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ON THE QUASICONFORMAL SELF-MAPPINGS OF THE UNIT BALL SATISFYING THE POISSON DIFFERENTIAL EQUATIONS

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Abstract. It is proved that the family of K quasiconformal mappings of the unit ball onto itself satisfying PDE $\Delta u = g$, $g \in L^{\infty}(B^n)$, $u(0) = 0$, is a uniformly Lipschitz family. In addition, it is showed that the Lipschitz constant tends to 1 as $K \to 1$ and $|g|_{\infty} \to 0$. This generalizes a similar two-dimensional case treated in [21] and solves the problem initialized in [16].

1. Introduction and statement of the main results

A twice differentiable function u defined in an open subset Ω of the Euclidean space $\mathbf{R}^n$ is said to be *harmonic* if it satisfies the differential equation

$$\Delta u(x) := D_{11}u(x) + D_{22}u(x) + \cdots + D_{nn}u(x) = 0.$$

In this paper we denote by B^n and by S^{n-1} the unit ball and unit sphere in $\mathbf{R}^n$, respectively. Also we will assume that $n > 2$ (the case $n = 2$ has been already treated in [21]). We will consider the vector norm $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$ and the matrix norm $|A| = \sup\{|Ax| \colon |x| = 1\}$.

A homeomorphism $u \colon \Omega \to \Omega'$ between two open subsets Ω and Ω' of the Euclidean space $\mathbf{R}^n$ will be called a K-*quasiconformal* ($K \geq 1$) or shortly a q.c. mapping if

(i) u is absolutely continuous function in almost every segment parallel to some of the coordinate axes, and there exist the partial derivatives which are locally L^n integrable functions on Ω (we will write $u \in ACL^n$), and

(ii) u satisfies the condition

$$|\nabla u(x)|^n/K \leq J_u(x) \leq Kl(\nabla u(x))^n$$

at almost every x in Ω where

$$l(\nabla u(x)) := \inf\{|\nabla u(x)\zeta| \colon |\zeta| = 1\}$$

and $J_u(x)$ is the Jacobian determinant of u (see [32]).

Notice that for a continuous mapping u the condition (i) is equivalent to the condition that u belongs to the Sobolev space $W^1_{n,\mathrm{loc}}(\Omega)$.

Let P be the Poisson kernel, i.e., the function

$$P(x, \eta) = \frac{1 - |x|^2}{|x - \eta|^n},$$

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and let G be the Green function, i.e., the function

$$G(x,y) = c_n\left(\frac{1}{|x-y|^{n-2}} - \frac{1}{(|x|y| - y/|y||)^{n-2}}\right) \tag{1.1}$$

where $c_n = \frac{1}{(n-2)\Omega_{n-1}}$, and Ω_{n-1} is the measure of S^{n-1}. Both functions P and G are harmonic for $|x| < 1$ with $x \neq y$.

Let $f\colon S^{n-1} \to \mathbf{R}^n$ be a bounded integrable function on the unit sphere S^{n-1}, and let $g\colon B^n \mapsto \mathbf{R}^n$ be a continuous function. The solution of the equation (in the sense of distributions) $\Delta u = g$ in the unit ball satisfying the boundary condition $u|_{S^{n-1}} = f \in L^1(S^{n-1})$ is given by

$$u(x) = P[f](x) - G[g](x) := \int_{S^{n-1}} P(x,\eta)f(\eta)\,d\sigma(\eta) - \int_{B^n} G(x,y)g(y)\,dy, \tag{1.2}$$

$|x| < 1$. Here $d\sigma$ is the Lebesgue $n-1$ dimensional measure of the Euclid sphere satisfying the condition: $P[1](x) \equiv 1$. It is well known that if f and g are continuous in S^{n-1} and in $\overline{B^n}$, respectively, then the mapping $u = P[f] - G[g]$ has a continuous extension $\tilde{u}$ to the boundary and $\tilde{u} = f$ on S^{n-1}. If $g \in L^\infty$, then $G[g] \in C^{1,\alpha}(\overline{B^n})$. See [12, Theorem 8.33] for this argument.

We will consider those solutions of the PDE $\Delta u = g$ that are quasiconformal as well and investigate their Lipschitz character. A mapping f of a set A in the Euclidean n-space $\mathbf{R}^n$ into $\mathbf{R}^n$, $n \geq 2$, is said to belong to the Lipschitz class $\mathrm{Lip}_\alpha(A)$, $\alpha > 0$, if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x-y|^\alpha \tag{1.3}$$

for all x and y in A. If D is a bounded domain in $\mathbf{R}^n$ and if f is quasiconformal in D with $f(D) \subset \mathbf{R}^n$, then f is in $\mathrm{Lip}_\alpha(A)$ for each compact $A \subset D$, where $\alpha = K_I(f)^{1/(1-n)}$ and $K_I(f)$ is the inner dilatation of f. Simple examples show that f need not be in $\mathrm{Lip}_\alpha(D)$ even when f is continuous in $\overline{D}$. However, Martio and Näkki in [27] (see also [26]) showed that if f induces a boundary mapping which belongs to $\mathrm{Lip}_\alpha(\partial D)$, then f is in $\mathrm{Lip}_\beta(D)$, where

$$\beta = \min(\alpha, K_I(f)^{1/(1-n)});$$

the exponent β is sharp.

We are interested in the condition under which the quasiconformal mapping is in $\mathrm{Lip}_1(B^n)$. It follows from our results that the conditions that u is quasiconformal and that $|\Delta u|$ is bounded, guaranty that the selfmapping of the unit ball is in $\mathrm{Lip}_1(B^n)$. In particular, the results hold for quasiconformal harmonic mappings. It seems that the family of q.c. harmonic mappings has first been considered in [25]. The papers [8], [14], [15]–[24] and [31] bring much light on the topic of quasiconformal harmonic mappings on the plane. In this paper we continue to study the same problem in the space $\mathbf{R}^n$ which was started in the paper [16]. See also [3], [4] and [5] for the related problem. The problem in the space is much more complicated because of the lack of the techniques of complex analysis.

It is well known that the harmonic extension (via Poisson integral) of a homeomorphism of the unit circle is always a diffeomorphism of the unit disk. In higher dimensions, however, the situation is quite different. Namely, Melas ([29], see also [22]) constructed a homeomorphism of the unit sphere S^{n-1} ($n \geq 3$) whose harmonic extension fails to be diffeomorphic. See [7] for related results on the class of smooth quasiconformal mappings. The questions arise, do there exist such examples, assuming both conditions, harmonicity and quasiconformality; in other words do some q.c. harmonic mappings have critical points, i.e., the points in which the Jacobian is zero? For $K < 2^{n-1}$, such examples do not exist, see [19] or [33] for this argument. In [19], the author and Mateljević proved that, under the condition $K < 2^{n-1}$, a harmonic quasiconformal mapping u of the unit ball onto itself is co-Lipschitz (meaning that u^{-1} is Lipschitz). If w is a harmonic univalent function on a domain in the complex plane, then by Lewy's theorem (see [23] and [13]), w has a non-vanishing Jacobian, and consequently, according to the inverse mapping theorem, w is a diffeomorphism. However, in the space we cannot use this argument. Indeed, Lewy's theorem fails in higher dimensions, as it is shown in [36]. For this problem concerning q.c. hyperbolic harmonic selfmappings of the unit ball see [33], and for q.c. harmonic mappings between complete Riemannian manifolds see [9].

The following theorem gives a positive answer to the question raised by the author in [16]: whether a q.c. harmonic self-mapping of the unit ball is Lipschitz continuous with Lipschitz constant depending only on a quasiconformality constant K? This is a generalization of an analogous theorem for the unit disk due to the author and Pavlović [21]. See also [16] and [30]. This is the main result of the paper as follows.

Theorem 1.1. *Let $K \geq 1$ be arbitrary, $n \in \mathbf{N}$ and let $g \in L^\infty(\overline{B^n})$. Then there exist constants $M_1'(n,K)$ and $M_2'(n,K)$ satisfying: if u is a K-quasiconformal self-mapping of the unit ball B^n satisfying the PDE (in the sense of distributions) $\Delta u = g$, with $u(0) = 0$, then:*

$$|u(x) - u(y)| \leq (M_1'(n,K) + M_2'(n,K)|g|_\infty)|x - y|, \quad x, y \in B^n. \tag{1.4}$$

Moreover, $M_1'(n,K) \to 1$ as $K \to 1$.

The example 1.2 given below shows that the condition $g \in L^\infty(\overline{B^n})$ of Theorem 1.1 is necessary for u being Lipschitz. See also Example e) of Section 4 (the mapping v).

Example 1.2. Let $f(x) = |x|^{-1+K^{1/(1-n)}}x$. Then f is a K-quasiconformal mapping of the space onto itself. Let φ be a Möbius transformation of the unit ball B^n onto the upper halfspace H^n; for example, $\varphi(x) = (x - S)/|x - S|^2 + S/2$, where $S = (0, \dots, 0, -1)$. Then the mapping $u(x) = \varphi^{-1} \circ f \circ \varphi$ is a C^∞ K-quasiconformal mapping of the unit ball onto itself, such that u is not Lipschitz, and therefore, $\Delta u \notin L^\infty(B^n)$.

It is important to notice that the class of harmonic functions (mappings) contains itself the class of holomorphic functions (mappings). Therefore, the class of holomorphic automorphisms of the unit ball is a subclass of quasiconformal harmonic self-mappings of the unit ball. On the other hand, according to Fefferman's theorem [10], every analytic bi-holomorphic mapping between two smooth domains has a smooth extension to the boundary, and therefore, the class of bi-holomorphic mappings between smooth domains is contained in the class of harmonic quasiconformal mappings. Therefore, our results can be considered as extensions of Fefferman's theorem. Some nontrivial examples of quasiconformal harmonic mappings are given in Section 4.

The proof of Theorem 1.1, given in Section 3, depends on the following result:

Proposition 1.3. [17] *Let $u\colon B^n \to \Omega$ be a twice differentiable q.c. mapping of the unit ball onto the bounded domain Ω with C^2 boundary satisfying the differential inequality*

$$|\Delta u| \le A|\nabla u|^2 + B$$

for some $A, B \ge 0$. Then ∇u is bounded and u is Lipschitz continuous.

One of the advantages of Theorem 1.1 in relation to Proposition 1.3 is that, in Theorem 1.1 the Lipschitz constant does not depend on the mapping u, contrary to the statement of Proposition 1.3. It also depends on Mori's theorem in the theory of quasiconformal mappings:

Proposition 1.4. [11] *If u is a K-quasiconformal self-mapping of the unit ball B^n with $u(0) = 0$, then there exists a constant $M_1(n,K)$, satisfying the condition $M_1(n,K) \to 1$ as $K \to 1$, such that*

$$|u(x) - u(y)| \le M_1(n,K)|x-y|^{K^{1/(1-n)}}. \tag{1.5}$$

See also [2] *with some constant that is not asymptotically sharp.*

The mapping $|x|^{-1+K^{1/(1-n)}}x$ shows that the exponent $K^{1/(1-n)}$ is optimal in the class of arbitrary K-quasiconformal homeomorphisms.

2. Auxiliary results

By S and T we denote the spherical coordinates:

$$S\colon Q_0^n = [0,1] \times [0,\pi] \times \cdots \times [0,\pi] \times [0,2\pi] \mapsto B^n$$

and

$$T\colon Q^{n-1} = [0,\pi] \times \cdots \times [0,\pi] \times [0,2\pi] \mapsto S^{n-1},$$

$(S(r,\theta_0,\dots,\theta_{n-2},\varphi) = rT(\theta_1,\dots,\theta_{n-2},\varphi))$, defined by $S = (x_1, x_2, \dots, x_{n-1})$,

$$\begin{aligned} x_1 &= r\cos\theta_1,\\ x_2 &= r\sin\theta_1\sin\theta_2,\\ &\ \vdots\\ x_{n-1} &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\cos\varphi,\\ x_n &= r\sin\theta_1\sin\theta_2\cdots\sin\theta_{n-2}\sin\varphi. \end{aligned}$$

Then we have

$$\det\nabla S(r,\theta_1,\dots,\theta_{n-2},\varphi) = r^{n-1}\sin^{n-2}\theta_1\cdots\sin\theta_{n-2}. \tag{2.1}$$

We will use notations $\theta = (\theta_1,\dots,\theta_{n-2},\varphi)$ and $\theta_{n-1} = \varphi$.

Lemma 2.1. *Let $u = P[f]$ be a harmonic function defined on the unit ball, and assume that its derivative $v = \nabla u$ is bounded on the unit ball (or equivalently, let u be Lipschitz continuous). Then there exists a mapping $A \in L^\infty(S^{n-1})$ defined on the unit sphere S^{n-1} such that $\nabla u(x) = P[A](x)$ and for almost every $\eta \in S^{n-1}$ there holds the relation*

$$\lim_{r\to 1-} \nabla u(r\eta) = A(\eta). \tag{2.2}$$

Moreover, the function $f\circ T$ is differentiable almost everywhere in Q^{n-1} and there holds

$$A(T(\theta)) \cdot T'(\theta) = (f\circ T)'(\theta).$$

Proof. For the proof of the first statement of the lemma, see for example [6, Theorem 6.13 and Theorem 6.39]. Next, since

$$|\frac{\partial}{\partial\theta_i}u(S(r,\theta))| = |r\nabla u(S(r,\theta))\frac{\partial}{\partial\theta_i}T(\theta)| \leq |r\nabla u(S(r,\theta))|\cdot|\frac{\partial}{\partial\theta_i}T(\theta)|$$
$$\leq \mathrm{essup}_\theta|A(T(\theta))|\cdot|\frac{\partial}{\partial\theta_i}T(\theta)| = M_i < \infty.$$

Now we make use of the following version Lebesgue Dominated Convergence Theorem. Suppose that f_n is a sequence of measurable functions in a measure space E, that $f_n \to f$ pointwise almost everywhere as $n \to \infty$, and that $|f_n| < g$ for all n, where g is integrable. Then f is integrable, and

$$\int_E f\,d\mu = \lim_{n\to\infty}\int_E f_n\,d\mu.$$

In our case we have $g(\theta) = M_i$, $f_n = \dfrac{\partial}{\partial\theta_i}u(S(r_n,\theta))$, where $r_n \to 1-0$. Thus

$$\begin{aligned}
f(T(\theta)) &= \lim_{n\to\infty} u(S(r_n,\theta))\\
&= \lim_{n\to\infty}\int_{\theta_i^0}^{\theta_i}\frac{\partial}{\partial\theta_i}u(S(r_n,\theta))\,d\theta_i + f(T(\theta^0))\\
&= \int_{\theta_i^0}^{\theta_i}\lim_{n\to\infty}\frac{\partial}{\partial\theta_i}u(rS(\theta))\,d\theta_i + f(T(\theta^0))\\
&= \int_{\theta_i^0}^{\theta_i}\lim_{n\to\infty} r_n\nabla u(S(r_n,\theta))\frac{\partial}{\partial\theta_i}T(\theta)\,d\theta_i + f(T(\theta^0))\\
&= \int_{\theta_i^0}^{\theta_i} A(T(\theta))\cdot\frac{\partial}{\partial\theta_i}T(\theta)\,d\theta_i + f(T(\theta^0)).
\end{aligned}\tag{2.3}$$

Differentiating in θ_i for every $i \in \{1,\dots,n-1\}$ we get

$$\frac{\partial}{\partial\theta_i}f(T(\theta)) = A(T(\theta))\cdot\frac{\partial}{\partial\theta_i}T(\theta)$$

almost everywhere in Q^{n-1}. Hence, we have

$$A(T(\theta))\cdot T'(\theta) = (f\circ T)'(\theta)$$

almost everywhere in Q^{n-1}. □

Lemma 2.2. *Let u be a harmonic Lipschitz continuous mapping defined in the unit ball B^n. Denote by ∇u the extension of the gradient up to the boundary $S^{n-1} = \partial B^n$, which exists almost everywhere in S^{n-1}. Then for $x \in B^n$*

$$|\nabla u(x)| \leq \text{ ess sup}_{|\eta|=1}|\nabla u(\eta)|,$$

where $|\cdot|$ is the matrix norm.

Proof. Let $u = (u_1,\dots u_n)$. For all pairs (i,j) the function $u_{i,j} = \frac{\partial u_i}{\partial x_j}$ is bounded and harmonic. Hence, there exists a bounded integrable function $g_{i,j}$ defined on the unit sphere such that $u_{i,j} = P[g_{i,j}]$. In other words,

$$\nabla u(x) = \int_{S^{n-1}} g(\eta)P(x,\eta)\,d\sigma(\eta),$$

where $g(\eta)$ is a $n\times n$ dimensional matrix $(g_{i,j}(\eta))_{i,j=1}^n$ and it coincides with $\nabla u(\eta)$. For the induced matrix norm we have

$$|A| = \max\{\langle Ah, k\rangle : |h| = |k| = 1\}.$$

Thus, for $|h| = |k| = 1$ we obtain

$$\begin{aligned}\langle \nabla u(x)h, k\rangle &= \int_{S^{n-1}} \langle g(\eta)h, k\rangle\, P(x,\eta)\, d\sigma(\eta) \le \int_{S^{n-1}} |g(\eta)| P(x,\eta)\, d\sigma(\eta)\\ &\le \operatorname{ess\,sup}_{|\eta|=1} |g(\eta)| \int_{S^{n-1}} P(x,\eta)\, d\sigma(\eta).\end{aligned}$$

The proof is completed. □

Lemma 2.3. *For every $\alpha < n$ the potential type integral*

$$I(x) = \int_{B^n} \frac{dy}{|x-y|^\alpha}$$

exists for every $x \in \mathbf{R}^n$, and achieves its maximum for $x = 0$. Furthermore,

$$I(0) = \frac{1}{n-\alpha}\,\Omega_{n-1}. \tag{2.4}$$

If $|x| = 1$ and $\alpha = n-1$, then

$$I(x) = \frac{2\Gamma(\frac{n}{2})}{(n-1)\sqrt{\pi}\Gamma(\frac{n-1}{2})}\,\Omega_{n-1}. \tag{2.5}$$

Moreover, there exists a decreasing function ϕ defined on $[0,+\infty)$ such that $I(x) = \phi(r)$ on the sphere $S^{n-1}(0,r)$ with $r > 0$.

Proof. Let $A = B^n \setminus B^n(x,1)$ and $B = B^n \cap B^n(x,1)$. Then $B^n = A \cup B$. If $y \in A$, then $|y-x| \ge |y|$. Thus

$$\int_A \frac{dy}{|x-y|^\alpha} \le \int_A \frac{dy}{|y|^\alpha}.$$

On the other hand, $B = -B + x$. Thus

$$\int_B \frac{dy}{|x-y|^\alpha} = \int_B \frac{dy}{|y|^\alpha}.$$

Hence,

$$I(x) = \int_{B^n} \frac{dy}{|x-y|^\alpha} \le I(0) = \int_{B^n} \frac{dy}{|y|^\alpha}.$$

Introducing the spherical coordinates centered at 0 and at a point x on the integrals $I(0)$ and $I(x)$, respectively we obtain the relations (2.4) and (2.5). Using the similar argument it follows that ϕ is decreasing.

Lemma 2.4. [16] *The integral*

$$I = \int_{S^{n-1}} |a-\eta|^\gamma\, d\sigma(\eta),$$

$a \in S^{n-1}$ converges if and only if $\gamma > 1-n$. If $\gamma = 2-n$, then $I = 1$. □

Lemma 2.5. *Let ρ be a bounded (absolutely) integrable function defined on a bounded domain $\Omega \subset \mathbf{R}^n$. Then the potential type integral*

$$I(x) = \int_\Omega \frac{\rho(y)\,dy}{|x-y|^\alpha}$$

belongs to the space $C^p(\mathbf{R}^n)$, $p \in \mathbf{N}$, such that $\alpha + p < n$. Moreover,

$$\nabla I(x) = \int_\Omega \nabla \frac{1}{|x-y|^\alpha} \rho(y)\,dy.$$

For the proof see for example [34, pp. 24–26].

Lemma 2.6. *If g is continuous on $\overline{B}^n$, then the mapping $G[g]$ has a bounded derivative, i.e., it is Lipschitz continuous. Moreover, $\nabla G[g]$ has a continuous extension to the boundary and there holds*

$$\nabla G[g](\eta)h = \int_{B^n} \frac{\langle \eta, h\rangle}{\Omega_{n-1}} \frac{1-|y|^2}{|\eta - y|^n} g(y)\,dy$$

for $\eta \in S^{n-1}$.

Proof. First of all for $x \neq y$ we have

$$G_x(x,y) = c_n \frac{(2-n)(x-y)}{|x-y|^n} + c_n \frac{(n-2)(|y|^2x - y)}{|\,x|y| - y/|y|\,|^n}.$$

Thus for $\eta \in S^{n-1}$ we have

$$\lim_{x \to \eta} G_x(x,y) = \frac{1}{\Omega_{n-1}} \frac{\eta(1-|y|^2)}{|\eta - y|^n}. \tag{2.6}$$

Let

$$G_1(x,y) := \frac{1}{\Omega_{n-1}} \frac{x-y}{|x-y|^n}, \tag{2.7}$$

and let

$$G_2(x,y) := \frac{1}{\Omega_{n-1}} \frac{y - |y|^2 x}{|\,x|y| - y/|y|\,|^n}. \tag{2.8}$$

The function G_2 is harmonic for $x \in B^n$. According to Lemma 2.5, it follows

$$\begin{aligned} \nabla G[g](x)h &= \int_{B^n} \langle G_x(x,y), h\rangle\, g(y)\,dy \\ &= \int_{B^n} \langle G_1(x,y), h\rangle\, g(y)\,dy + \int_{B^n} \langle G_2(x,y), h\rangle\, g(y)\,dy. \end{aligned} \tag{2.9}$$

The last statement of the lemma follows from relations (2.6) and (2.9) and Lebesgue Dominated Convergence Theorem. □

Lemma 2.7. *Let u be a solution of the PDE $\Delta u = g$ ($g \in C(\overline{B^n})$) that maps the unit ball onto itself properly ($|u(x)| \to 1$ as $|x| \to 1$). Let in addition u be Lipschitz continuous. Let $\chi(\theta) = f(T(\theta)) := f(t)$, $t \in S^{n-1}$. Then there exist almost everywhere in S^{n-1}*

$$\nabla u(t) := \lim_{r \to 1-} \nabla u(rt) \tag{2.10}$$

and

$$J_u(t) := \lim_{r\to 1-} J_u(rt), \tag{2.11}$$

$t \in S^{n-1}$. *Furthermore, the following relation holds:*

$$\begin{aligned} J_u(t) &= \frac{D_\chi}{D_T}\int_{S^{n-1}} \frac{|f(t)-f(\eta)|^2}{|\eta - t|^n}\, d\sigma(\eta) \\ &+ \frac{D_\chi}{D_T}\int_0^1 r^{n-1}\left(\int_{S^{n-1}} P(r\eta, t)\,\langle g(rt), f(\eta)\rangle\, d\sigma(\eta)\right) dr, \quad t\in S^{n-1}. \end{aligned} \tag{2.12}$$

Here D_χ *and* D_T *denote the square roots of Gram determinants of* $\nabla\chi$ *and* ∇T, *respectively. If* u *is biharmonic* ($\Delta\Delta u = 0$), *then there holds*

$$J_u(t) = \frac{D_\chi}{D_T}\int_{S^{n-1}} \frac{|f(t)-f(\eta)|^2}{|\eta - t|^n}\, d\sigma(\eta) + \frac{D_\chi}{D_T}\int_0^1 r^{n-1}\left\langle g(r^2t), f(t)\right\rangle dr, \tag{2.13}$$

$t \in S^{n-1}$. *For arbitrary continuous function* g *and* $|g|_\infty = \max_{|x|\le 1}|g(x)|$ *there holds the inequality*

$$|J_u(t) - \frac{D_\chi}{D_T}\int_{S^{n-1}} \frac{|f(t)-f(\eta)|^2}{|\eta - t|^n}\, d\sigma(\eta)| \le \frac{D_\chi}{D_T}\frac{|g|_\infty}{n}, \quad t\in S^{n-1}. \tag{2.14}$$

Proof. First of all, according to Lemma 2.6, $G[g]$ has a bounded derivative, and there exists the function $\nabla G[g](\eta)$, $\eta\in S^{n-1}$ which is continuous and satisfies the limit relation $\lim_{x\to\eta}\nabla G[g](x) = \nabla G[g](\eta)$. Since $u = P[f] - G[g]$ has a bounded derivative, according to the Lemma 2.1, it follows that there exists $\lim_{r\to 1-}\nabla P[f](r\eta) = \nabla P[f](\eta)$. Thus, $\lim_{r\to 1-}\nabla u(r\eta) = \nabla u(\eta)$. It follows that the mapping χ defines the outer normal vector field $\mathbf{n}_\chi$ almost everywhere in S^{n-1} at the point $\chi(\theta) = f(T(\theta)) = (\chi_1, \chi_2, \dots, \chi_n)$ by the formula

$$\mathbf{n}_\chi(\chi(\theta)) = \chi_{\theta_1}\times\cdots\times\chi_{\theta_{n-2}}\times\chi_\varphi. \tag{2.15}$$

Since $\mathbf{n}_\chi(\chi(\theta))$ is the normal vector to the unit sphere, there holds the equality

$$\mathbf{n}_\chi(\chi(\theta)) = D_\chi\cdot f(T(\theta)). \tag{2.16}$$

Let $u(S(r,\theta)) = (y_1, y_2, \dots, y_n)$, where S are spherical coordinates. According to Lemma 2.1, we obtain

$$\lim_{r\to 1-} y_{i\varphi}(r,\theta) = \chi_{i\varphi}(\theta), \ \ i\in\{1,\dots,n\}, \tag{2.17}$$

$$\lim_{r\to 1-} y_{i\theta_j}(r,\theta) = \chi_{i\theta_j}(\theta), \ \ i\in\{1,\dots,n\}, \ \ j\in\{1,\dots,n-2\}. \tag{2.18}$$

On the other hand, for almost every $t\in S^{n-1}$ we have

$$\frac{\chi_i(\theta) - y_i(r,\theta)}{1-r} = y_{ir}(\rho_{r,\theta}, \theta),$$

where $r < \rho_{r,\theta} < 1$. Thus we get

$$\lim_{r\to 1-} y_{ir}(r,\theta) = \lim_{r\to 1-}\frac{\chi_i(\theta) - y_i(r,\theta)}{1-r}, \ \ i\in\{1,\dots,n\}. \tag{2.19}$$

Hence, we find that

$$\begin{aligned}
\lim_{r\to 1-} J_{u\circ S}(r,\theta) &= \lim_{r\to 1-}\left\langle \frac{\chi - P[f]}{1-r}, \chi_{\theta_1}\times\cdots\times\chi_{\theta_{n-2}}\times\chi_{\varphi}\right\rangle + \Lambda\\
&= \lim_{r\to 1-}\int_{S^{n-1}} \frac{1+r}{|\eta - rt|^n}\left\langle \chi - f(\eta), \chi_{\theta_1}\times\cdots\times\chi_{\theta_{n-2}}\times\chi_{\varphi}\right\rangle d\sigma(\eta) + \Lambda\\
&= \lim_{r\to 1-}\int_{S^{n-1}} \frac{1+r}{|\eta - S(r,\theta)|^n}\left\langle f(T(\theta)) - f(\eta), \mathbf{n}_{f\circ T}(T(\theta))\right\rangle d\sigma(\eta) + \Lambda\\
&= \lim_{r\to 1-} D_{\chi}(\theta)\int_{S^{n-1}} \frac{1+r}{|\eta - S(r,\theta)|^n}\left\langle f(T(\theta)) - f(\eta), f(T(\theta))\right\rangle d\sigma(\eta) + \Lambda\\
&= \lim_{r\to 1-} \frac{1+r}{2} D_{\chi}(\theta)\int_{S^{n-1}} \frac{|f(T(\theta)) - f(\eta)|^2}{|\eta - S(r,\theta)|^n} d\sigma(\eta) + \Lambda,
\end{aligned} \tag{2.20}$$

where $\Lambda = \lim_{r\to 1-}\left\langle \frac{G[g]}{1-r}, \chi_{\theta_1}\times\cdots\times\chi_{\theta_{n-2}}\times\chi_{\varphi}\right\rangle$.

In order to estimate Λ, observe first that

$$G(x,y) = c_n \frac{\big|x|y| - y/|y|\big|^{n-2} - |x-y|^{n-2}}{|x-y|^{n-2}\cdot \big|x|y| - y/|y|\big|^{n-2}}. \tag{2.21}$$

Next, we have

$$\begin{aligned}
&\big|x|y| - y/|y|\big|^{n-2} - |x-y|^{n-2}\\
&= \left(\big|x|y| - y/|y|\big| - |x-y|\right)\sum_{k=1}^{n-2}\left(\big|x|y| - y/|y|\big|\right)^{n-2-k}\cdot |x-y|^{k-1}
\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}
\big|x|y| - y/|y|\big| - |x-y| &= \frac{\big|x|y| - y/|y|\big|^2 - |x-y|^2}{\big|x|y| - y/|y|\big| + |x-y|}\\
&= \frac{(1+|x|^2|y|^2 - 2\langle x,y\rangle) - (|x|^2+|y|^2 - 2\langle x,y\rangle)}{\big|x|y| - y/|y|\big| + |x-y|}\\
&= \frac{(1-|x|^2)(1-|y|^2)}{\big|x|y| - y/|y|\big| + |x-y|}.
\end{aligned} \tag{2.23}$$

Inserting (2.22) and (2.23) into (2.21), we obtain for a.e. $t\in S^{n-1}$

$$\lim_{x\to t}\frac{G(x,y)}{1-|x|} = \frac{1}{\Omega_{n-1}}P(y,t). \tag{2.24}$$

On the other hand, we have

$$\begin{aligned}
&\frac{1}{\Omega_{n-1}}\int_{B^n} P(y,t)\left\langle g(y), f(t)\right\rangle dy\\
&= \int_0^1 r^{n-1}\Big(\int_{S^{n-1}} P(r\eta,t)\left\langle g(r\eta), f(t)\right\rangle d\sigma(\eta)\Big)\, dr.
\end{aligned} \tag{2.25}$$

Next, there holds

$$J_{u\circ S}(r,\theta) = r^{n-1}J_u(rT(\theta))\cdot D_T(\theta). \tag{2.26}$$

Combining (2.20), (2.24), (2.25) and (2.26) we obtain (2.12). Relations (2.13) and (2.14) follow from (2.12) and (1.2). If u is biharmonic, then g is harmonic and thus,

$$\int_{S^{n-1}} P(r\eta,t)\,\langle g(r\eta), f(t)\rangle\, d\sigma(\eta) = \left\langle g(r^2 t), f(t)\right\rangle .$$

This yields the relation (2.13). □

Assume that A is an $n\times n$ matrix with entries from $\mathbf{R}$. Define the (i,j)-minor $M_{i,j}$ of A as the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting i'th row and j'th column of A, and the (i,j)-cofactor of A as

$$C_{ij} = (-1)^{i+j} M_{ij}\,.$$

Then the adjugate of A is the $n\times n$ matrix

$$\tilde{A} = (C_{ji})_{i,j=1}^{n}\,.$$

If A is an invertible matrix, then

$$A^{-1} = \det(A)^{-1}\,\tilde{A}.$$

That is, the adjugate of A is the transpose of the cofactor matrix $(C_{ij})_{i,j=1}^{n}$ of A.

Lemma 2.8. *Let $A\colon \mathbf{R}^n\to\mathbf{R}^n$ be a linear operator such that $A = [a_{ij}]_{i,j=1,\dots,n}$. If A is K-quasiconformal, then there holds the following double inequality:*

$$K^{1-n}|A|^{n-1}|x_1\times\cdots\times x_{n-1}| \le |Ax_1\times\cdots\times Ax_{n-1}| \le |A|^{n-1}|x_1\times\cdots\times x_{n-1}|. \tag{2.27}$$

Both inequalities in (2.27) *are sharp.*

Proof. Let $e_1 = (1,0,\dots,0),\dots e_n = (0,\dots,0,1)$. Let $x_i = \sum_{j=1}^n x_{ij}e_j$, $i = 1,\dots n-1$. Then

$$Ax_1\times\cdots\times Ax_{n-1} = \sum_{\sigma} \varepsilon_\sigma x_{1,\sigma_1}\dots x_{n-1\sigma_{n-1}} Ae_{\sigma_1}\times\cdots\times Ae_{\sigma_{n-1}},$$

where σ runs through all permutations of $\{1,2,\dots,n-1\}$. It follows that

$$Ax_1\times\cdots\times Ax_{n-1} = \tilde{A}x_1\times\cdots\times x_{n-1}. \tag{2.28}$$

As A is K-quasiconformal, $\tilde{A}$ is quasiconformal as well. Let $\lambda_1^2\le\cdots\le\lambda_n^2$ be the eigenvalues of the matrix A^TA. Since A is K-quasiconformal, then

$$\frac{\lambda_n}{\lambda_1}\le K. \tag{2.29}$$

From $\tilde{A} = \det A\cdot A^{-1}$ it follows that

$$\tilde{\lambda}_k^2 = (\det A)^2\cdot\frac{1}{\lambda_k^2},\quad k=1,\dots n,$$

are eigenvalues of the matrix $\tilde{A}^T\tilde{A}$. Moreover,

$$\tilde{\lambda}_n\le\tilde{\lambda}_{n-1}\le\cdots\le\tilde{\lambda}_1$$

and consequently,

$$\frac{\tilde{\lambda}_1}{\tilde{\lambda}_n}\le K.$$

From (2.28) we obtain

$$|Ax_1\times\cdots\times Ax_{n-1}| \le |\tilde{A}|\cdot|x_1\times\cdots\times x_{n-1}|. \tag{2.30}$$

Furthermore

$$|\tilde{A}| = \tilde{\lambda}_1 = \frac{\det A}{\lambda_1} = \prod_{k=2}^{n} \lambda_k \leq \lambda_n^{n-1} = |A|^{n-1}. \tag{2.31}$$

The relations (2.30) and (2.31) yield the right inequality of (2.27).

In order to obtain the left inequality of (2.27), again we make use of (2.28). From (2.28) it follows that

$$|Ax_1 \times \cdots \times Ax_{n-1}| \geq \tilde{\lambda}_n \cdot |x_1 \times \cdots \times x_{n-1}|. \tag{2.32}$$

On the other hand,

$$\tilde{\lambda}_n = \frac{\det A}{\lambda_n} = \prod_{k=1}^{n-1} \lambda_k \geq K^{1-n} \lambda_n^{n-1}.$$

This inequality completes the proof of lemma. □

Lemma 2.9. *Let u be a solution of the PDE $\Delta u = g$, $g \in C(\overline{B}^n)$, that is Lipschitz continuous. Denote by ∇u its extension up to the boundary $S^{n-1} = \partial B^n$, which exists almost everywhere in S^{n-1}. Then for $x \in B^n$,*

$$|\nabla u(x)| \leq \text{ ess sup}_{|\eta|=1} |\nabla u(\eta)| + \left(1 + \frac{2\Gamma(\frac{n}{2})}{(n-1)\sqrt{\pi}\Gamma(\frac{n-1}{2})}\right) |g|_\infty, \tag{2.33}$$

where $|\cdot|$ is any norm of matrices and $|g|_\infty = \max\{|g(x)|, x \in \overline{B}^n\}$.

Proof. By using the notation of Lemma 2.6, we have

$$\begin{aligned}\nabla u &= \nabla P[f](x) - \nabla G[g](x)\\ &= \nabla P[f](x) - \int_{B^n} G_1(x,y) g(y)\, dy - \int_{B^n} G_2(x,y) g(y)\, dy.\end{aligned}$$

Thus

$$\nabla u(x) + \int_{B^n} G_1(x,y) g(y)\, dy = \nabla P[f](x) - \int_{B^n} G_2(x,y) g(y)\, dy =: h(x).$$

Applying Lemma 2.2 to the harmonic mapping h, we have

$$\begin{aligned}|\nabla u(x) + \int_{B^n} G_1(x,y) g(y)\, dy| &\leq \text{ ess sup}_{|t|=1} |h(t)|\\ &\leq \text{ ess sup}_{|t|=1} |\nabla u(t)| + \sup_{|t|=1} |\int_{B^n} G_1(t,y) g(y)\, dy|.\end{aligned}$$

Hence, for $x \in B^n$ we have

$$\begin{aligned}|\nabla u(x)| \leq &\text{ ess sup}_{|t|=1} |\nabla u(t)| + \text{ ess sup}_{|x|\leq 1} |\int_{B^n} G_1(x,y) g(y)\, dy|\\ &+ \text{ ess sup}_{|t|=1} \int_{B^n} |G_1(t,y)||g(y)|\, dy.\end{aligned}$$

Using now Lemma 2.3, we have

$$|\nabla u(x)| \leq \text{ ess sup}_{|t|=1} |\nabla u(t)| + (1 + \frac{2\Gamma(\frac{n}{2})}{(n-1)\sqrt{\pi}\Gamma(\frac{n-1}{2})}) |g|_\infty.$$ □

Remark 2.10. It is known that harmonic and subharmonic functions satisfy the maximum principle. However, if $u \in C^2(B^n) \cap C^1(\overline{B^n})$ satisfies the PDE $\Delta u = g$, with

$$g \in C^1(\Omega), \quad \langle \nabla u, \nabla g \rangle \leq \frac{|g|_\infty^2}{n}, \tag{2.34}$$

then the mapping ∇u satisfies the maximum principle

$$\sup_{B^n} |\nabla u(x)| = \sup_{S^{n-1}} |\nabla u(x)|. \tag{2.35}$$

This estimate is better than the estimate (2.33), but the condition (2.34) is an essential one. For the details, see [12, Theorem 15.1].

Lemma 2.11. *If $x \geq 0$ is a solution of the inequality $x \leq ax^\alpha + b$, where $a \geq 1$ and $0 \leq a\alpha < 1$, then*

$$x \leq \frac{a + b - \alpha a}{1 - \alpha a}. \tag{2.36}$$

Observe that for $\alpha = 0$, (2.36) coincides with $x \leq a + b$, i.e., $x \leq ax^\alpha + b$.

Proof. We will use the Bernoulli's inequality $x \leq ax^\alpha + b = a(1 + x - 1)^\alpha + b \leq a(1 + \alpha(x - 1)) + b$. The relation (2.36) now easily follows. $\square$

3. The main results

Theorem 3.1. *Let $K \geq 1$ be arbitrary, $n \in \mathbf{N}$, and let $g \in C(\overline{B^n})$. Then there exists a constant $M' = M'(n, K)$ such that if u is K-quasiconformal self-mapping of the unit ball B^n satisfying the PDE $\Delta u = g$, with $u(0) = 0$, then*

$$|u(x) - u(y)| \leq M'|x - y|, \quad x, y \in B^n, \tag{3.1}$$

where $M' = M_1'(n, K) + M_2'(n, K)|g|_\infty$. Moreover, if u is harmonic, then $M'(n, K) \to 1$ as $K \to 1$.

Proof. Let $u(S(r, \theta)) = (y_1, y_1, \ldots, y_n)$, where S are the spherical coordinates. Combining Proposition 1.3 and Lemma 2.7, in the special case where the co-domain is the unit ball, we obtain that there exists ∇u and J_u almost everywhere in S^{n-1} and there holds the following inequality:

$$J_u(t) \leq \frac{D_\chi}{D_T} \left(\int_{S^{n-1}} \frac{|f(t) - f(\eta)|^2}{|\eta - t|^n} d\sigma(\eta) + \frac{|g|_\infty}{n} \right), \quad t \in S^{n-1}. \tag{3.2}$$

Now from

$$|\nabla u(S(r, \theta))|^n \leq K J_u(S(r, \theta))$$

we obtain

$$\lim_{r \to 1-} |\nabla u(S(r, \theta))|^n \leq \lim_{r \to 1-} K J_u(S(r, \theta)) \tag{3.3}$$

almost everywhere in Q^{n-1}. From Lemma 2.1 we deduce that

$$\begin{aligned} &\lim_{r \to 1-} \frac{\partial u \circ S}{\partial \theta_1}(r, \theta) \times \cdots \times \frac{\partial u \circ S}{\partial \theta_{n-2}}(r, \theta) \times \frac{\partial u \circ S}{\partial \varphi}(r, \theta) \\ &= \frac{\partial f \circ T}{\partial \theta_1}(\theta) \times \cdots \times \frac{\partial f \circ T}{\partial \theta_{n-2}}(\theta) \times \frac{\partial f \circ T}{\partial \varphi}(\theta) \end{aligned}$$

almost everywhere in Q^{n-1}. Since

$$\frac{\partial u\circ S}{\partial \theta_i}(r,\theta)=ru'(S(r,\theta))\frac{\partial T}{\partial \theta_i},$$

using (2.27) we obtain that

$$D_\chi(\theta)\le \lim_{r\to 1-}|\nabla u(S(r,\theta))|^{n-1}D_T(\theta). \tag{3.4}$$

From (3.2)–(3.4) we infer that

$$|\nabla u(T(\theta))|^n\le K|\nabla u(T(\theta)|^{n-1}\left(\int_{S^{n-1}}\frac{|f(T(\theta))-f(\eta)|^2}{|\eta-T(\theta)|^n}\,d\sigma(\eta)+\frac{|g|_\infty}{n}\right),$$

i.e.,

$$|\nabla u(T(\theta))|\le K\left(\int_{S^{n-1}}\frac{|f(T(\theta))-f(\eta)|^2}{|\eta-T(\theta)|^n}\,d\sigma(\eta)+\frac{|g|_\infty}{n}\right). \tag{3.5}$$

In view of Lemma 2.9, for every $\varepsilon>0$ there exists $\theta_\varepsilon\in Q^{n-1}$ such that

$$\begin{aligned}M&:=\operatorname{ess\,sup}\{|\nabla u(x)|\colon |x|<1\}\\ &\le (1-\varepsilon)^{-1}\left(|\nabla u(T(\theta_\varepsilon))|+(1+\frac{2\Gamma(\frac{n}{2})}{(n-1)\sqrt{\pi}\Gamma(\frac{n-1}{2})})|g|_\infty\right).\end{aligned} \tag{3.6}$$

The mean value theorem yields

$$|u(x)-u(y)|\le \operatorname*{ess\,sup}_{t\in B^n}|\nabla u(t)|\cdot|x-y|. \tag{3.7}$$

Let $\mu=K^{1/(1-n)}$. It is clear that $0<\mu\le 1$. Let $\gamma=1-n+\mu^2$, and let $\nu=1-\mu$. Now applying the relation (3.5) for $\theta=\theta_\varepsilon$, and using (1.5), (3.6) and (3.7), we obtain

$$\begin{aligned}&(1-\varepsilon)M-(1+\frac{2\Gamma(\frac{n}{2})}{(n-1)\sqrt{\pi}\Gamma(\frac{n-1}{2})})|g|_\infty\\ &\le K\left(M^\nu\int_{S^{n-1}}|\eta-T(\theta_\varepsilon)|^\gamma\frac{|f(T(\theta_\varepsilon))-f(\eta)|^{2-\nu}}{|T(\theta_\varepsilon)-\eta|^{\mu^2+\mu}}\,d\sigma(\eta)+\frac{|g|_\infty}{n}\right)\\ &\le KM^\nu M_1(n,K)^{1+\mu}\int_{S^{n-1}}|\eta-T(\theta_\varepsilon)|^\gamma\,d\sigma(\eta)+K\frac{|g|_\infty}{n}.\end{aligned}$$

Letting $\varepsilon\to 0$ we obtain

$$M\le M_2(n,K)M^\nu+M_3(n,K)|g|_\infty,$$

where

$$M_2(n,K)=KM_1(n,K)^{1+\mu}\int_{S^{n-1}}|\eta-T(\theta_0)|^\gamma\,d\sigma(\eta),$$

θ_0 is a fixed vector in Q^{n-1}, and

$$M_3(n,K)=(1+\frac{2\Gamma(\frac{n}{2})}{(n-1)\sqrt{\pi}\Gamma(\frac{n-1}{2})})+\frac{K}{n}.$$

First of all, there holds

$$\begin{aligned}M\le M_4&:=(M_2(n,K)+M_3(n,K)|g|_\infty)^{1/(1-\nu)}\\ &=(M_2(n,K)+M_3(n,K)|g|_\infty)^{K^{1/(n-1)}}.\end{aligned} \tag{3.8}$$

If $\nu M_2(K) < 1$, from Lemma 2.11 we get

$$M \leq M_5 := \frac{M_2(n,K) + M_3(n,K)|g|_\infty - \nu M_2(n,K)}{1 - \nu M_2(n,K)}. \tag{3.9}$$

Therefore, the inequality (3.1) does hold for

$$M' = \min(\{M_4\} \cup \{M_5 \colon \nu M_2(K) < 1\}).$$

Using (1.5), Lemma 2.8 and Lemma 2.4, it follows that $\lim_{K\to 1} M'(n,K) = 1$ if $g = 0$. □

Concerning the co-Lipschitz character of these mappings we have the following partial result.

Theorem 3.2. *Let $K < 2^{n-1}$ and assume that u is a K-q.c. solution of PDE $\Delta u = g$ that maps the unit ball onto itself satisfying the following conditions:*

i) $u \in C^1(\overline{B^n})$,

ii) $g \in C(\overline{B^n})$ *such that* $|g|_\infty < M_0(n,K)$, *where* $M_0(n,K)$ *is given in* (3.14).

Then u is co-Lipschitz.

Proof. From (2.14) we obtain

$$J_u(t) \geq \frac{D_\chi}{D_T}\int_{S^{n-1}} \frac{|f(t)-f(\eta)|^2}{|\eta - t|^n}\,d\sigma(\eta) - \frac{D_\chi}{D_T}\frac{|g|_\infty}{n}, \quad t \in S^{n-1}. \tag{3.10}$$

Using (2.27) we obtain

$$K^{1-n}\lim_{r\to 1-}|\nabla u(S(r,\theta))|^{n-1} \leq \frac{D_\chi}{D_T} \leq \lim_{r\to 1-}|\nabla u(S(r,\theta))|^{n-1}. \tag{3.11}$$

Combining (3.10) and (3.11) it follows that

$$\begin{aligned}\lim_{r\to 1-}|\nabla u(S(r,\theta))|^n \geq K^{-n}|\lim_{r\to 1-}|\nabla u(S(r,\theta))|^{n-1}\int_{S^{n-1}} \frac{|f(t)-f(\eta)|^2}{|\eta - t|^n}\,d\sigma(\eta)\\ - \lim_{r\to 1-}|\nabla u(S(r,\theta))|^{n-1}\frac{|g|_\infty}{n}, \quad t \in S^{n-1},\end{aligned}$$

i.e.,

$$\lim_{r\to 1-}|\nabla u(S(r,\theta))| \geq K^{-n}\int_{S^{n-1}} \frac{|f(t)-f(\eta)|^2}{|\eta - t|^n}\,d\sigma(\eta) - \frac{|g|_\infty}{n}, \quad t \in S^{n-1}. \tag{3.12}$$

As u^{-1} is K-q.c., using (1.5) and (3.12) we get

$$\lim_{r\to 1-}|\nabla u(S(r,\theta))| \geq \frac{M_0(n,K) - |g|_\infty}{n}, \quad t \in S^{n-1}, \tag{3.13}$$

where

$$M_0(n,K) = \int_{S^{n-1}} \frac{nK^{1-n}(M_1(n,K))^{2K^{1/(1-n)}}}{|\eta - t|^{n-2K^{1/(n-1)}}}\,d\sigma(\eta). \tag{3.14}$$

The rest of the proof follows from the condition i) and [33, Lemma 4.5]. □

4. Examples of quasiconformal self-mappings of B^n satisfying the Poisson equation

In this section we give some examples of quasiconformal and harmonic mappings on the space that are not trivial, i.e., that are not linear transformation of the space. We also deduce some sufficient conditions for radial homeomorphisms to be quasiconformal solutions of a Poisson's equation.

a) Let $B^{2n} \subset \mathbf{C}^n$ and let f be a holomorphic automorphism of the unit ball B^{2n} onto itself. Then f is a q.c. harmonic mapping. To prove this fact, observe that $\bar{\partial} f = 0 \Rightarrow \partial\bar{\partial} f = 0$. Also, f has a holomorphic extension up to the boundary. This means that it is bi-Lipschitz. Therefore, f is a q.c. harmonic mapping. In this setting it is interesting to note that the composition of harmonic and holomorphic mapping is itself harmonic.

b) Define $I_\delta(x) = x + \delta(x)$, where $x \in B^n$ and $\delta(x) \in B^n$, and take

$$\phi_\delta = I_\delta / |J_\delta|,$$

where

$$|J_\delta|^2 = 1 + 2\,\langle x, \delta(x)\rangle + |\delta(x)|^2.$$

Let $\Phi_\delta(x) = P[\phi_\delta](x)$, where ϕ_δ is C^2 smooth perturbation of the identity mapping of the unit sphere onto itself. It was shown in [19] that $\Phi_\delta(x)$ is a quasiconformal harmonic mapping if δ is close enough to zero mapping in C^2 norm.

In the example (c) given below it is shown that for the class of radial twice differentiable q.c. selfmappings of the unit ball (which is quite large), Theorem 3.1 yields also a sufficient condition. In its particular case (d) it is shown that the condition $K < 2^n$ of Theorem 3.2 is the best possible.

c) Consider now $u(x) = h(|x|)x$, where $r \mapsto rh(r)$ is a twice differentiable diffeomorphism of $[0, 1)$ onto itself. Then for $r = |x|$,

$$J_u(x) = h^n(r)\left(1 + \frac{h'(r)}{h(r)} r\right), \tag{4.1}$$

and

$$|\nabla u(x)|^n = h^n(r)\left(1 + \frac{h'(r)}{h(r)} r\right)^n. \tag{4.2}$$

From (4.1) and (4.2) we obtain

$$\frac{|\nabla u(x)|^n}{J_u(x)} = \left(1 + \frac{h'(r)}{h(r)} r\right)^{n-1}.$$

Thus, u is a self-mapping of the unit ball satisfying PDE

$$\Delta u(x) = g(x) := \left(h''(r) + \frac{(n+1)h'(r)}{r}\right) x,$$

and it is quasiconformal if and only if

$$\limsup_{r\to 1} h'(r) < \infty, \tag{4.3}$$

or what is the same if and only if $|\nabla u(x)|$ is bounded.

d) Take $u(x) = |x|^\alpha x$ with $\alpha \geq 1$. Then

$$J_u(x) = (1+\alpha)|x|^{n\alpha}, \tag{4.4}$$

and

$$|\nabla u(x)| = (\alpha + 1)|x|^{\alpha}. \tag{4.5}$$

By (4.4) and (4.5) it follows that

$$\frac{|\nabla u(x)|^n}{J_u(x)} = (\alpha + 1)^{n-1}.$$

Therefore, u is twice differentiable $(1+\alpha)^{n-1}$-quasiconformal self-mapping of the unit ball with $J_u(0) = 0$. This means that the constant 2^{n-1} is the best possible.

e) It was suggested to me by professor Mateljević [28] that the Kalvin transform of the identity, i.e., the mapping

$$f(x) = |x|^{-n}x,$$

is a quasiconformal harmonic mappings of the extended space to itself with maximal dilatation equal to $n - 1$. Namely, for $|h| = 1$,

$$f'(x)h = \frac{h}{|x|^n} - \frac{nx\,\langle x, h\rangle}{|x|^{n+2}},$$

and this implies that

$$|f'(x)h|^2 = \frac{|h|^2}{|x|^{2n}} + (n^2 - 2n)\frac{\langle x, h\rangle^2}{|x|^{2n+2}}.$$

Therefore,

$$\frac{1}{|x|^{2n}} \leq |f'(x)h|^2 \leq \frac{(n-1)^2}{|x|^{2n}}. \tag{4.6}$$

In view of (2.29), this implies the fact that f is a $n-1$-quasiconformal mapping. We conclude that the Kelvin transform of every quasiconformal harmonic mapping is a quasiconformal harmonic mapping. Let φ be a Möbius transformation of the unit ball B^n onto the upper halfspace H^n. Then the mapping $u(x) = \varphi^{-1} \circ f \circ \varphi$ is a C^∞ $n - 1$-quasiconformal mapping of the unit ball onto itself. Namely,

$$\varphi(x) = A[(x - b)^* - (a - b)^*]$$

for some constant conformal matrix A, a, b are some vectors from the unit sphere. Here $y^* := y/|y|^2$ for $y \neq 0$, see [1]. An example of a mapping of the unit ball onto the upper half-space is the mapping

$$\varphi(x) = (x - S)^* + S/2,$$

where $S = (0, 0, \ldots, 0, -1)$ is the south pole of the sphere. It satisfies the condition $\varphi(S) = \infty$, $\varphi(N) = 0$, where N is the north pole and

$$\psi(y) = \varphi^{-1}(y) = (y - S/2)^* + S.$$

Now we have

$$u'(x) = \psi'(f) \cdot f'(\varphi) \cdot \varphi'(x).$$

On the other hand,

$$|\varphi'(x)| = \frac{1}{|x - S|^2},$$

and

$$|\psi'(f)| = \frac{1}{|f - S/2|^2}.$$

From (4.6) it follows that

$$|u'(x)| \le (n-1)\frac{1}{|f(\varphi(x)) - S/2|^2} \cdot \frac{1}{|\varphi(x)|^n} \cdot \frac{1}{|x-S|^2}. \tag{4.7}$$

Since $|f(\varphi(x)) - S/2| \ge 1/2$ for $x \in B^n$, the only points at which the right hand side of (4.7) is possibly unbounded are the points S and N. However, it is straightforward that these points are removable singularities of the right hand side of (4.7), namely $u'(S) = 0$ and $u'(N) = 0$. Thus, u is Lipschitz but it is not bi-Lipschitz. It seems that $\Delta u \in L^\infty(B^n)$, but I didn't verify this fact.

On the other hand, the mapping $v = u^{-1} = \varphi^{-1} \circ f^{-1} \circ \varphi$ is a $(n-1)$-quasiconformal mapping of the unit ball onto itself such that it is not Lipschitz and therefore, $\Delta v \notin L^\infty(B^n)$. Recall that we have a priori the assumption that $n > 2$. The case $n = 2$ is excluded because the mappings u and v are anti-conformal.

Remark 4.1. The condition (4.3) together with the condition $r \mapsto rh(r)$ is a twice differentiable diffeomorphism of $[0,1)$ onto itself, implies that $g = \Delta u(x) \in L^1(B^n)$ but not $g \in L^\infty(B^n)$ which is a subject of our a priori conditions throughout the paper. The question arises, can the condition $g \in L^\infty(B^n)$ be replaced by $g = \Delta u(x) \in L^p(B^n)$ (for some p in our context).

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