measure $\mu_r * \mu_r$. Since all the similitudes have the same ratio of contraction, it is easily verified that it satisfies an identity as the one in (1.1), with IFS $\big\{rx, rx + 1 - r, rx + 2(1 - r)\big\}$ and weights $(1/4, 1/2, 1/4)$. Thus it is a measure of pure type (see [PSS00], Proposition 3.1).

The Fourier transform is a useful tool to determine whether a measure is not absolutely continuous. Recall that the Fourier transform of a finite Borel measure μ is defined by

$$\hat{\mu}(x) = \int e^{itx}\, d\mu(t)$$

(see [Mat95], Chapter 12) and that by the Riemann–Lebesgue lemma, a necessary condition for absolute continuity of a measure is that its Fourier transform vanishes at infinity. It is well known that $\hat{\mu}_r$ does not tend to 0 at infinity if and only if $1/r$ is a Pisot number different from 2 (Pisot numbers are a special class of algebraic integers); see [Sal63]. By a general property of convolutions, $\widehat{\mu_r * \mu_r} = \hat{\mu}_r \cdot \hat{\mu}_r$, whence $\mu_r * \mu_r$ is singular if r is the reciprocal of a Pisot number; however it may happen that $\mathscr{L}(A_r + A_r) > 0$. For example, this is the case when $r = 1/3$. Lau et al. ([FLN00], [HL01]) studied the multifractal structure of the m-th convolution of the measure $\mu_{1/3}$, which is singular by the above argument. Nazarov et al. [NPS09] determined that the correlation dimension of $\mu_r * \mu_s$ is $\min\bigl(d_r + d_s, 1\bigr)$ whenever $\log r/\log s$ is irrational.

Shmerkin informed us that in a joint work with Hochman [HS09] they generalize the work on sums of Cantor sets [PS09] and their methods imply that the dimension of the convolution $\mathscr{H}_p * \mathscr{H}_{p'}$ is $\min\bigl(\dim C_p + \dim C_{p'}, 1\bigr)$ whenever p/p' is irrational. This in turns implies that for these parameters formula (1) holds. However, when the sum of the dimensions is greater than 1 they do not obtain results on the absolute continuity of the convolution.

1.4. Some open questions. 1) Let ϑ_p be the invariant measure associated to the IFS $\{f_{p,0}, f_{p,1}\}$ with probabilities $(1/2, 1/2)$. This measure is equivalent to $\mathscr{H}_p$ (see Proposition 18), whence $\vartheta_p * \vartheta_{p'}$ is of pure type if and only if so is $\mathscr{H}_p * \mathscr{H}_{p'}$. Then, from results in Section 5, we know that $\vartheta_p * \vartheta_{p'}$ is of pure type for *almost everywhere* p given p', but we do not know if is always true. Moreover, in general it is unknown if the convolution of two invariant measures associated to regular Cantor sets is of pure type.

2) For which values $p > 1$ does the Fourier transform of ϑ_p vanish at infinity? If h_p is the diffeomorphism between $A_{2^{-p}}$ and C_p (which exists by Theorem 2), then the relation

$$\vartheta_p = \mu_{2^{-p}} \circ h_p^{-1}$$

holds by uniqueness of the invariant measure. Although we know this identity, the nonlinearity of the diffeomorphism h_p does not allows us to transfer the information from $\mu_{2^{-p}}$ to ϑ_p in order to estimate the decay of its Fourier transform (recall that $\hat{\mu}_r \to 0$ if and only if r is not the reciprocal of a Pisot number).

2. Basic definitions and notation

In this section we provide the basic definitions and notation that we will use later.

The symbolic space. Given $n \geq 1$, let Ω_n be the set of binary strings of length n, that is

$$\Omega_n = \{\omega_1 \dots \omega_n \colon \omega_i = 0, 1 \text{ with } 1 \leq i \leq n\}.$$

Set $\Omega_0 = \{e\}$ with e the empty string and let $\Omega^* = \bigcup_{n\geq 0} \Omega_n$. Define $\Omega = \{\omega_1\omega_2 \dots \colon \omega_i = 0, 1 \text{ with } i \in \mathbf{N}\}$, the set of binary infinite strings. The length of $\omega \in \Omega^* \cup \Omega$ is denoted by $|\omega|$. Elements in Ω have infinite length. Given $\omega \in \Omega^* \cup \Omega$ with $|\omega| \geq k$, its *k-truncation* is $\omega|_k = \omega_1 \dots \omega_k$. The infinite string with all entries 0 is denoted by $\bar{0}$; analogously, we define $\bar{1}$. Moreover, if $\omega \in \Omega^*$ and $\tau \in \Omega^* \cup \Omega$ then $\omega\tau$ denotes the string obtained by juxtaposing the elements of ω and τ. Furthermore, for $\omega \in \Omega_n$ denote by $\ell(\omega)$ the binary representation

$$\ell(\omega) = \sum_{j=1}^{n} \omega_j 2^{n-j}.$$

Note that this is a bijection from Ω_n to $\{0, 1, \dots, 2^n - 1\}$.

Given $\beta > 1$, we define a metric on Ω by

$$d_\beta(\omega,\tau) = \begin{cases} \beta^{-|\omega\wedge\tau|} & \text{if } \omega \neq \tau, \\ 0 & \text{if } \omega = \tau, \end{cases}$$

where $|\omega\wedge\tau| = \min\{k \colon \omega_k \neq \tau_k\}$. The space (Ω, d_β) is a compact, perfect and totally disconnected metric space.

Cantor set associated to a sequence. Let $a = \{a_j\}$ be a positive and summable sequence and let I_a be the closed interval $[0, \sum_j a_j]$. We define the zero Lebesgue measure Cantor set C_a associated to the sequence a as follows. In the first step, we remove from I_a an open interval L_1 of length a_1, termed *gap*, resulting in two closed intervals I_0^1 and I_1^1. The position of the gap will be uniquely determined by the sequence, as it will become clear from the construction (see Remark below). Having constructed the k-th step, we obtain the 2^k closed intervals I_ω^k, $\omega \in \Omega_k$, contained in I_a. The next step consists in removing from I_ω^k the gap $L_{2^k+\ell(\omega)}$ of length $a_{2^k+\ell(\omega)}$, obtaining the closed intervals $I_{\omega 0}^{k+1}$ and $I_{\omega 1}^{k+1}$. Then we define

$$C_a := \bigcap_{k=1}^{\infty} \bigcup_{\omega\in\Omega_k} I_\omega^k.$$

The intervals I_ω^k are the *basic intervals* of C_a. Sometimes it will be convenient to label these intervals with integers instead of finite strings, so we define $I_j^k = I_\omega^k$, where $j = \ell(\omega)$.

Remark. In the above construction there is a unique way of removing the open intervals at each step. Also notice that the lengths of the closed intervals of the same step not necessarily coincide. In fact, for $\omega \in \Omega_k$ we have by construction that

$$I_\omega^k = I_{\omega 0}^{k+1} \cup L_{2^k+\ell(\omega)} \cup I_{\omega 1}^{k+1}; \tag{2.1}$$

then, applying this identity recursively to each closed interval of the right hand side, the length of the intervals is given by

$$|I_\omega^k| = \sum_{n\geq k} \sum_{\lambda\in\Omega_{n-k}} a_{2^n+\ell(\omega\lambda)}, \tag{2.2}$$

or

$$|I_j^k| = \sum_{h\geq 0} \sum_{i=j2^h}^{(j+1)2^h-1} a_{2^{k+h}+i} \tag{2.3}$$

using the other notation.

Given $\omega \in \Omega^* \cup \Omega$, with $|\omega| \geq k$, we define

$$I_\omega^k = I_{\omega|_k}^k.$$

Observe that for $\omega \in \Omega$, the family $\{I_\omega^k\}_k$ is a nested sequence of closed intervals whose intersection is a single point. Thus we define the *projection map* $\pi_a \colon \Omega \to C_a$ by

$$\{\pi_a(\omega)\} = \bigcap_{k\geq 1} I_\omega^k. \tag{2.4}$$

Endowed with the lexicographical order $\prec$ on Ω, this map is an order preserving homeomorphism and provides a natural way to code the Cantor set. For notational convenience we will identify the point $\omega \in \Omega$ with the single element of $\bigcap_{k\geq 1} I_\omega^k \subset C$.

By the *endpoints of a Cantor set* C_a we mean the set of endpoints of all the intervals I_ω^k with $\omega \in \Omega_k$, $k \geq 1$. The next proposition says that endpoints correspond to points of the form $\omega\bar{u}$, where $\omega \in \Omega^*$ and $u = 0, 1$.

Proposition 3. *For $\omega \in \Omega_k$ we have that*

$$I_\omega^k = \big[\pi_a(\omega\bar{0}), \pi_a(\omega\bar{1})\big] \quad \text{and} \quad L_{2^k+\ell(\omega)} = \big(\pi_a(\omega 0\bar{1}), \pi_a(\omega 1\bar{0})\big).$$

Proof. The result follows from the definition of π and its order preserving property. We omit the details. $\square$

Recall that C_p is the Cantor set associated to $\{1/n^p\}_n$. In Section 5 we will work with the more general set $C_{p,q}$, that is the one associated to $\{(\log n)^q/n^p\}_n$ (the term corresponding to $n = 1$ is defined as 1). Here $p > 1$ and $q \in \mathbf{R}$. It is known that $\dim C_{p,q} = 1/p$, but $\mathscr{H}^{1/p}(C_{p,q}) = 0$ if $q < 0$ and $\mathscr{H}^{1/p}(C_{p,q}) = +\infty$ if $q > 0$; see [GMS07].

The next lemma states the bounds for the basic intervals of $C_{p,q}$ that will be used throughout the paper.

Lemma 4. *If I_j^k is a k-step interval of C_p, then*

$$\frac{2^p}{2^p-2}\left(\frac{1}{2^k+j+1}\right)^p \leq |I_j^k| \leq \frac{2^p}{2^p-2}\left(\frac{1}{2^k+j}\right)^p. \tag{2.5}$$

Moreover, if $I_j^{p,q}$ is a k-step interval of $C_{p,q}$, then

$$c_{p,q}\frac{k^q}{2^{kp}} \leq |I_j^{p,q}| \leq c'_{p,q}\frac{k^q}{2^{kp}}, \tag{2.6}$$

where $c_{p,q}$ and $c'_{p,q}$ depend continuously on p and q and are finite and positive.

Proof. Estimate (2.5) is given in [CMPS05], Lemma 3.2. The lower bound in (2.6) holds since $I_j^{p,q} \supset L_{2^k+j}$ and $|L_{2^k+j}| > |L_{2^{k+1}}|$. The remaining bound is obtained using (2.3):

$$|I_j^k| = \sum_{h\geq 0}\sum_{i=j2^h}^{(j+1)2^h-1}\frac{(\log(2^{k+h}+i))^q}{(2^{k+h}+i)^p} \leq \sum_{h\geq 0} 2^h\frac{(\log(2^h(2^k+j+1)))^q}{(2^h(2^k+j))^p} \leq c'_{p,q}\frac{k^q}{2^{kp}}. \quad \square$$

Regular Cantor sets. For simplicity let $I = [0,1]$. Consider an IFS of diffeomorphisms $\{f_0, f_1\}$ defined on I such that

$$0 < f_i'(x) < 1 \text{ for all } x \in [0,1], \quad 0 = f_0(0) < f_0(1) < f_1(0) < f_1(1) = 1,$$

and the derivatives are η-Hölder continuous, i.e.,

$$|f_i'(x) - f_i'(y)| \leq c|x-y|^\eta \quad \text{for all } x, y \in I.$$

Such an IFS is called *$\mathscr{C}^{1+\eta}$-regular.*

The first condition implies that the attractor is already a Cantor set of zero Lebesgue measure. If we only required differentiability to the system, the Hausdorff and box dimensions of the attractor coincide (see [PT93], Chapter 4), but the addition of the Hölder condition assures that, in the corresponding dimensional parameter, the Hausdorff and packing measures are positive and finite (see [Fal97], Theorem 5.3 and its proof).

Given $\omega \in \Omega_k$ we set $f_\omega = f_{\omega_1} \circ \cdots \circ f_{\omega_k}$. It is easily seen that the attractor of a regular system is given by

$$C = \bigcap_{k\geq 0} \bigcup_{\omega\in\Omega_k} f_\omega(I).$$

3. C_p is a regular Cantor set

In this section we show that C_p is a regular Cantor set. A sufficient condition for an IFS $\{f_0, f_1\}$ to have C_p as its attractor is that

$$f_\omega(I) = I^k_{\ell(\omega)} \quad \text{for all } \omega \in \Omega_k \quad \text{and} \quad k \geq 1. \tag{3.1}$$

Thus, in order to prove our theorem it is enough to find functions that satisfy the above properties. The existence of such functions is not evident, especially if we want them to be smooth. The proof of our theorem is motivated by the following necessary condition for the derivatives of the functions of an IFS at the points of its attractor.

Proposition 5. *Assume that C is the attractor of an IFS $\{f_0, f_1\}$, where f_0, f_1 have continuous positive derivatives. Given $x \in C$, let $\omega \in \Omega$ be such that $x = \pi(\omega)$. Then the derivative at x is given by the limit*

$$f_i'(x) = \lim_{n\to\infty} \frac{|f_{i\omega|_n}(I)|}{|f_{\omega|_n}(I)|}, \quad i = 0, 1. \tag{3.2}$$

Proof. By the mean value theorem we have that

$$|f_{i\omega|_n}(I)| = |f_i(f_{\omega|_n}(I))| = f_i'(\xi_n)|f_{\omega|_n}(I)|, \tag{3.3}$$

where $\xi_n \in f_{\omega|_n}(I)$. As n goes to infinite, ξ_n tends to the unique point $x \in C$ which is in the intersection $\bigcap_{n\geq 1} f_{\omega|_n}(I)$. Thus (3.2) follows from the positiveness and continuity of f'. $\square$

Therefore this proposition provides us with the starting point. The proof of Theorem 1 has essentially two parts. First we prove that for each endpoint $\omega \in \Omega$, the sequence of quotients

$$\left\{|I^{n+1}_{\ell(i\omega|_n)}|/|I^n_{\ell(\omega|_n)}|\right\}_n \tag{3.4}$$

converges and we find an expression for the limit. Thus by (3.2) these limits should be the values, at the endpoints of our Cantor set, of the derivatives of the functions of an IFS that satisfies (3.1). Then, with these values, in the second part we are able to extend the derivatives to the whole interval I so that (3.1) holds and thus the system has C_p as its attractor.

Notice that if the derivatives are positive then f_i is order preserving, so $f_0(0) = 0$ and $f_1(|I|) = |I|$. From this, once we have constructed the derivatives F_0 and F_1, we define

$$f_0(x) = \int_0^x F_0 \quad \text{and} \quad f_1(x) = \int_0^x F_1 + c, \tag{3.5}$$

with $c = |I^1_0| + 1$, since 1 is the length of the first gap.

3.1. Definition of the derivatives on C_p and properties. Recall that endpoints of C_p correspond to strings of the form $\omega\bar{u}$, with $u = 0$ or 1 and $\omega \in \Omega_k$, $k \geq 1$. We have the following result.

Proposition 6. *At each endpoint $\omega\bar{u}$ of C_p the limit of $\left\{|I^{n+1}_{\ell(i(\omega\bar{u})|_n)}|/|I^n_{\ell((\omega\bar{u})|_n)}|\right\}_n$ exists. It is given by the formula*

$$G_i(\omega\bar{u}) = \left(\frac{2^k + \ell(\omega) + u}{2^{k+1} + i2^k + \ell(\omega) + u}\right)^p, \quad \omega \in \Omega_k,\ u = 0, 1. \tag{3.6}$$

Proof. Let $\omega \in \Omega_k$ with $k \geq 1$. It follows from (2.5) that

$$\left(\frac{2^n + \ell((\omega\bar{u})|_n)}{2^{n+1} + \ell(i(\omega\bar{u})|_n) + 1}\right)^p \leq \frac{\left|I^{n+1}_{\ell(i(\omega\bar{u})|_n)}\right|}{\left|I^n_{\ell((\omega\bar{u})|_n)}\right|} \leq \left(\frac{2^n + \ell((\omega\bar{u})|_n) + 1}{2^{n+1} + \ell(i(\omega\bar{u})|_n)}\right)^p.$$

From equalities

$$\ell((\omega\bar{u})|_n) = \sum_{j=1}^{k} \omega_j 2^{n-j} + u(2^{n-k} - 1) = 2^{n-k}(\ell(\omega) + u(1 - 1/2^{n-k}))$$

and

$$\ell(i(\omega\bar{u})|_n) = i2^n + \ell((\omega\bar{u})|_n),$$

we get

$$\frac{\left|I^{n+1}_{\ell(i(\omega\bar{u})|_n)}\right|}{\left|I^n_{\ell((\omega\bar{u})|_n)}\right|} \leq \left(\frac{2^k + \ell(\omega) + u(1 - 1/2^{n-k}) + 1/2^{n-k}}{2^{k+1} + i2^k + \ell(\omega) + u(1 - 1/2^{n-k})}\right)^p,$$

with a similar lower bound. Since $\ell(\omega)$ is independent of n, the limit of the sequence (3.4) exists and is given by (3.6). □

Remark. In fact, the limit in the above proposition exists not only at the endpoints but in all of C_p. For our purposes however it is enough to know the values at the endpoints.

Let us denote with E_p the set of endpoints of C_p. The functions of the previous proposition have the following properties.

Lemma 7. *Let G_i, $i = 0, 1$ be defined on E_p by formula (3.6). Then*

(a) *each function G_i takes the same value at the endpoints of a single gap; that is, at the endpoints of $L_{2^k+\ell(\omega)}$ we have that $G_i(\omega 0\bar{1}) = G_i(\omega 1\bar{0})$, $\omega \in \Omega_k$, $k \geq 0$, $i = 0, 1$,*

(b) *both functions G_0 and G_1 are non-decreasing,*

(c) *for every $\omega \in \Omega_k$, $u = 0, 1$,*

$$\left(\frac{1}{2}\right)^p \leq G_0(\omega\bar{u}) \leq \left(\frac{2}{3}\right)^p \quad \text{and} \quad \left(\frac{1}{3}\right)^p \leq G_1(\omega\bar{u}) \leq \left(\frac{1}{2}\right)^p.$$

Proof. (a) Since $\ell(\omega 1) = \ell(\omega 0) + 1$, the statement is a consequence of the definition of G_i.

(b) By the previous item and the continuity of G_i, it is enough to show that this function is increasing when restricted to the left endpoints. Let $\omega\bar{0}$ and $\tau\bar{0}$ be left endpoints with $\pi(\omega\bar{0}) < \pi(\tau\bar{0})$. Then $\omega\bar{0} \prec \tau\bar{0}$ since π is order preserving. We may assume that $\omega, \tau \in \Omega_k$ because left endpoints of one level are left endpoints of all higher levels. By (3.6) we must see that

$$\left(\frac{2^{k+1}+\ell(\omega 1)}{2^{k+2}+i2^{k+1}+\ell(\omega 1)}\right)^p < \left(\frac{2^{k+1}+\ell(\tau 1)}{2^{k+2}+i2^{k+1}+\ell(\tau 1)}\right)^p,$$

but this is equivalent to

$$\ell(\omega 1) < \ell(\upsilon 1),$$

which holds because ω is lexicographically smaller than τ.

(c) is consequence of (b) and the values of the functions at the endpoints of I. $\square$

Note that item (c) emphasizes that the derivatives are strictly less than 1 in C_p. Below we establish the Hölder regularity of G_i on E_p.

Proposition 8. *Let G_i be as above. Then $G_i \in \mathscr{C}^{1/p}(E_p)$, but $G_i \notin \mathscr{C}^{\eta}(E_p)$ for any $\eta > 1/p$.*

Proof. Firstly assume that x and y are endpoints of the same interval of the m-step. By Proposition 3, there exists $\omega \in \Omega_m$ such that $x = \omega\bar{0}$ and $y = \omega\bar{1}$. Applying formula (3.6), we have $G_i(\omega\bar{0}) = (a/b)^p$ and $G_i(\omega\bar{1}) = ((a+1)/(b+1))^p$, with $a = 2^k + \ell(\omega)$ and $b = 2^{k+1} + i2^k + \ell(\omega)$. Then

$$G_i(\omega\bar{1}) - G_i(\omega\bar{0}) = \frac{((a+1)b)^p - ((b+1)a)^p}{(b(b+1))^p} = h(b) - h(a),$$

where $h(t) = \left(\frac{ab+t}{b(b+1)}\right)^p$. By the mean value theorem there exists $a < \xi < b$ such that

$$G_i(\omega\bar{1}) - G_i(\omega\bar{0}) = \frac{p(ab+\xi)^{p-1}(b-a)}{(b(b+1))^p} = p\left(\frac{a+\xi/b}{b+1}\right)^p \frac{(b-a)}{b}\frac{1}{a+\xi/b}.$$

Since

$$1/4 \le \frac{a+\xi/b}{b+1}, \frac{(b-a)}{b} \le 1,$$

by inequalities (2.5) there are positive and finite quantities c_1 and c_2 depending only on p such that

$$c_1|I^k_{\ell(\omega)}|^{1/p} \le G_i(\omega\bar{1}) - G_i(\omega\bar{0}) \le c_2|I^k_{\ell(\omega)}|^{1/p}. \tag{3.7}$$

The last inequality says that G_i is $1/p$-Hölder continuous at the endpoints of each basic interval with constant independent of the interval. On the other hand, the first inequality shows that the exponent $1/p$ cannot be improved. In fact, if there is an $\epsilon > 0$ such that $G_i(\omega\bar{1}) - G_i(\omega\bar{0}) \le c|I^k_{\ell(\omega)}|^{1/p+\epsilon}$, then $0 < c_1c^{-1} \le |I^k_{\ell(\omega)}|^{\epsilon}$ for all k, which is impossible because $|I^k_{\ell(\omega)}| \to 0$ as k increases. Therefore, the second claim is proved.

To complete the proof of the first claim we need the following result of [CMPS05] (Lemma 3.5):

Let J be an open interval and let $k \in \mathbf{N}$. Then $\sum_{j\,:\, I^k_j \subset J} |I^k_j|^{1/p} \le 4|J|^{1/p}$.

Let x and y be arbitrary endpoints and $\epsilon > 0$. We define $J_\epsilon = (x-\epsilon, y+\epsilon)$. As a consequence of the construction note that x and y are endpoints of the k-step for some k, so let $x = x_0 < \ldots < x_N = y$ be all the endpoints of the k-step between x

and y. By Lemma 7 (a) we have that $G_i(x_{n+1}) - G_i(x_n) = 0$ if (x_n, x_{n+1}) is a gap. Thus, using inequality (3.7) and the above lemma we obtain

$$\begin{aligned}|G_i(x) - G_i(y)| &= \left|\sum_{k=0}^{N-1} G_i(x_k) - G_i(x_{k+1})\right| \leq \sum_{\omega\colon I^k_{\ell(\omega)}\subset J_\epsilon} |G_i(\omega\bar{1}) - G_i(\omega\bar{0})| \\ &\leq c_2 \sum_{j\colon I_j^k\subset J_\epsilon} |I_j^k|^{1/p} \leq 4c_2|J_\epsilon|^{1/p},\end{aligned}$$

and the result follows letting $\epsilon \to 0$. □

Remark. Once we have constructed an IFS with continuous derivatives that satisfies (3.1), it follows from the last proposition and denseness of E_p that the derivatives are $1/p$-Hölder continuous on all C_p.

The following lemma will be useful to prove the Hölder continuity of the extension.

Lemma 9. *Let $f\colon (a,b) \to \mathbf{R}$ and let $a < c < b$ be such that f restricted to the intervals $(a,c]$ and $[c,b)$ is α-Hölder continuous with constants C_1 and C_2 respectively. Then f is α-Hölder continuous in (a,b) with constant $C = 2\max\{C_1, C_2\}$.*

Proof. Let $x \in (a,c)$ and $y \in (c,b)$. Then

$$\begin{aligned}|f(y) - f(x)| &\leq C_2(y-c)^\alpha + C_1(c-x)^\alpha \leq 2\max\{C_2(y-c)^\alpha, C_1(c-x)^\alpha\} \\ &\leq C\max\{(y-c+c-x)^\alpha, (c-x+y-c)^\alpha\} = C(y-x)^\alpha,\end{aligned}$$

and the lemma is proved. □

3.2. Construction of the derivatives. Here we define the derivatives F_i of the desired maps f_i extending the functions G_i to the whole interval I in such a way that $1/p$-Hölder continuity is preserved and that equation (3.1) holds. Firstly we give an equivalent condition to this equation in terms of the lengths of gaps.

Lemma 10. *Condition (3.1) is equivalent to $f_0(0) = 0$, $f_1(|I|) = |I|$ and*

$$|L_{2^{n+1}+\ell(i\omega)}| = \int_{L_{2^n+\ell(\omega)}} F_i \tag{3.8}$$

for any $\omega \in \Omega_n$, $n \geq 0$.

Proof. Suppose that (3.8) holds. For $\omega \in \Omega_n$ let $\tilde{\omega} = \omega_2 \ldots \omega_n$. Then by (2.2) we get

$$\begin{aligned}|I^n_{\ell(\omega)}| &= \sum_{k\geq n}\sum_{\lambda\in\Omega_{k-n}} |L_{2^k+\ell(\omega\lambda)}| = \sum_{k\geq n}\sum_{\lambda\in\Omega_{k-n}} \int_{L_{2^{k-1}+\ell(\tilde{\omega}\lambda)}} F_{\omega_1} \\ &= \int_{I^{n-1}_{\ell(\tilde{\omega})}} F_{\omega_1} = |f_{\omega_1}(I^{n-1}_{\ell(\tilde{\omega})})|.\end{aligned}\tag{3.9}$$

For $n = 1$ this implies that $|f_i(I)| = |I_i^1|$, and since both intervals have a common endpoint, then they are the same. Inductively, if for $n \geq 1$ equality $f_\omega(I) = I^n_{\ell(\omega)}$ holds for all $\omega \in \Omega_n$, then

$$|f_{\omega i}(I)| = |f_{\omega_1}(I^n_{\ell(\tilde{\omega}i)})| = |I^{n+1}_{\ell(\omega i)}|,$$

where we used (3.9) in the last equality. Hence each interval in the dynamical $n+1$-step has the same length as its corresponding interval associated to the sequence. Moreover, we know that $I_i^1 = f_i(I)$, whence

$$I^n_{\ell(\omega)} = f_\omega(f_0(I) \cup L_1 \cup f_1(I)) = f_{\omega 0}(I) \cup f_\omega(L_1) \cup f_{\omega 1}(I).$$

Recall by definition that

$$I^n_{\ell(\omega)} = I^{n+1}_{\ell(\omega 0)} \cup L_{2^n+\ell(\omega)} \cup I^{n+1}_{\ell(\omega 1)}, \tag{3.10}$$

then $f_{\omega i}(I)$ has a common endpoint with $I^{n+1}_{\ell(\omega i)}$ since f_ω is increasing. Therefore $f_\omega(I) = I^n_{\ell(\omega)}$ for all $\omega \in \Omega_n$, $n \geq 1$.

On the other hand, if (3.1) holds then $f_0(0) = 0$ and $f_1(|I|) = |I|$; also by hypothesis

$$I^n_{\ell(\omega)} = f_{\omega_1}(I^{n-1}_{\ell(\tilde{\omega})}(I)) = I^{n+1}_{\omega 0} \cup f_{\omega_1}(L_{2^{n-1}+\ell(\tilde{\omega})}) \cup I^{n+1}_{\omega 1},$$

hence $f_{\omega_1}\left(L_{2^{n-1}+\ell(\tilde{\omega})}\right) = L_{2^n+\ell(\omega)}$ by (3.10), and equality (3.8) follows. □

Obviously one can define on each gap a smooth function that satisfies the end-point condition (3.6) and also (3.8), but we need to do this with a *uniform* bound of the Hölder constants on all gaps. Below we show that this can be realized if, for any gap in a sufficiently large step, the graph of F_i on this gap coincides with the equal sides of an isosceles triangle as it is shown in Figure 1. This construction will be possible whenever the triangle is above the x-axis, since we want the derivatives to be positive.

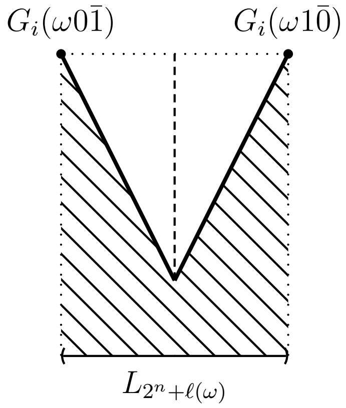

Figure 1. The shared area must be $|L_{2^{n+1}+\ell(i\omega)}|$ by Lemma 10.

Remark. The values of G_i at the endpoints of each gap coincide (Lemma 7(a)), but if we define F_i on $L_{2^n+\ell(\omega)}$ as the constant value $G_i(\omega 1\bar{0})$, then (3.8) does not hold because this function has too much area over this gap.

Let us denote with $h^i_{2^n+\ell(\omega)}$ the height of the triangle over the gap $L_{2^n+\ell(\omega)}$.

Lemma 11. *There is an integer n_p such that on each gap $L_{2^n+\ell(\omega)}$, with $\omega \in \Omega_n$ and $n \geq n_p$, it is possible to define a positive function g_ω through the isosceles triangle as in Figure 1 so that it satisfies (3.8). Moreover, for these gaps we have that $h^i_{2^n+\ell(\omega)} \leq \frac{p}{2^n}$. Furthermore, the $1/p$-Hölder constants of these functions are uniformly bounded.*

Proof. Let us define $j = \ell(\omega)$, so that $\ell(\omega 1) = 2j + 1$ and $\ell(i\omega) = i2^n + j$. Let R be the area of the rectangle with base $L_{2^n+\ell(\omega)}$ and height $G_i(\omega 1\bar{0})$ (the dotted rectangle in Figure 1). The area under the triangle decreases continuously as the vertex approaches the x-axis and is equal to $\frac{1}{2}R$ when they intersect. So, by condition (3.8), it is necessary to verify that $\frac{1}{2}R < |L_{2^{n+1}+\ell(i\omega)}|$ for all n big enough; that is

$$\frac{1}{2}\left(\frac{1}{2^n+j}\right)^p\left(\frac{2^{n+1}+2j+1}{2^{n+2}+i2^{n+1}+2j+1}\right)^p < \left(\frac{1}{2^{n+1}+i2^n+j}\right)^p.$$

Writing $a = 2^n + j$, the last inequality is equivalent to

$$\left(\left(\frac{2a+1}{2a}\right)\left(\frac{a+2^n(1+i)}{a+2^n(1+i)+1/2}\right)\right)^p < 2.$$

Each factor in the product tends to 1 as n increases, thus the inequality holds for every $n \geq n_p$, where n_p is an integer depending on p.

For $n \geq n_p$ we know the area of the triangle so we can compute its height:

$$\begin{aligned} h^i_{2^n+j} &= 2\frac{|L_{2^n+j}| \cdot G_i(\omega 1\bar{0}) - |L_{2^{n+1}+i2^n+j}|}{|L_{2^n+j}|} \\ &= 2\left[\left(\frac{2^{n+1}+2j+1}{2^{n+2}+i2^{n+1}+2j+1}\right)^p - \left(\frac{2^n+j}{2^{n+1}+i2^n+j}\right)^p\right]. \end{aligned}$$

Applying the mean value theorem $(0 < \xi < 1/2)$ we obtain

$$\begin{aligned} h^i_{2^n+j} &= 2\left[\left(\frac{a+1/2}{a+2^n(1+i)+1/2}\right)^p - \left(\frac{a}{a+2^n(1+i)}\right)^p\right] \\ &= p\left(\frac{a+\xi}{a+\xi+2^n(1+i)}\right)^{p-1}\frac{2^n(1+i)}{(a+2^n(1+i)+\xi)^2} < \frac{p}{2^n}. \end{aligned}$$

For the last statement, let $\omega \in \Omega_n$ and $j = \ell(\omega)$. Let s be the midpoint of L_{2^k+j} and take x and y in this gap. The absolute value of the slope of the side of the triangle is

$$m^i_{2^k+j} = 2h^i_{2^k+j}/|L_{2^k+j}|.$$

First assume that $s \leq x, y$. Then

$$|g_\omega(x) - g_\omega(y)| = m^i_{2^k+j}|x-y| \leq m^i_{2^k+j}|L_{2^k+j}|^{1-1/p}|x-y|^{1/p} \leq 4p|x-y|^{1/p}.$$

Hence the Hölder constant is independent of ω. The case $x, y \leq s$ is symmetric, and for $x < s < y$ the inequality follows using Lemma 9 given earlier. □

Now we proceed to define the derivatives F_i, that will be the limit of a sequence of functions $\{F_i^n\}$. Each F_i^n interpolates suitably the values of G_i at the endpoints of the basic intervals of the n-step.

To begin with, on each gap L_k, with $1 \leq k \leq 2^{n_p} - 1$ and n_p as in Lemma 11, we define $F_i^{n_p}$ joining the values of G_i at the endpoints of the gap so that it be $\mathscr{C}^1$, positive and its area under the gap be given by (3.8). On the remaining intervals, that is, on the closed intervals of the n_p-step, we interpolate linearly so that $F_i^{n_p}$ is a continuous function. For $n > n_p$ we define F_i^n inductively: on the gap L_k, $1 \leq k < 2^{n-1}$, F_i^n coincides with F_i^{n-1}; on the remaining gaps of the n-step, that is, on $L_{k'}$, with $2^{n-1} \leq k' < 2^n$, we define the graph of F_i^n as the sides of the isosceles triangle mentioned above; finally we complete the definition with linear interpolation.

The sequence $\{F_i^k\}_k$ has the following property.

Lemma 12. $\{F_i^k\}_k$ *is a uniform Cauchy sequence.*

Proof. It is enough to prove that $\|F_i^{k+1} - F_i^k\|_\infty = O\left(\frac{1}{2^k}\right)$ for every $k \geq n_p$. For this, notice that F_i^k and F_i^{k+1} coincide on the complementary gaps of the k-step, so we need to estimate their difference for points in the closed intervals of that step. Let $x \in I^k_{\ell(\omega)} = [\omega\bar{0}, \omega\bar{1}]$, with $\omega \in \Omega_k$. The functions are increasing in C_p so (see Figure 2)

$$G_i(\omega\bar{0}) \leq F_i^k(x) \leq G_i(\omega\bar{1})$$

and

$$G_i(\omega\bar{0}) - h^i_{2^k+\ell(\omega)} \leq F_i^{k+1}(x) \leq G_i(\omega\bar{1}).$$

Then

$$|F_i^{k+1}(x) - F_i^k(x)| \leq G_i(\omega\bar{1}) - G_i(\omega\bar{0}) + h^i_{2^k+\ell(\omega)}. \tag{3.11}$$

Therefore the result follows as a consequence of the estimate in Lemma 11, inequality (3.7) in the proof of Proposition 8 and since $|I^k_{\ell(\omega)}| \leq C2^{-kp}$. $\square$

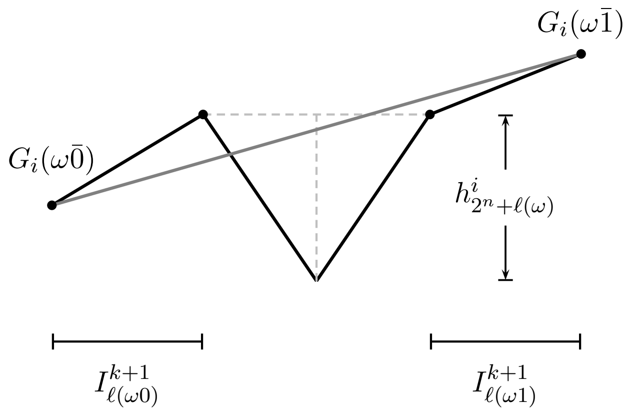

Figure 2. F_i^k in grey and F_i^{k+1} in black.

The previous lemma allows us to define F_i as the (uniform) limit of $\{F_i^k\}_k$, which results a continuous function. Integrating we obtain the system $\{f_{p,0}, f_{p,1}\}$ that has C_p as attractor.

Remark. Because of the freedom to extend the derivatives on each gap it is obvious that there is no uniqueness in the construction of the system.

End of proof of Theorem 1. It remains to show that F_i is $1/p$-Hölder continuous on I. Continuity and Proposition 8 imply this on C_p (see the remark after that proposition). Also, by definition and Lemma 11, F_i is $1/p$-Hölder continuous on each gap with constant independent of the gap. Let C be the maximum between this constant and the one given by Proposition 8. Take x and y in I with $x < y$. If these points are in different gaps, let e_x and e_y denote the right and left endpoints of the respective gaps. Then

$$\begin{aligned}|F_i(x) - F_i(y)| &\leq |F_i(x) - F_i(e_x)| + |F_i(e_x) - F_i(e_y)| + |F_i(e_y) - F_i(y)| \\ &\leq C\left(|x - e_x|^{1/p} + |e_x - e_y|^{1/p} + |e_y - y|^{1/p}\right) \leq 3C|x - y|^{1/p},\end{aligned}$$

which is what we need. The other possibilities for x and y in I follow in the same way. $\square$

4. Conjugations

In this section we prove Theorem 2 stated in the introduction, which shows how the sets C_p and $A_{2^{-p}}$ are related. Since the attractors of conjugate (smooth) systems satisfy $\widetilde{C} = h(C)$, then they are diffeomorphic. In particular they have the same Hausdorff, packing and box dimensions, since these quantities are invariant under bi-Lipschitz maps. Moreover, at their critical dimension, Hausdorff and packing measures are positive and finite. Nevertheless, these facts are not sufficient to ensure that the sets are smoothly conjugated.

Next we define the scaling function of a regular Cantor set, that is a complete invariant for this class of sets and is due to Sullivan [Sul88].

Let Δ be the unit simplex in $\mathbf{R}^3$, i.e.,

$$\Delta = \big\{(a,b,c)\colon a+b+c=1,\ a,b,c\geq 0\big\}.$$

Given $\omega\in\Omega_k$ denote with $\omega^\star$ the reverse string $\omega_k\dots\omega_1$. For a $\mathscr{C}^{1+\epsilon}$-regular Cantor C and for each $\omega\in\Omega$, we define a function $R_n\colon\Omega\to\triangle$ by

$$R_n(\omega) = \big(|I_{(\omega|_n)^\star 0}|, |L_{(\omega|_n)^\star}|, |I_{(\omega|_n)^\star 1}|\big)\,/|I_{(\omega|_n)^\star}|.$$

These functions converge uniformly on Ω with an order of convergence $O(\beta^{n\epsilon})$, where on Ω we consider the metric d_β given in Section 2.

Definition. The scaling function $R\colon\Omega\to\operatorname{int}(\Delta)$ is defined by

$$R(\omega) := \lim_{n\to\infty} R_n(\omega).$$

With the metric d_β, the scaling function is Hölder continuous with exponent ϵ.

Theorem. (Sullivan) *Two $\mathscr{C}^{k+\epsilon}$-regular Cantor sets are $\mathscr{C}^{k+\epsilon}$-conjugated if and only if they have the same scaling function.*

We are ready to prove Theorem 2.

Proof of Theorem 2. By Sullivan's Theorem we must verify that both scaling functions coincide. Since $A_{2^{-p}}$ has contraction ratio 2^{-p}, it follows that its scaling function is

$$R(\alpha) = \left(\frac{1}{2^p}, \frac{2^p-2}{2^p}, \frac{1}{2^p}\right).$$

Let us see that this is also the scaling function of C_p. Recall that for $\omega\in\Omega$,

$$\frac{2^p}{2^p-2}\left(\frac{1}{2^n+\ell((\omega|_n)^\star)+1}\right)^p \leq |I^n_{\ell((\omega|_n)^\star)}| \leq \frac{2^p}{2^p-2}\left(\frac{1}{2^n+\ell((\omega|_n)^\star)}\right)^p.$$

Then, by the identity $\ell((\omega|_n)^\star i) = 2\ell((\omega|_n)^\star)+i$ for $i=0,1$, we obtain

$$\frac{|I_{(\omega|_n)^\star i}|}{|I_{(\omega|_n)^\star}|} \leq \left(\frac{2^n+\ell((\omega|_n)^\star)+1}{2^{n+1}+\ell((\omega|_n)^\star i)}\right)^p \leq \frac{1}{2^p}\left(1+\frac{1}{2^n+\ell((\omega|_n)^\star)}\right)^p \longrightarrow \frac{1}{2^p},$$

with a similar lower bound, thus $|I_{(\omega|_n)^\star i}|/|I_{(\omega|_n)^\star}| \to 1/2^p$. Since the sum of the coordinates of the scaling function is 1, we obtain the coincidence of these functions. □

The scaling function also exists if weaker conditions on the derivatives of the functions are required. For example, if they satisfy the Dini condition (see, for example, [FJ99]), or more generally, a bounded distortion property. We finish this section illustrating that, even when the functions of the IFS are only $\mathscr{C}^1$, the scaling function may exist, and moreover, it can be a constant function.

Example 13. The Cantor set $C_{p,1}$ associated to the sequence $\{(\log n)/n^p\}$ satisfies:

1) It is the attractor of an IFS $\{f_0, f_1\}$ with $f_i \in \mathscr{C}^1$.
2) The derivatives are not ϵ-Hölder continuous for any $\epsilon > 0$ (actually, they do not satisfy the bounded distortion property).
3) Its scaling function is constant, with value $\left(\frac{1}{2^p}, \frac{2^p-2}{2^p}, \frac{1}{2^p}\right)$; in particular, this function is ϵ-Hölder continuous, for any $\epsilon > 0$.

Proof. 1) First, for all $0 \le j < 2^k$, $k \ge 1$ we have

$$(4.1)\quad (\tilde{c}_p + c_p \log(2^k + j))\frac{1}{(2^k + j + 1)^p} \le |I_j^k| \le (\tilde{c}_p + c_p \log(2^k + j + 1))\frac{1}{(2^k + j)^p},$$

where $c_p = \sum_{h\ge 0} \frac{1}{2^{(p-1)h}}$ and $\tilde{c}_p = \sum_{h\ge 0} \frac{\log 2^h}{2^{(p-1)h}}$; this is obtained in the same way as the bounds of Lemma 4.

Now we show that $C_{p,1}$ is the attractor of a system $\{f_0, f_1\}$ with continuous derivatives such that $f_\omega(I) = I^k_{\ell(\omega)}$ for all $\omega \in \Omega_k$, $k \ge 1$. Given $\omega \in \Omega_k$, by estimate (4.1) we have that

$$\lim_{n\to\infty} \frac{\left|I^{n+1}_{\ell(i(\omega\bar{u})|_n)}\right|}{\left|I^{n}_{\ell((\omega\bar{u})|_n)}\right|} = \left(\frac{2^k + \ell(\omega) + u}{2^{k+1} + i2^k + \ell(\omega) + u}\right)^p \quad \text{for } u = 0, 1.$$

By Proposition 5, this limit gives the values of the derivatives at the endpoints. Notice that these are the same values than the one obtained in the C_p case; in particular, they coincide at the endpoints of any gap (Lemma 7 (b)).

As before we must subtract some area over each gap, which can be done with triangles because the same bounds as in Lemma 11 hold.

2) It was shown in [GMS07] that $\dim_H C_{p,1} = 1/p$ and moreover, that

$$\mathscr{H}^{1/p}(C_{p,1}) = +\infty,$$

whence this set cannot be the attractor of a system whose functions have ϵ-Hölder continuous derivatives, for any $\epsilon > 0$ (nor can the derivatives satisfy the bounded distortion property).

3) Given $\omega \in \Omega$ we have

$$\frac{|I_{(\omega|_n)^\star i}|}{|I_{(\omega|_n)^\star}|} \le \frac{\tilde{c}_p + c_p \log(2^{n+1} + \ell((\omega|_n)^\star i) + 1)}{\tilde{c}_p + c_p \log(2^n + \ell((\omega|_n)^\star))} \cdot \left(\frac{2^n + \ell((\omega|_n)^\star) + 1}{2^{n+1} + \ell((\omega|_n)^\star i)}\right)^p \longrightarrow \frac{1}{2^p},$$

since one can show that the second factor in the product goes to 1. The lower bound is similar. Hence the scaling function is $R(\omega) = \left(\frac{1}{2^p}, \frac{2^p-2}{2^p}, \frac{1}{2^p}\right)$ for all $\omega \in \Omega$. □

Corollary 14. *The scaling functions of* $(C_p, \{f_{p,1}, f_{p,0}\})$ *and* $(C_{p,1}, \{f_0, f_1\})$ *coincide. Nevertheless these Cantor sets are not even bi-Lipschitz conjugate. Therefore in Sullivan's Theorem the regularity hypothesis cannot be weakened to* $\mathscr{C}^1$.

Proof. Since $0 < \mathscr{H}^{1/p}(C_p) < +\infty$ and $\mathscr{H}^{1/p}(C_{p,1}) = +\infty$, these Cantor sets cannot be bi-Lipschitz conjugate. □

5. Sums and convolutions

In this section we provide results on sums of two Cantor sets in the family $\{C_p\}$ and on the convolution of measures supported on these sets. We begin giving an estimate of the thickness of C_p, which is used to obtain conditions so that the sumset has nonempty interior. Subsequently, for a given compact $K \subset \mathbf{R}$, we adapt a result of Peres and Schlag [PS00] to study the size of the set of parameters where the convolution measure $\mathscr{H}^{1/p}|_{C_p} * \mathscr{H}^{1/p'}|_{C_{p'}}$ is not absolutely continuous (Corollary 20) and also where the formula $\dim(K + C_p) = \min(\dim K + \dim C_p, 1)$ does not hold.

Let L be a bounded gap of a Cantor set C. A *bridge* B of L is a maximal interval that has a common endpoint with L and does not intersect any gap whose length is at least that of L. We say that (B, L) is a *bridge/gap* pair of C. The *thickness of* C is defined by

$$\tau(C) = \inf \left\{ \frac{|B|}{|L|} \colon (B, L) \text{ is a bridge/gap pair} \right\}.$$

An important consequence of Newhouse's gap lemma is that the sum $C_1 + C_2$ of two Cantor sets is a finite union of intervals if $\tau(C_1) \cdot \tau(C_2) \geq 1$ (see [PT93]). Moreover, if none of the translates of either of the Cantor sets are contained in a (bounded) gap of the other, then $C_1 + C_2$ is an interval.

In the classical case we have $\tau(A_r) = 2r/(1 - 2r)$. If 2^{-p} takes the place of r one would expect that $C_p + C_{p'}$ be an interval when $\frac{1}{2^p - 2} \frac{1}{2^{p'} - 2} \geq 1$. But the thickness of C_p is bigger than expected because of the nonlinearity of the set. Nevertheless, a slightly weaker result can be attained if we consider a local version of thickness instead.

Given $x \in C$ let $\omega \in \Omega$ be such that $\pi(\omega) = x$. The Cantor sets $C_x^k := C \cap I^k_{\ell(\omega^k)}$ decrease to $\{x\}$ as $k \to \infty$. Then the *local thickness of* C *at* x is

$$\tau_{\text{loc}}(C, x) := \varlimsup_{k \to \infty} \tau(C_x^k).$$

It can be shown, following the proof of Newhouse's lemma, that if $x_1 \in C_1$ and $x_2 \in C_2$ are such that $\tau_{\text{loc}}(C_1, x_1) \cdot \tau_{\text{loc}}(C_2, x_2) > 1$, then $C_1 + C_2$ contains a nonempty open interval. Moreover, if

$$\inf_{x \in C_1, y \in C_2} \{\tau_{\text{loc}}(C_1, x) \cdot \tau_{\text{loc}}(C_2, y)\} > 1$$

then $C_1 + C_2$ is a finite union of intervals.

In general, regular Cantor sets have constant local thickness (see [PT93]). In our case it is easy to compute this value.

Proposition 15. *We have*

$$\frac{1}{2^p - 2} \frac{1}{2^p} \leq \tau(C_p) \leq \frac{1}{2^p - 2} \left(\frac{2}{3} \right)^p. \tag{5.1}$$

Moreover,

$$\tau_{\text{loc}}(C_p) = \frac{1}{2^p - 2}. \tag{5.2}$$

Proof. Since lengths of bounded gaps of C_p are lexicographically decreasing, the bridge for $L_{2^k + j}$ is the closed interval of the $k + 1$-step which is at its right, that is

I_{2j+1}^{k+1}. Therefore inequalities (2.5) imply

$$\frac{1}{2^p-2}\left(\frac{2^k+j}{2^k+j+1}\right)^p \le \frac{|I_{2j+1}^{k+1}|}{|L_{2^k+j}|} \le \frac{1}{2^p-2}\left(\frac{2^k+j}{2^k+j+1/2}\right)^p. \tag{5.3}$$

Thus the bounds for the quotient bridge/gap increase to $1/(2^p-2)$ as k and j increase. From this, for $k=j=0$, the first inequality in (5.3) gives a lower bound for all quotients bridge/gap and therefore the lower bound in (5.1). Moreover, $k=j=0$ gives the smallest of all upper bounds in (5.3), that is the second inequality in (5.1).

Note that for the local thickness every bridge/gap pair of $C_{p,x}^k$ is one of C_p; hence by (5.3) we have

$$\frac{1}{2^p-2}\left(\frac{2^k}{2^k+1}\right)^p \le \tau(C_{p,x}^k) \le \frac{1}{2^p-2}\left(\frac{2^{k+1}-1}{2^{k+1}-1/2}\right)^p,$$

and (5.2) follows letting $k\to\infty$. □

As a consequence of the above we have the following result.

Corollary 16. *For* $\frac{1}{2^p(2^p-2)}\frac{1}{2^q(2^q-2)} \ge 1$ *the set* C_p+C_q *is an interval. Moreover, if* $\frac{1}{2^p-2}\cdot\frac{1}{2^q-2} > 1$ *then* C_p+C_q *is a finite union of intervals.*

Finally we concentrate on the measure theoretic results of the dimension of C_p+C_q and the corresponding problem of convolution measures given in the introduction.

Let μ_0 denote the uniform product measure on Ω. A probability measure ϑ_p on C_p is defined by

$$\vartheta_p = \mu_0\circ\Gamma_p^{-1}.$$

Remark 17. It can be easily verified that ϑ_p is the invariant measure of the regular i.f.s. that generates C_p with weights $(1/2,1/2)$.

Note that in view of the next proposition, the convolution measure $\mathscr{H}_p * \mathscr{H}_q$ is equivalent to $\vartheta_p*\vartheta_q$, hence in this section we will work with the later.

Proposition 18. *ϑ_p is equivalent to $\mathscr{H}_p$.*

Proof. Recall that $h_p(A_{2^{-p}}) = C_p$. Given $B\subset[0,1]$ we have that

$$\vartheta_p(B) = \mu_{2^{-p}}(h_p^{-1}(B)) = \mathscr{H}^{1/p}(h_p^{-1}(B)\cap A_{2^{-p}}) = \mathscr{H}^{1/p}(h_p^{-1}(B\cap C_p)).$$

Since h_p is a bi-Lipschitz function, there is a constant $c>0$ such that

$$c^{-1}\mathscr{H}^{1/p}(B\cap C_p) \le \mathscr{H}^{1/p}(h_p^{-1}(B\cap C_p)) \le c\mathscr{H}^{1/p}(B\cap C_p),$$

whence the measures are equivalent. □

Indeed we will work with a more general parametric family. Let $\{C_{p,q}\}_{p>1,q\in\mathbf{R}}$ be the family of Cantor sets associated to the sequence $\{\log^q n/n^p\}_n$ (the term $n=1$ is defined to be 1). Notice that $\dim C_{p,q} = 1/p$; see [GMS07]. We regard q as a $\mathscr{C}^\infty$ function of p on $(1,+\infty)$, so from now onwards $\{C_{p,q}\}_p$ is an uniparametric family with q depending on p.

Recall that a finite measure η with compact support is a Frostman measure with exponent $s>0$ if

$$\eta(B_r(x)) \le Cr^s \quad \text{for } x\in\mathbf{R},\ \ r>0.$$

By Frostman's Lemma, given a compact set K and $s<\dim K$ there is a Frostman measure supported on K with exponent s; see Mattila [Mat95].

A probability measure on $C_{p,q}$ is defined by $\vartheta_{p,q} = \mu_0 \circ \Gamma_{p,q}^{-1}$, where μ_0 is defined as above and $\Gamma_{p,q} \colon \Omega \to C_{p,q}$ is the projection defined in (2.4). Note that $\vartheta_p = \vartheta_{p,0}$, and this is a Frostman measure with exponent $1/p$ because it is equivalent to $\mathscr{H}_p$ and this is a Frostman measure; seed Theorem 5.3 in [Fal97].

The main theorems of this part are stated below. Let us denote with $\nu \in L^2$ ($\nu \notin L^2$) the fact that the measure ν has (does not have) a density in $L^2(\mathbf{R})$.

Theorem 19. *Let η be a Frostman measure with exponent $s \in (0,1)$ and let $\bar{p}$ be such that $s + 1/\bar{p} = 1$. Given $J \subset (1, \bar{p})$ a closed interval, $J = [p_0, p_1]$ we have*

$$\dim\bigl(\{p \in J \colon \eta * \vartheta_{p,q} \notin L^2\}\bigr) \le 2 - \Bigl(s + \frac{1}{p_1}\Bigr). \tag{5.4}$$

In particular, the measure $\eta * \vartheta_{p,q}$ has a density in L^2 for $\mathscr{L}$-a.e. $p \in (1, \bar{p})$.

Let us denote by $\mu \ll \nu$ if μ is absolutely continuous with respect to ν.

Corollary 20. *For a fixed $p' > 1$ we have $\vartheta_p * \vartheta_{p'} \ll \mathscr{L}$ ($\mathscr{H}^{1/p}|_{C_p} * \mathscr{H}^{1/p'}|_{C_{p'}} \ll \mathscr{L}$) with density in $L^2(\mathbf{R})$ for $\mathscr{L}$-a.e. p such that $\dim C_p + \dim C_{p'} > 1$.*

Remark. If $\dim C_p + \dim C_{p'} < 1$, the convolution $\vartheta_{p'} * \vartheta_p$ is singular with respect to $\mathscr{L}$, since $\operatorname{supp}(\vartheta_p * \vartheta_{p'}) = C_p + C_{p'}$.

For sumsets we have the following result, that is analogous to Theorem 5.12 for homogeneous Cantor sets in [PS00].

Theorem 21. *Let $K \subset \mathbf{R}$ be a compact set and $J = [p_0, p_1] \subset (1, +\infty)$. Then*

$$\dim\bigl\{p \in J \colon \dim(K + C_{p,q}) < \dim K + \dim C_{p,q}\bigr\} \le \dim K + \dim C_{p_0, q(p_0)} \tag{5.5}$$

and

$$\dim\bigl\{p \in J \colon \mathscr{H}^1(K + C_{p,q}) = 0\bigr\} \le 2 - (\dim K + \dim C_{p_1, q(p_1)}). \tag{5.6}$$

Note that (5.6) follows from (5.4) choosing $s < \dim K$, taking a Frostman measure on K with exponent s and then letting $s \nearrow \dim K$.

5.1. Proof of Theorems 19 and 21. These theorems are a consequence of a projection theorem of Peres and Schlag [PS00] (see also [PSS00]) and their proofs follow closely that of Theorem 5.12 in that paper. We need to state a one dimensional version of the projection theorem, and for this we require some definitions and notation.

Definition. The Sobolev dimension of a finite measure on $\mathbf{R}^n$ with compact support is defined by

$$\dim_s(\nu) = \sup\Bigl\{\alpha \colon \int (1 + |x|)^{\alpha - n} |\hat{\nu}(x)|^2 \, dx < +\infty\Bigr\}.$$

The properties of Sobolev dimension that we will use are stated below; see Mattila [Mat04], Proposition 5.1.

Proposition 22. *Let ν be finite measure on $\mathbf{R}^n$ with compact support.*

1. *If $0 \le \dim_s \nu \le n$, then $\dim_s \nu \le \dim(\operatorname{supp} \nu)$.*
2. *If $\dim_s \nu \ge n$, then $\nu \in L^2(\mathbf{R}^n)$.*

The general setting of the projection theorem consists in a compact metric space (Θ, d) together with a continuous map $\Pi \colon L \times \Theta \to \mathbf{R}$, where $L \subset \mathbf{R}$ is an open interval. For this map it is assumed that for any compact $J \subset L$ and $m \in \mathbf{N}$ there exists $c_{m,J}$ such that

$$\left|\frac{d^m}{dp^m}\Pi(p,\theta)\right| \le c_{m,J}$$

for every $p \in J$ and $\theta \in \Theta$. The functions $\Pi_p(\cdot) := \Pi(p,\cdot)$ can be seen as a family of projections parameterized by p. Given a finite measure μ on Θ, consider the family of projected measures $\nu_p = \mu \circ \Pi_p^{-1}$. Peres and Schlag [PS00] related the smoothness of the measures ν_p to the α-energy of the measure μ, defined by

$$\mathscr{E}_\alpha(\mu) = \int_\Theta \int_\Theta \frac{d\mu(\theta_1)\, d\mu(\theta_2)}{d(\theta_1,\theta_2)^\alpha}.$$

For this it is crucial that Π verifies the *transversality condition*, which is a kind of non degeneracy condition.

Definition. For any distinct $\theta_1, \theta_2 \in \Theta$ and $p \in J$ let

$$\Phi_{\theta_1,\theta_2}(p) = \frac{\Pi(p,\theta_1) - \Pi(p,\theta_2)}{d(\theta_1,\theta_2)}.$$

For any $\beta \in [0,1)$ we say that J is an *interval of transversality of order β for* Π if there is a constant C_β such that for all $p \in J$ and for all $\theta_1, \theta_2 \in \Theta$ condition

$$|\Phi_{\theta_1,\theta_2}(p)| \le C_\beta d(\theta_1,\theta_2)^\beta$$

implies

$$\left|\frac{d}{dp}\Phi_{\theta_1,\theta_2}(p)\right| \ge C_\beta d(\theta_1,\theta_2)^\beta. \tag{5.7}$$

In addition, we say that Π is *regular on* J if under the same condition and for all positive integer m there is a constant $C_{\beta,m}$ such that

$$\left|\frac{d^m}{dp^m}\Phi_{\theta_1,\theta_2}(p)\right| \le C_{\beta,m} d(\theta_1,\theta_2)^{-\beta m}. \tag{5.8}$$

Next we state (incompletely) the Peres–Schlag projection theorem.

Theorem 23. [[PS00], Theorem 2.8] *Let* Θ, J *and* Π *be as above. Suppose that* J *is an interval of transversality of order* β *for* Π *for some* $\beta \in (0,1]$ *and that* Π *is regular on* J*. Let* μ *be a finite measure on* Θ *with finite* α*-energy for some* $\alpha > 0$*. Then, for any* $\sigma \in (0,\alpha]$ *we have*

$$\dim\bigl\{p \in J\colon\ \dim_s(\nu_p) \le \sigma\bigr\} \le 1 + \sigma - \frac{\alpha}{1 + a_0\beta}, \tag{5.9}$$

where a_0 *is some absolute constant. Moreover, for any* $\sigma \in (0, \alpha - 3\beta]$ *we have*

$$\dim\bigl\{p \in J\colon\ \dim_s(\nu_p) < \sigma\bigr\} \le \sigma. \tag{5.10}$$

Now we apply the above to prove Theorems 19 and 21.

Recall that $a_n^p = (\log n)^q/n^p$, where q is a $\mathscr{C}^\infty$ function of p. For notational convenience we will denote the code map of $C_{p,q}$ by Γ_p instead of $\Gamma_{p,q}$. For $\omega \in \Omega$ we define $\Gamma_\omega(p) = \Gamma_p(\omega)$. Note that $\Gamma_\omega \in \mathscr{C}^\infty((1,+\infty))$ for each $\omega \in \Omega$. In fact, since any point in a Cantor set is the sum of the length of all gaps which lie to its left, we obtain $\Gamma_p(\omega) = \sum_{n\geq 1} a_n^p(\omega)$, where

$$a_n^p(\omega) = \begin{cases} a_n^p, & \text{if the gap } L_n \text{ is to the left of } \Gamma_p(\omega); \\ 0, & \text{otherwise.} \end{cases}$$

Notice that the values of n for which $a_n^p(\omega) = 0$ depend only on ω and not on p.

Now fix a compact set $K \subset \mathbf{R}$ and a Frostman measure η supported on K with exponent s. In our context transversality fails globally but we will be able to apply Peres-Schlag Theorem on small pieces of the Cantor sets. More precisely, given $\delta \in \Omega_m$, consider the cylinder $[\delta] = \{\delta\tau \colon \tau \in \Omega\}$. Let $\Theta_\delta = K \times [\delta]$. The projection map $\Pi \colon (1,\infty) \times \Theta_\delta \to \mathbf{R}$ is defined by

$$\Pi(p, x, \omega) = x + \Gamma_p(\omega).$$

Given $J = [p_0, p_1] \subset (1,\infty)$ we define a metric on $[\delta]$ by

$$\tilde{d}(\omega, \tau) = \begin{cases} d_k, & \text{if } |\omega \wedge \tau| = k, \\ 0, & \text{if } \omega = \tau, \end{cases}$$

where $d_k = \max_{|\gamma|=k} |I_\gamma^{p_0}|$, with I_γ^p the corresponding interval of the k-step of $C_{p,q}$. Thus, the metric on Θ_δ is

$$d((x,\omega),(y,\tau)) = |x-y| + \tilde{d}(\omega,\tau).$$

Consider on the space Θ_δ the measure $\mu_\delta = \eta \times \mu_0|_{[\delta]}$, with μ_0 the uniform product measure on Ω.

Remark 24. Let $C_{p,q,\delta} := C_{p,q} \cap I_\delta^p$ and $\vartheta_{p,q,\delta} = \vartheta_{p,q}|_{C_{p,q,\delta}}$. It can be verified directly from the definition that the projected measure $\nu_{p,q,\delta} := \mu_\delta \circ \Pi_{p,q}^{-1}$ coincides with the convolution $\eta * \vartheta_{p,q,\delta}$. We will find $M > 0$ such that Peres–Schlag Theorem can be applied to $\eta * \vartheta_{p,q,\delta}$ for all $\delta \in \Omega_M$. Hence the desired conclusion in Theorem 19 for $\eta * \vartheta_{p,q}$ follows easily from the corresponding conclusion for each $\eta * \vartheta_{p,q,\delta}$.

Let us begin with the energy estimate for μ_δ.

Lemma 25. *Let $J = [p_0, p_1] \subset (1,\infty)$. Then the α-energy of μ_δ is finite provided*

$$\alpha < s + \dim C_{p_0,q}.$$

Proof. Note that

$$\begin{aligned}
\mathscr{E}_\alpha(\mu_\delta) &= \int_{[\delta]}\int_{[\delta]}\int_K\int_K \frac{d\eta(x)\ d\eta(y)\ d\mu_0(\omega)\ d\mu_0(\tau)}{(|x-y| + \tilde{d}(\omega,\tau))^\alpha} \\
&\leq \int_\Omega\int_\Omega\int_K\int_K \frac{d\eta(x)\ d\eta(y)\ d\mu_0(\omega)\ d\mu_0(\tau)}{(|x-y| + \tilde{d}(\omega,\tau))^\alpha} \\
&= \int_K\int_K \sum_{k\geq 0} \frac{1}{2^k} \frac{d\eta(x)\ d\eta(y)}{(|x-y| + d_k)^\alpha} \\
&= \int_K\int_K \sum_{k\colon |x-y|\leq d_k} + \int_K\int_K \sum_{k\colon |x-y|>d_k} \\
&= I + II.
\end{aligned}$$

We are going to estimate I and II separately. We have

$$I \leq \sum_{k\geq 0} \frac{1}{2^k} \frac{1}{d_k^\alpha} (\eta \times \eta)\big\{|x-y| \leq d_k\big\} \leq c \sum_{k\geq 0} \frac{1}{2^k} \frac{1}{d_k^{\alpha-s}},$$

where the last inequality holds since η is a Frostman measure with exponent s. Observe that $2^k d_k^t \to +\infty$ for any $t < \dim \Lambda_{p_0}$ since $2^k d_k^t$ is an upper bound for the t-dimensional cover of Λ_{p_0} with the intervals of the k-step. Therefore I is bounded by a convergent geometrical series.

For the second term, choose $\epsilon > 0$ such that $t := \alpha - s + \epsilon < \dim \Lambda_{p_0}$. Then $d_k \geq c 2^{-k/t}$ for all k and for some $c > 0$, since $2^k d_k^t \to +\infty$. If

$$\kappa(x, y) := \min\{k \colon |x-y| > d_k\},$$

then

$$II \leq c' \int_K \int_K \frac{1}{2^{\kappa(x,y)}} \frac{d\eta(x)\ d\eta(y)}{|x-y|^\alpha} \leq c'' \int_K \int_K \frac{d\eta(x)\ d\eta(y)}{|x-y|^{\alpha-t}} < +\infty,$$

the last inequality is because $\alpha - t = s - \epsilon$ is smaller than the exponent of η (see [Mat95], Chapter 8). □

A sufficient condition for transversality is given below. We say that the interval J is of transversality of order β relative to $[\delta]$ if it is of transversality of order β for $\Pi \colon (1, \infty) \times \Theta_\delta \to \mathbf{R}$.

Lemma 26. *The closed interval $J \subset (1, +\infty)$ is of transversality of order β relative to $[\delta]$ provided there exists a constant c_β such that*

$$|\Gamma'_\omega(p) - \Gamma'_\tau(p)| \geq c_\beta d_k^{\beta+1}, \quad \text{if } |\omega \wedge \tau| = k, \tag{5.11}$$

for all ω, $\tau \in [\delta]$, $p \in J$. Moreover, Π is regular on J if

$$\Big|\frac{d^m}{dp^m}\Gamma_\omega(p) - \frac{d^m}{dp^m}\Gamma_\tau(p)\Big| \leq c_{\beta,m} d_k^{1-\beta m} \tag{5.12}$$

for some constant $c_{\beta,m}$.

Proof. Suppose

$$|\Phi_{\theta_1,\theta_2}(p)| \leq c_\beta d(\theta_1, \theta_2)^\beta \text{ for all } \theta_1, \theta_2 \in \Theta_\delta, \tag{5.13}$$

for some small enough constant c_β. Now fix $\omega_1 = (x, \omega)$, $\omega_2 = (y, \tau) \in \Theta_\delta$. We may assume $\omega \neq \tau$, otherwise is trivial. Let

$$r := d((x, \omega), (y, \tau)) = |x-y| + d_k.$$

Observe that transversality and regularity follow easily from (5.11) and (5.12) if we can show that $r < 2d_k$. This is a consequence of (5.13). In fact, if $|x-y| \geq 2d_k$, then $u = |x-y|/d_k > 2$, which implies

$$|\Phi_{\omega_1,\omega_2}(p)| \geq \frac{|x-y| - d_k}{|x-y| + d_k} = \frac{u-1}{u+1} \geq C > 0.$$

This contradicts (5.13) if $c_\beta < C$. □

Proof of Theorem 19. Note that

$$\frac{d}{dp} \frac{(\log n)^q}{n^p} = (q'(p) \log\log n - \log n) \frac{(\log n)^q}{n^p},$$

whence $d(a_n^p)/dp$ is comparable to $-(\log n)^{q+1}/n^p$ for $n \geq n_0$, for some n_0 depending on the function q. That is, there exist positive and finite quantities c and c' depending on J such that

$$c\frac{(\log n)^{q+1}}{n^p} \leq -\frac{d}{dp}a_n^p \leq c'\frac{(\log n)^{q+1}}{n^p}. \tag{5.14}$$

Choose $M > 0$ such that at least the first n_0 gaps where removed to construct the M-step of $C_{p,q}$. Let $\delta \in \Omega_M$ and $\omega, \tau \in [\delta]$, with $|\omega \wedge \tau| = k$, and assume that $\omega \prec \tau$. Note that

$$|\Gamma'_\omega(p) - \Gamma_\tau(p)'| = \sum_{n\in\Lambda} -\frac{d}{dp}a_n^p,$$

where Λ is the set of subindex of all the gaps of $C_{p,q}$ which lie between $\Gamma_{p,q}(\omega)$ and $\Gamma_{p,q}(\tau)$. Observe that Λ only depends on the relative positions of ω and τ in Ω. It follows by (5.14) that $|\Gamma'_\omega(p) - \Gamma_\tau(p)'|$ is comparable to $|\Gamma_{p,q+1}(\omega) - \Gamma_{p,q+1}(\tau)|$, which is comparable to the length of the basic interval $I_{\omega\wedge\tau}$ of the Cantor set $C_{p,q+1}$. Hence, from Lemma 4 we get

$$c_p\frac{k^{q+1}}{2^{kp}} \leq |\Gamma'_\omega(p) - \Gamma_\tau(p)'| \leq c'_p\frac{k^{q+1}}{2^{kp}}. \tag{5.15}$$

Then transversality relative to $[\delta]$ holds in smaller subintervals of J. That is, given $\beta \in (0,1)$ decompose $J = \bigcup_{i=1}^N J_i$, with $J_i = [p_i, p_{i+1}]$, so that

$$\beta > \frac{p_{i+1}}{p_i} - 1.$$

Choose $\epsilon > 0$ such that $\beta - \epsilon$ satisfies the above inequality. With ω and τ as above and letting $\tilde{q} = \min_{p\in J}\{q(p)\}$, we have from (5.15)

$$|\Gamma'_\omega(p) - \Gamma'_\tau(p)| \geq c_J\frac{k^{\tilde{q}+1}}{2^{kp_{i+1}}} > c_J\frac{k^{\tilde{q}+1}2^{kp_i\epsilon}}{2^{kp_i(\beta+1)}} \geq c_{J,\beta}\frac{k^{q(p_i)}}{2^{kp_i(\beta+1)}} \geq c'_{J,\beta}d_k^{\beta+1},$$

the last inequality follows from Lemma 4. Hence each J_i is an interval of β transversality relative to $[\delta]$ by Lemma 26.

Next we check regularity relative to $[\delta]$ on each J_i. As above, we can verify that $d^m(a_n^p)/dp^m$ is comparable to $(-1)^m(\log n)^{q+m}/n^p$ for sufficiently large values of n (depending on q and m). In particular, $|d^m(a_n^p)/dp^m| \leq c_{J,m}(\log n)^{q+m}/n^p$ for all n, for some constant $c_{J,m}$. With the same notation as above we have

$$\left|\frac{d^m}{dp^m}\Gamma_\omega(p) - \frac{d^m}{dp^m}\Gamma_\tau(p)\right| = \sum_{n\in\Lambda}|\frac{d^m}{dp^m}a_n^p| \leq c_{J,m}\sum_{n\in\Lambda}\frac{(\log n)^{q+m}}{n^p}.$$

The last sum is equivalent to the length of the basic interval $I_{\omega\wedge\tau}$ of $C_{p,q+m}$, thus it is bounded by a constant $c'_{J,m}k^{q+m}/2^{kp}$. From this regularity follows.

Since on J_i the α-energy of μ_δ is finite provided $\alpha < s + 1/p_i$, then from Proposition 22, Remark 24 and (5.9) in Theorem 23 we obtain

$$\begin{aligned}\dim\big(\{p \in J_i \colon \eta * \vartheta_{p,q,\delta} \notin L^2\}\big) &\leq \dim\big(\{p \in J_i \colon \dim_s(\eta * \vartheta_{p,q,\delta}) \leq 1\}\big)\\ &\leq 2 - \frac{s+1/p_i}{1+a_0\beta} \leq 2 - \frac{s+1/p_1}{1+a_0\beta}.\end{aligned}$$

Finally, note that $\dim(\{p \in J_i \colon \upsilon_{p,q} \notin L^2\}) \leq \max_{\delta\in\Omega_M}\dim(\{p \in J_i \colon \upsilon_{p,q,\delta} \notin L^2\})$, since $\upsilon_{p,q} = \sum_{\delta\in\Omega_M}\upsilon_{p,q,\delta}$. Therefore (5.4) follows letting $\beta \to 0$. $\square$

To prove (5.5) in Theorem 21 we proceed as in the above proof but we use (5.10) in Theorem 23 instead. Details are omitted.

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