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QUASISYMMETRICALLY MINIMAL MORAN SETS AND HAUSDORFF DIMENSION

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Abstract. In this paper, we prove that a large class of Moran sets on the line with Hausdorff dimension 1 are 1-dimensional quasisymmetrically minimal. We also obtain a general theorem on the Hausdorff dimension of Moran set on the line.

1. Introduction

Let (X, d_X), (Y, d_Y) be metric spaces. A topological homeomorphism $f\colon X \to Y$ is called quasisymmetric if there is a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that

$$\frac{d_Y\big(f(x), f(a)\big)}{d_Y\big(f(x), f(b)\big)} \le \eta\left(\frac{d_X(x,a)}{d_X(x,b)}\right)$$

for all triples a, b, x of distinct points in X. In particular, we also say that f is an n-dimensional quasisymmetric mapping when $X = Y = \mathbf{R}^n$. Quasisymmetry is an important notion in the theory of analysis on metric spaces ([9]) and complex analysis ([2]). It is the generalization of quasiconformality in Euclidean spaces to general metric spaces (see [19]).

Unlike bi-Lipschitz mappings, quasisymmetric mappings do not preserve Hausdorff dimension. So there is a natural problem that *how the quasisymmetric mappings change the Hausdorff dimension*. Many efforts have been devoted to this problem, especially in Euclidean spaces. For example, if $\dim_{\mathrm{H}} E = 0$, then $\dim_{\mathrm{H}} f(E) = 0$ for any quasisymmetric mapping f, since f is Hölder continuous (see [1]); if $0 < \dim_{\mathrm{H}} E < n$, Bishop [3] showed that for any $\varepsilon > 0$ there is an n-dimensional quasisymmetric mapping f such that $\dim_{\mathrm{H}} f(E) > n - \varepsilon$; Tyson and Wu [21] obtained that the Sierpinski

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gasket can be mapped by a 2-dimensional quasisymmetric mapping onto a set with Hausdorff dimension arbitrarily close to one.

We call a set $E \subset \mathbf{R}^n$ *quasisymmetrically minimal* if

$$\dim_{\mathrm{H}} f(E) \geq \dim_{\mathrm{H}} E$$

for any n-dimensional quasisymmetric map f. Let $E \subset \mathbf{R}^n$ and f be an n-dimensional quasisymmetric mapping. The following are some known facts.

(1) Tyson [20] proved that for all $\alpha \in [1, n]$, there exists a set $E \subset \mathbf{R}^n$ which is quasisymmetrically minimal with $\dim_{\mathrm{H}} E = \alpha$.
(2) Kovalev [14] pointed out that there is *no* quasisymmetrically minimal set E with $0 < \dim_{\mathrm{H}} E < 1$.
(3) If $n \geq 2$, Gehring et al. [6, 7] showed that n-dimensional quasisymmetric mappings preserve sets of Hausdorff dimension n.
(4) However, when $n = 1$, Tukia [18] obtained that there exists $E \subset \mathbf{R}$ of Hausdorff dimension 1 with $\dim_{\mathrm{H}} f(E) < 1$ for some 1-dimensional quasisymmetric mapping f.

From the above results, we will focus attention on the question that *which sets in* $\mathbf{R}$ *of Hausdorff dimension* 1 *are minimal*. There are some related result:

(1) The first known examples of minimal subsets of $\mathbf{R}$ are quasisymmetrically thick sets. Recall from [17], a set $E \in \mathbf{R}$ is called a *quasisymmetrically thick set* if $f(E)$ has positive Lebesgue measure for all quasisymmetric mapping f.
(2) Hakobyan [8] proved that middle interval Cantor sets of Hausdorff dimension 1 are all minimal. It was shown that these sets need not be quasisymmetrically thick sets.
(3) Recently, Hu and Wen [10] extended the results in [8]. They proved that uniform Cantor sets of Hausdorff dimension 1 are minimal.

It is worth noting that all minimal sets appearing in [8, 10] are some special kinds of Moran sets—homogeneous Cantor sets (see Definition 1 and 2).

In this paper, we will show that the results of [8, 10] are no accidents. In fact, a large class of Moran sets on the line with Hausdorff dimension 1 are minimal (Theorem 1). The main tool in the proof of Theorem 1 is some Gibbs-like measures. Moveover, the measures are also useful to determine the Hausdorff dimension of Moran sets. With this measure in hand, we can generalize some classic results in [13] on the Hausdorff dimension of Moran sets (Theorem 2).

This paper is organized as follows. In the rest of Section 1, we state Theorems 1 and 2 in Section 1.1, before the introduction to the Moran sets (Section 1.2). In Section 2, we introduce the so called Gibbs-like measures. The proof of Theorem 1 is given in Section 3, which based on some ideas of Hakobyan [8]. In Section 4.1, we prove Theorem 2. Some remarks on the Hausdorff dimension of Moran sets appear in Section 4.2.

1.1. Main results. With the technical notations and definition of Moran set in Section 1.2, we state our main results. The first one concerns the quasisymmetrically minimal sets on the line.

Theorem 1. *Let $E \in \mathscr{M}(J, \{n_k\}, \{c_{k,j}\})$. If $s_* = 1$, $\sup_k n_k < \infty$ and there exists a constant $\alpha \in (0,1)$ such that*

$$\liminf_{n\to\infty} \frac{\operatorname{card}\{1 \le k \le n \colon D_k \le \alpha\}}{n} > 0, \tag{1.1}$$

then $\dim_{\mathrm{H}} E = 1$ and E is minimal for 1-dimensional quasisymmetric mapping.

Remark 1. Theorem 1 includes the results in [8, 10]. In fact, [8] requires $n_k \equiv 2$, $c_{k,1} = c_{k,2} = D_k < 1/2$ for all k, [10] requires $\sup_k n_k < \infty$, $c_{k,1} = \cdots = c_{k,n_k} = D_k \le 1/2$ for all k, and all the minimal sets in [8, 10] are homogeneous Cantor sets (see Section 1.2) with $s_* = 1$. In above cases, $s_* = 1$, $\sup_k n_k < \infty$ and (1.1) holds for $\alpha = 1/2$. Therefore the conditions in Theorem 1 is much weaker than those in [8, 10].

As far as we know, there is no theorem to ensure $\dim_{\mathrm{H}} E = 1$ under the conditions of Theorem 1. This enlightens a more general theorem on the Hausdorff dimension of Moran sets in the Moran class $\mathscr{M}(J, \{n_k\}, \{c_{k,j}\})$.

Theorem 2. *Let $E \in \mathscr{M}(J, \{n_k\}, \{c_{k,j}\})$. If*

$$\lim_{k\to\infty} \frac{\log(kn_k)}{-\sum_{i=1}^{k} \log D_i} = 0, \tag{1.2}$$

then $\dim_{\mathrm{H}} E = s_*$. Moreover, when $s_* = 1$, the condition that

$$\sum_{k=1}^{\infty} n_k \left(\prod_{i=1}^{k} D_i\right)^{\delta} < +\infty \quad \textit{for some } \delta \in (0,1) \tag{1.3}$$

ensures $\dim_{\mathrm{H}} E = 1$.

Remark 2. The condition (1.2) is equivalent to

$$\sum_{k=1}^{\infty} n_k \left(\prod_{i=1}^{k} D_i\right)^{\delta} < +\infty \quad \text{for all } \delta > 0. \tag{1.4}$$

So the condition (1.3) is weaker than the condition (1.2).

Remark 3. By Theorem 2, it is plain to see that $\dim_{\mathrm{H}} E = 1$ under the conditions of Theorem 1. In fact, $\sup_k n_k < \infty$ implies $\log(kn_k) \sim \log k$ as $k \to \infty$, and conation (1.1) implies $\sum_{i=1}^{k} \log D_i = O(k)$ as $k \to \infty$. And so the condition of Theorem 2 follows.

Remark 4. In the proof of Theorem 2, we can loosen the restriction $n_k \ge 2$ and $c_{k,j} \in (0,1)$ in the definition of Moran sets, here we permit the case that $n_k = 1$ or $c_{k,j} = 1$.

1.2. Definition of Moran sets. Let $\{n_k\}_{k\ge1}$ be a sequence of positive integers and $\{c_{k,j}\}_{k\ge1, 1\le j\le n_k}$ a sequence of positive numbers satisfying $n_k \ge 2$ and $c_{k,j} \in (0,1)$ for all $k \ge 1$, $1 \le j \le n_k$. Write

$$D_k = \max_{1\le j\le n_k} c_{k,j}, \quad d_k = \min_{1\le j\le n_k} c_{k,j}, \quad c_* = \inf_{k,j} c_{k,j} \tag{1.5}$$

and

$$s_* = \liminf_{k\to\infty} s_k \quad \text{and} \quad s^* = \limsup_{k\to\infty} s_k, \tag{1.6}$$

where s_k is defined by the equation

$$\prod_{i=1}^{k}\sum_{j=1}^{n_i} c_{i,j}^{s_k} = 1. \tag{1.7}$$

Let $\Omega_0 = \{\emptyset\}$, where $\emptyset$ is the empty word. For any positive integer k, let

$$\Omega_k = \big\{(\sigma_1, \ldots, \sigma_k) \colon \sigma_j \in [1, n_j] \cap \mathbf{N} \text{ for } 1 \le j \le k\big\}.$$

Define $\Omega = \bigcup_{k\ge 0} \Omega_k$. For any integers l, k with $l > k \ge 1$, let

$$\Omega_{k,l} = \big\{(\tau_{k+1}, \ldots, \tau_l) \colon \tau_j \in [1, n_j] \cap \mathbf{N} \text{ for } k+1 \le j \le l\big\}.$$

Define $\sigma * \tau = (\sigma_1, \ldots, \sigma_k, \tau_{k+1}, \ldots, \tau_l) \in \Omega_l$ for any $\sigma \in \Omega_k$ and $\tau \in \Omega_{k,l}$. The length of $\sigma \in \Omega_k$ will be denoted by $|\sigma| (= k)$ and the diameter of set $A \subset \mathbf{R}^n$ will be denoted by $|A|$. For convenience, we also use $\sigma_1 \ldots \sigma_i$ to denote $(\sigma_1, \ldots, \sigma_i)$.

Definition 1. (Moran set) Suppose that $J \subset \mathbf{R}$ is a closed interval. For a collection $\mathscr{F} = \{J_\sigma \colon \sigma \in \Omega\}$ of closed subintervals of J with $J_\emptyset = J$, we say $\mathscr{F}$ has Moran structure, if for all $k \ge 1$, there are constants $c_{k,1}, \cdots, c_{k,n_k}$ such that for any $\sigma \in \Omega_{k-1}$, $J_{\sigma*1}, J_{\sigma*2}, \cdots, J_{\sigma*n_k}$ are subintervals of J_σ with their interiors pairwise disjoint, and for any $1 \le j \le n_k$,

$$|J_{\sigma*j}|/|J_\sigma| = c_{k,j}.$$

A Moran set determined by $\mathscr{F}$ is defined by

$$E(\mathscr{F}) := \bigcap_{k\ge 1} \bigcup_{\sigma\in\Omega_k} J_\sigma.$$

Here any J_σ in $\mathscr{F}$ is called a basic element of E. Denote by $\mathscr{M}(J, \{n_k\}, \{c_{k,j}\})$ the class of all Moran sets associated with J, $\{n_k\}$ and $\{c_{k,j}\}$.

Definition 2. (Homogenous Cantor set) We call $E(\mathscr{F})$ a homogeneous Cantor set, if furthermore the Moran structure $\mathscr{F}$ satisfies the following conditions:

(i) for any k, $c_{k,j}$ take the same value c_k independent of j;
(ii) for any $k \ge 1$ and $\sigma \in \Omega_{k-1}$, the gaps between $J_{\sigma*j}$ and $J_{\sigma*(j+1)}$ $(1 \le j < n_k)$ are equal;
(iii) for any $k \ge 1$ and $\sigma \in \Omega_{k-1}$, the left endpoint of $J_{\sigma*1}$ is the same as that of J_σ, and the right endpoint of $J_{\sigma*n_k}$ is the same as that of J_σ.

Some special cases of Moran sets were first studied by Moran [15]. The later works [5, 11, 12, 13, 16, 22] developed the theory on the geometrical structure and dimensions of Moran sets systematically. Roughly speaking, the Moran sets generalize the classic self-similar sets from the following points (see Definition 1):

- the placements of the basic sets at each step of the construction can be arbitrary;
- the contraction ratios may be different at each step;
- the lower limit of the contraction ratios permits zero.

Sometimes, these generalizations make it possible to find a Moran subset B in a given fractal set A with $\dim B = \dim A$. As a result, the theory on the dimensions of Moran sets has become a powerful tool in dimension computation.

When $c_* > 0$, Hua et al. [11, 12] showed that

$$\dim_{\mathrm{H}} E = s_* \quad \text{and} \quad \dim_{\mathrm{P}} E = \overline{\dim}_{\mathrm{B}} E = s^* \tag{1.8}$$

for all $E \in \mathscr{M}(J, \{n_k\}, \{c_{k,j}\})$ (for s_* and s^*, recall (1.6)). When $c_* = 0$, Hua et al. [13] also obtained two sufficient conditions under which the dimension formula (1.8) holds (see Theorems A and B in Section 4.2). However, except for the two sufficient conditions, it is little known about the dimensions of Moran sets in the case of $c_* = 0$. We even don't know what conditions ensure that all Moran sets in $\mathscr{M}(J, \{n_k\}, \{c_{k,j}\})$ assume the same Hausdorff (or packing) dimension. It seems very difficult to give a complete answer to this problem. In Theorem 2, we give a more general sufficient condition compared with the two conditions in [13] (see Section 4.2 for details). We hope this result leads to some deep discoveries on the dimensions of Moran sets.

2. The Gibbs-like measures

Let $E \in \mathscr{M}(J, \{n_k\}, \{c_{k,j}\})$. For every $d \in (0,1)$, by the extension theorem of measures, there exists a unique probability measure μ supported on E such that $\mu(J_\emptyset) = \mu(E) = 1$ and

$$\mu(J_{\sigma * j'}) = \mu(J_\sigma) \cdot \frac{|J_{\sigma * j'}|^d}{\sum_{j=1}^{n_k} |J_{\sigma * j}|^d} \tag{2.1}$$

for all $k \geq 1$, $\sigma \in \Omega_{k-1}$ and $1 \leq j' \leq n_k$. Similarly, for very $d \in (0,1)$ and every 1-dimensional quasisymmetric mapping f, there also exists a unique probability measure ν supported on $f(E)$ such that $\nu(f(J_\emptyset)) = \nu(f(E)) = 1$ and

$$\nu(f(J_{\sigma * j'})) = \nu(f(J_\sigma)) \cdot \frac{|f(J_{\sigma * j'})|^d}{\sum_{j=1}^{n_k} |f(J_{\sigma * j})|^d} \tag{2.2}$$

for all $k \geq 1$, $\sigma \in \Omega_{k-1}$ and $1 \leq j' \leq n_k$.

The measures μ and ν are so called Gibbs-like measures. We have some lemmas on the properties of measures μ and ν.

Lemma 1. *Let $k \geq 1$. Suppose that $\sigma \in \Omega_k$ and $d \leq s_k$, then*

$$\frac{|J_\sigma|^d}{\mu(J_\sigma)} \geq \prod_{i=1}^{k} D_i^{d - s_k},$$

where μ is the Gibbs-like measure defined by (2.1).

Proof. By (2.1), we have

$$\begin{aligned}\mu(J_\sigma) &= |J_\sigma|^d \frac{|J_{\sigma_1 \ldots \sigma_{k-1}}|^d}{|J_{\sigma_1 \ldots \sigma_{k-1} * 1}|^d + \cdots + |J_{\sigma_1 \ldots \sigma_{k-1} * n_k}|^d} \cdots \frac{1}{|J_1|^d + \cdots + |J_{n_1}|^d} \\ &= |J_\sigma|^d \frac{1}{c_{k,1}^d + \cdots + c_{k,n_k}^d} \cdots \frac{1}{c_{1,1}^d + \cdots + c_{1,n_1}^d}.\end{aligned}$$

By the definition of s_k (see (1.7)), we have

$$\frac{|J_\sigma|^d}{\mu(J_\sigma)} = \prod_{i=1}^{k} \sum_{j=1}^{n_i} c_{i,j}^d = \prod_{i=1}^{k} \frac{\sum_{j=1}^{n_i} c_{i,j}^d}{\sum_{j=1}^{n_i} c_{i,j}^{s_k}} \geq \prod_{i=1}^{k} D_i^{d - s_k},$$

since $D_i = \max_{1 \leq j \leq n_i} c_{i,j}$ and $d \leq s_k$. $\square$

Let $d \in (0,1)$ and f be a 1-dimensional quasisymmetric mapping. For $k > 1$ and $\sigma \in \Omega_{k-1}$, write

$$\phi_\sigma = \frac{\sum_{j=1}^{n_k} |f(J_{\sigma * j})|^d}{\left(\sum_{j=1}^{n_k} |f(J_{\sigma * j})|\right)^d} \quad \text{and} \quad \varphi_\sigma = \frac{\sum_{j=1}^{n_k} |f(J_{\sigma * j})|}{|f(J_\sigma)|}. \tag{2.3}$$

For $\sigma = \emptyset$, write

$$\phi_\emptyset = \frac{\sum_{j=1}^{n_1} |f(J_j)|^d}{\left(\sum_{j=1}^{n_1} |f(J_j)|\right)^d} \quad \text{and} \quad \varphi_\emptyset = \sum_{j=1}^{n_1} |f(J_j)|. \tag{2.4}$$

By a similar argument as in the proof of Lemma 1, we have

Lemma 2. *Let $d \in (0,1)$ and f be a 1-dimensional quasisymmetric mapping, then for all $k \geq 1$ and $\sigma \in \Omega_k$,*

$$\frac{|f(J_\sigma)|^d}{\nu(f(J_\sigma))} = \phi_\emptyset \varphi_\emptyset^d \cdot \prod_{i=1}^{k-1} \phi_{\sigma_1 \dots \sigma_i} \varphi_{\sigma_1 \dots \sigma_i}^d,$$

where ν is the Gibbs-like measure defined by (2.2).

We need the following lemma to estimate ϕ_σ and φ_σ, which is an invariant formulation of Lemma 1 in Wu [23].

Lemma 3. *Let f be a 1-dimensional quasisymmetric mapping. Then*

$$\gamma \frac{|J|^q}{|I|^q} \leq \frac{|f(J)|}{|f(I)|} \leq 4 \frac{|J|^p}{|I|^p}$$

for all intervals I, J with $J \subset I$, where γ, p, q are three constants dependent on f with $\gamma > 0$, $0 < p \leq 1 \leq q$.

Lemmas 4 and 5 are devoted to the lower bound of ϕ_σ and φ_σ.

Lemma 4. *Let $k \geq 1$, $\sigma \in \Omega_{k-1}$ and ϕ_σ as in (2.3) and (2.4), then $\phi_\sigma > 1$. Moreover, if $\sum_{j=1}^{n_k} c_{k,j} \geq \beta > \alpha \geq D_k$ for some constants α, β with $1 > \beta > \alpha > 0$, then $\phi_\sigma > 1 + \epsilon$, where $\epsilon > 0$ is a constant dependent on α, β, n_k and f.*

Proof. Since $d \in (0,1)$, it is obvious that $\phi_\sigma > 1$. Now suppose that $\sum_{j=1}^{n_k} c_{k,j} \geq \beta > \alpha \geq D_k$. Without loss of generality, we assume that

$$M_\sigma = \max_{1 \leq j \leq n_k} |f(J_{\sigma * j})| = |f(J_{\sigma * 1})|.$$

Write

$$x = \frac{\sum_{j=2}^{n_k} |f(J_{\sigma * j})|}{M_\sigma}.$$

By Lemma 3,

$$\frac{\sum_{j=2}^{n_k} |f(J_{\sigma * j})|}{|f(J_\sigma)|} \geq \gamma \sum_{j=2}^{n_k} c_{k,j}^q \geq \gamma (n_k - 1)^{1-q} \left(\sum_{j=2}^{n_k} c_{k,j} \right)^q \geq \frac{\gamma (\beta - \alpha)^q}{n_k^q},$$

and $M_\sigma \cdot |f(J_\sigma)|^{-1} \leq 4 D_k^p \leq 4\alpha^p$. Therefore,

$$x \geq \frac{\gamma (\beta - \alpha)^q}{4 \alpha^p n_k^q} > 0. \tag{2.5}$$

We return to estimate ϕ_σ. Since $|f(J_{\sigma*j})|\cdot M_\sigma^{-1}\le 1$ for $1\le j\le n_k$, we have

$$\phi_\sigma=\frac{\sum_{j=1}^{n_k}|f(J_{\sigma*j})|^d\cdot M_\sigma^{-d}}{\left(\sum_{j=1}^{n_k}|f(J_{\sigma*j})|\cdot M_\sigma^{-1}\right)^d}\ge\frac{\sum_{j=1}^{n_k}|f(J_{\sigma*j})|\cdot M_\sigma^{-1}}{\left(\sum_{j=1}^{n_k}|f(J_{\sigma*j})|\cdot M_\sigma^{-1}\right)^d}=(1+x)^{1-d}.$$

Together with (2.5), we complete the proof. □

Lemma 5. *Let $k>1$, $\sigma\in\Omega_{k-1}$ and φ_σ as in (2.3). Let γ,p,q be as in Lemma 3, then $\varphi_\sigma\ge\gamma n_k^{1-q}\big(\sum_{j=1}^{n_k}c_{k,j}\big)^q$. Moreover, when $\sum_{j=1}^{n_k}c_{k,j}$ is sufficiently close to 1, the following lower bound is useful:*

$$\varphi_\sigma\ge 1-8n_k\Big(1-\sum_{j=1}^{n_k}c_{k,j}\Big)^p. \tag{2.6}$$

Proof. By Lemma 3 and (2.3), it follows that

$$\varphi_\sigma\ge\gamma\sum_{j=1}^{n_k}c_{k,j}^q\ge\gamma n_k^{1-q}\Big(\sum_{j=1}^{n_k}c_{k,j}\Big)^q.$$

We now turn to the other lower bound. Let $\Delta_0,\dots,\Delta_{n_k}$ (some Δ_j may be empty) be the connected components of $J_\sigma\setminus\bigcup_{j=1}^{n_k}J_{\sigma*j}$ and $\delta_j=|\Delta_j|/|J_\sigma|$. Then $\sum_{j=1}^{n_k}c_{k,j}+\sum_{j=0}^{n_k}\delta_j=1$. Together with Lemma 3, we have

$$\begin{aligned}\varphi_\sigma&=\frac{\sum_{j=1}^{n_k}|f(J_{\sigma*j})|}{|f(J_\sigma)|}=\frac{|f(J_\sigma)|-\sum_{j=0}^{n_k}|f(\Delta_j)|}{|f(J_\sigma)|}\ge 1-4\sum_{j=0}^{n_k}\delta_j^p\\&\ge 1-4(n_k+1)\Big(\sum_{j=0}^{n_k}\delta_j\Big)^p\ge 1-8n_k\Big(1-\sum_{j=1}^{n_k}c_{k,j}\Big)^p.\end{aligned}$$

□

3. 1-dimensional quasisymmetrically minimal set

This section is devoted to the proof of Theorem 1. We begin with some lemmas.

Lemma 6. *Suppose that $\lim_{k\to\infty}s_k=1$ and $\sup_k n_k<\infty$. Then*

(a) $\lim_{k\to\infty}k^{-1}\sum_{i=1}^k\log\sum_{j=1}^{n_i}c_{i,j}=0$;

(b) $\lim_{k\to\infty}k^{-1}\,\mathrm{card}\big\{1\le i\le k\colon\ \sum_{j=1}^{n_i}c_{i,j}<\beta\big\}=0$ *for any* $\beta\in(0,1)$;

(c) $\lim_{k\to\infty}k^{-1}\sum_{i=1}^k\big(1-\sum_{j=1}^{n_i}c_{i,j}\big)^p=0$ *for any* $p>0$.

Proof. (a) By the Jensen's inequality,

$$n_i^{-1}\sum_{j=1}^{n_i}c_{i,j}\ge\Big(n_i^{-1}\sum_{j=1}^{n_i}c_{i,j}^{s_k}\Big)^{1/s_k}\quad\text{for all } 1\le i\le k,$$

since $s_k\le 1$. By (1.7), a simple computation shows that

$$\frac1k\sum_{i=1}^k\log\sum_{j=1}^{n_i}c_{i,j}\ge\frac{\sum_{i=1}^k\log n_i}{k}\left(1-s_k^{-1}\right)\ge\left(1-s_k^{-1}\right)\cdot\log\sup_{i\ge1}n_i.$$

Together with $\sum_{j=1}^{n_i}c_{i,j}\le 1$ and $\lim_{k\to\infty}s_k=1$, we have

$$0\ge\lim_{k\to\infty}\frac1k\sum_{i=1}^k\log\sum_{j=1}^{n_i}c_{i,j}\ge\lim_{k\to\infty}\left(1-s_k^{-1}\right)\cdot\log\sup_{i\ge1}n_i=0.$$

(b) It follows immediately from (a).

(c) For any $\beta \in (0,1)$, we have

$$\frac{1}{k}\sum_{i=1}^{k}\Big(1-\sum_{j=1}^{n_i} c_{i,j}\Big)^p \le (1-\beta)^p + \frac{1}{k}\operatorname{card}\Big\{1\le i\le k\colon \sum_{j=1}^{n_i} c_{i,j} < \beta\Big\}.$$

Then (c) follows from (b) and the arbitrariness of β. □

Lemma 7. *Let $d \in (0,1)$ and f be a 1-dimensional quasisymmetric mapping. Let ν be the Gibbs-like measure defined by (2.2). Suppose that the conditions in Theorem 1 hold, then there exists a constant $\zeta > 0$ such that for sufficiently large k,*

$$\frac{|f(J_\sigma)|^d}{\nu(f(J_\sigma))} > (1+\zeta)^k \quad \text{for all } \sigma \in \Omega_k.$$

Proof. Let α be as in (1.1). Pick a $\beta \in (\alpha, 1)$. Then by (1.1) and Lemma 6 (b),

$$\liminf_{k\to\infty} \frac{1}{k}\operatorname{card}\Big\{1\le i\le k\colon \sum_{j=1}^{n_i} c_{i,j} \ge \beta > \alpha \ge D_i\Big\} > \lambda > 0.$$

Together with Lemma 4 and $\sup_{i\ge 1} n_i < \infty$, it follows that there exists $\epsilon > 0$ such that for sufficiently large k,

$$\prod_{i=0}^{k-1} \phi_{\sigma_1\ldots\sigma_i} > (1+\epsilon)^{\lambda k} \quad \text{for all } \sigma \in \Omega_k. \tag{3.1}$$

By Lemma 2 and (3.1), to complete the proof, it suffices to show that for every $\varepsilon > 0$,

$$\varepsilon > \frac{1}{k}\sum_{i=0}^{k-1} \log \varphi_{\sigma_1\ldots\sigma_i} > -\varepsilon, \tag{3.2}$$

for sufficiently large k and all $\sigma \in \Omega_k$.

The left inequality is obvious since $\varphi_\tau \le 1$ for all $\tau \ne \emptyset$. For the right one, we apply Lemma 5. Since $\sup_i n_i < \infty$, we can pick a $\beta' \in (0,1)$ such that

$$8\sup_{i\ge 1} n_i \cdot (1-\beta')^p < \frac{1}{2}, \tag{3.3}$$

where p is as in (2.6). For $k \ge 1$, write

$$\Lambda_k = \Big\{1\le i\le k\colon \sum_{j=1}^{n_i} c_{i,j} \ge \beta'\Big\} \quad \text{and} \quad \Lambda_k^* = \Big\{1\le i\le k\colon \sum_{j=1}^{n_i} c_{i,j} < \beta'\Big\}.$$

Then by Lemma 5, for every $\sigma \in \Omega_k$, we have

$$\begin{aligned} \frac{1}{k}\sum_{i\in\Lambda_k^*} \log \varphi_{\sigma_1\ldots\sigma_{i-1}} &\ge \frac{1}{k}\sum_{i\in\Lambda_k^*} \log\left(\gamma n_i^{1-q}\Big(\sum_{j=1}^{n_i} c_{i,j}\Big)^q\right) \\ &\ge \frac{1}{k}\operatorname{card}\Lambda_k^* \cdot \Big(\log\gamma + (1-q)\log\sup_{i\ge 1} n_i\Big) + \frac{q}{k}\sum_{i=1}^{k}\log\sum_{j=1}^{n_i} c_{i,j} \to 0, \end{aligned} \tag{3.4}$$

as $k \to +\infty$, according to (b) and (a) of Lemma 6. Also by Lemma 5, for every $\sigma \in \Omega_k$,

$$
\begin{aligned}
\frac{1}{k}\sum_{i\in\Lambda_k}\log\varphi_{\sigma_1\ldots\sigma_{i-1}} &\geq \frac{1}{k}\sum_{i\in\Lambda_k}\log\left(1-8n_i\Big(1-\sum_{j=1}^{n_i}c_{i,j}\Big)^p\right)\\
&\geq -\frac{16}{k}\sum_{i\in\Lambda_k}n_i\Big(1-\sum_{j=1}^{n_i}c_{i,j}\Big)^p \geq -16\sup_{i\geq1}n_i\cdot\frac{1}{k}\sum_{i=1}^{k}\Big(1-\sum_{j=1}^{n_i}c_{i,j}\Big)^p \to 0,
\end{aligned}
\tag{3.5}
$$

as $k \to +\infty$, according to (c) of Lemma 6. The second inequality of above estimation follows from (3.3) and the fact that $\log(1-t) \geq -2t$ for all $t \in (0, 1/2)$. Inequalities (3.4) and (3.5) implies the right inequality of (3.2), and so the proof is completed. □

The following lemma concerns the geometrical structure of $f(E)$.

Lemma 8. *Let U be an interval. For $k \geq 1$, write*

$$\Theta_k^U = \big\{\sigma \in \Omega_k\colon f(J_\sigma) \subset U \text{ and } f(J_{\sigma_1\ldots\sigma_{k-1}}) \not\subset U\big\},$$

then $\operatorname{card}\Theta_k^U \leq 2n_k$.

Proof. Write $\Theta = \{\tau \in \Omega_{k-1}\colon f(J_\tau)\cap U \neq \varnothing \text{ and } f(J_\tau) \not\subset U\}$, then

$$\Theta_k^U \subset \{\tau * j\colon \tau \in \Theta \text{ and } 1 \leq j \leq n_k\}.$$

Therefore, we only need to show that $\operatorname{card}\Theta \leq 2$. If otherwise, suppose that $\tau^1, \tau^2, \tau^3 \in \Theta$ and the position of $f(J_{\tau^1})$, $f(J_{\tau^2})$ and $f(J_{\tau^3})$ are from left to right. Since $f(J_{\tau^1})\cap U \neq \varnothing$, $f(J_{\tau^3})\cap U \neq \varnothing$ and U is an interval, we must have $f(J_{\tau^2}) \subset U$. Contradiction! □

We are now ready for the proof of Theorem 1.

The proof of Theorem 1. It suffices to prove that $\dim_{\mathrm{H}} f(E) \geq d$ for all $d \in (0,1)$ and all 1-dimensional quasisymmetric mapping f. To this end, fix d and f, let ν be the Gibbs-like measure defined by (2.2). We will show that there exists a constant $c > 0$ such that

$$\nu(U) \leq c|U|^d \quad \text{for all interval } U.$$

Then the conclusion $\dim_{\mathrm{H}} f(E) \geq d$ follows from the mass distribution principle (see [4, Proposition 2.1]).

By Lemma 7, there exists a constant $c_0 > 0$ such that

$$\nu\big(f(J_\sigma)\big) \leq c_0\cdot\frac{|f(J_\sigma)|^d}{(1+\zeta)^{|\sigma|}} \quad \text{for all } \sigma \in \Omega.$$

Let U be an interval and Θ_k^U as in Lemma 8, then

$$
\begin{aligned}
\nu(U) = \nu\bigg(\bigcup_{k\geq0}\bigcup_{\sigma\in\Theta_k^U}f(J_\sigma)\bigg) &= \sum_{k\geq0}\sum_{\sigma\in\Theta_k^U}\nu\big(f(J_\sigma)\big) \leq c_0\cdot\sum_{k\geq0}\sum_{\sigma\in\Theta_k^U}\frac{|f(J_\sigma)|^d}{(1+\zeta)^k}\\
&\leq c_0\cdot\sum_{k\geq0}\frac{\big(\operatorname{card}\Theta_k^U\big)^{1-d}\big(\sum_{\sigma\in\Theta_k^U}|f(J_\sigma)|\big)^d}{(1+\zeta)^k}
\end{aligned}
$$

$$\le c_0\Big(2\sup_{i\ge1} n_i\Big)^{1-d}\sum_{k\ge0}\frac{\big(\sum_{\sigma\in\Theta_k^U}|f(J_\sigma)|\big)^d}{(1+\zeta)^k}$$
$$\le c_0\Big(2\sup_{i\ge1} n_i\Big)^{1-d}\Big(\sum_{k\ge0}(1+\zeta)^{-k/(1-d)}\Big)^{1-d}\Big(\sum_{k\ge0}\sum_{\sigma\in\Theta_k^U} f(J_\sigma)\Big)^d \le c|U|^d.$$

The second and the fourth inequality of the above estimation follow from the Hölder inequality. Thus we complete the proof. □

4. The Hausdorff dimension of Moran set

In this section, we will prove Theorem 2 and give some remarks on the Hausdorff dimension of Moran set.

4.1. The proof of Theorem 2. We begin with a lemma similar to Lemma 8.

Lemma 9. *Let U be an interval. For $k\ge1$, write*

$$\Xi_k^U=\big\{\sigma\in\Omega_k\colon J_\sigma\subset U \text{ and } J_{\sigma_1\dots\sigma_{k-1}}\not\subset U\big\},$$

then $\operatorname{card}\Xi_k^U\le 2n_k$.

Proof of Theorem 2. According to [13, Proposition 2.1], the statement $\dim_{\mathrm H} E\le s_*$ is true. This implies that Theorem 2 holds when $s_*=0$. So we may assume $s_*>0$ and only need to show that $\dim_{\mathrm H} E\ge d$ for all $d\in(0,s_*)$. For this, fix $d\in(0,s_*)$, let μ be the Gibbs-like measure defined by (2.1). We will show that

$$\mu(U)\le c|U|^d \quad \text{for all interval } U \text{ with } |U| \text{ small enough},$$

where $c>0$ is a constant. Then by the mass distribution principle, we obtain $\dim_{\mathrm H} E\ge d$ and the proof is completed.

Since $d<s_*$, for every $\delta\in(0,1)$, there is an integer K_δ such that $\delta(s_*-d)\le s_k-d$ for all $k\ge K_\delta$. Together with Lemma 1, it follows that

$$\mu(J_\sigma)\le|J_\sigma|^d\cdot\prod_{i=1}^{|\sigma|}D_i^{\delta(s_*-d)}\quad\text{for all }\sigma\in\bigcup_{k\ge K_\delta}\Omega_k.$$

Let U be an interval with $|U|<\min_{\sigma\in\Omega_{K_\delta}}|J_\sigma|$ and Ξ_k^U as in Lemma 9, then

$$\mu(U)=\mu\Big(\bigcup_{k>K_\delta}\bigcup_{\sigma\in\Xi_k^U}J_\sigma\Big)=\sum_{k>K_\delta}\sum_{\sigma\in\Xi_k^U}\mu(J_\sigma)\le\sum_{k>K_\delta}\sum_{\sigma\in\Xi_k^U}\Big(|J_\sigma|^d\prod_{i=1}^k D_i^{\delta(s_*-d)}\Big)$$
$$\le\sum_{k>K_\delta}\Big(\big(\operatorname{card}\Xi_k^U\big)^{1-d}\cdot\Big(\sum_{\sigma\in\Xi_k^U}|J_\sigma|\Big)^d\cdot\prod_{i=1}^k D_i^{\delta(s_*-d)}\Big)$$
$$\le 2^{1-d}\sum_{k>K_\delta}\Big(n_k^{1-d}\prod_{i=1}^k D_i^{\delta(s_*-d)}\cdot\Big(\sum_{\sigma\in\Xi_k^U}|J_\sigma|\Big)^d\Big)$$

$$\le 2^{1-d}\Bigg(\sum_{k>K_\delta} n_k \prod_{i=1}^{k} D_i^{\frac{\delta(s_*-d)}{1-d}}\Bigg)^{1-d}\Bigg(\sum_{k>K_\delta}\sum_{\sigma\in\Xi_k^U}|J_\sigma|\Bigg)^{d}$$

$$\le 2^{1-d}\Bigg(\sum_{k>K_\delta} n_k \prod_{i=1}^{k} D_i^{\frac{\delta(s_*-d)}{1-d}}\Bigg)^{1-d}\cdot |U|^{d}.$$

The Hölder inequality is used in the second and the fourth inequality of above computation.

When $s_* = 1$, taking δ as in (1.3), we obtain $\mu(U) \le c|U|^d$ for U with $|U|$ small enough. For general s_*, since $s_* - d$ can be arbitrarily small, we must require the condition (1.4) to ensure $\mu(U) \le c|U|^d$.

It remains to show that the condition (1.4) is equivalent to (1.2). When (1.2) holds,

$$\sum_{k=1}^{\infty} n_k\Bigg(\prod_{i=1}^{k} D_i\Bigg)^{\delta} < c_\delta \sum_{k=1}^{\infty} n_k\cdot (kn_k)^{-2} < \infty \quad \text{for all } \delta > 0.$$

Conversely, suppose that (1.4) is true. Then $n_k \prod_{i=1}^{k} D_i^{\delta} \to 0$ for all $\delta > 0$, and so

$$\lim_{k\to\infty} \frac{\log n_k}{-\sum_{i=1}^{k}\log D_i} = 0.$$

Noting that the sequence $\big\{\prod_{i=1}^{k} D_i^{\delta}\big\}_k$ is decreasing and summable for all $\delta > 0$, so we have $\prod_{i=1}^{k} D_i^{\delta} < k^{-1}$ when k large enough. It follows that

$$\lim_{k\to\infty} \frac{\log k}{-\sum_{i=1}^{k}\log D_i} = 0.$$

Therefore, (1.4) implies (1.2). □

4.2. Some remarks. When $c_* = 0$, there are two theorems which concern the dimensions of Moran sets obtained in [13].

Theorem A. *Let $\mathscr{M} = \mathscr{M}(J, \{n_k\}, \{c_{k,j}\})$ be a Moran class. Suppose that*

(i) $\sup_k n_k < \infty$;

(ii) $0 < \inf_i D_i \le \sup_i D_i < 1$.

Then for all $E \in \mathscr{M}$, dimension formula (1.8) *holds.*

Theorem B. *Let $\mathscr{M} = \mathscr{M}(J, \{n_k\}, \{c_{k,j}\})$ be a Moran class. Suppose that*

$$\lim_{k\to\infty} \frac{\log d_k}{\sum_{i=1}^{k}\log D_i} = 0,$$

then for all $E \in \mathscr{M}$, dimension formula (1.8) *holds.*

We will show that the conditions in Theorem A and B both imply the condition (1.2). For the conditions of Theorem A, it is easy to check. For Theorem B, since $n_k d_k \le \sum_{j=1}^{n_k} c_{k,j} \le 1$, we have $\lim_{k\to\infty} \frac{\log n_k}{-\sum_{i=1}^{k}\log D_i} = 0$. So it remains to show that

$$\lim_{k\to\infty} \frac{\log k}{-\sum_{i=1}^{k}\log D_i} = 0.$$

If otherwise, suppose that there exist a constant $c > 0$ and a sequence $\{k_j\}_{j\geq 1}$ such that

$$\frac{\log k_j}{-\sum_{i=1}^{k_j} \log D_i} > c. \tag{4.1}$$

Then we can find $K > 0$ such that

$$\frac{\log d_k}{\sum_{i=1}^{k} \log D_i} < \frac{c}{2} < \frac{\log k_j}{-2\sum_{i=1}^{k_j} \log D_i} \quad \text{for all } k \geq K \text{ and all } k_j.$$

Thus for all $K \leq k \leq k_j$, we have $d_k \geq k_j^{-1/2}$, and so $D_k \leq 1 - d_k \leq 1 - k_j^{-1/2}$. It follows that for all $k_j \geq 2K$,

$$\frac{\log k_j}{-\sum_{i=1}^{k_j} \log D_i} \leq \frac{\log k_j}{-\sum_{i=K+1}^{k_j} \log D_i} \leq \frac{\log k_j}{-\sum_{i=K+1}^{k_j} \log(1 - k_j^{-1/2})} \leq \frac{2\log k_j}{k_j^{1/2}} \to 0,$$

as $k_j \to \infty$. This contradicts (4.1).

By above discussion, we see that Theorem 2 generalizes Theorem A and B in some sense.

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