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A MEAN-VALUE THEOREM FOR SOME EIGENFUNCTIONS OF THE LAPLACE–BELTRAMI OPERATOR ON THE UPPER-HALF SPACE

Sirkka-Liisa Eriksson and Heikki Orelma

Tampere University of Technology, Department of Mathematics
P.O. Box 553, 33101 Tampere, Finland; sirkka-liisa.eriksson@tut.fi

Tampere University of Technology, Department of Mathematics
P.O. Box 553, 33101 Tampere, Finland; heikki.orelma@tut.fi

Abstract. In this paper we study a mean-value property for solutions of the eigenvalue equation of the Laplace–Beltrami operator

$$\Delta_{lb} h = -(n-1)h$$

with respect to the volume and the surface integrals on the Poincaré upper-half space $\mathbf{R}^{n+1}_+ = \{(x_0, \dots, x_n) \in \mathbf{R}^{n+1} \colon x_n > 0\}$ with the Riemannian metric $ds^2 = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_n^2}$.

1. Preliminaries

In this section we recall the Laplace–Beltrami operator in the Poincaré upper-half space and formulate its connections with the so called hypermonogenic functions. Let us denote $\mathbf{R}^{n+1}_+ = \{(x_0, x_1, \dots, x_n) \in \mathbf{R}^{n+1} : x_n > 0\}$. The Poincaré half-space is the Riemannian manifold $(\mathbf{R}^{n+1}_+, ds^2)$, where the Riemannian metric is

$$ds^2 = \frac{dx_0^2 + dx_1^2 + \cdots + dx_n^2}{x_n^2}.$$

The Laplace–Beltrami operator on the Poincaré upper-half space is the operator (details are available for example in [7])

$$\Delta_{lb} f = x_n^2 \Delta f - (n-1) x_n \frac{\partial f}{\partial x_n},$$

where $f \colon \Omega \to \mathbf{R}$ is a smooth enough function defined on an open subset Ω of $\mathbf{R}^{n+1}_+$ and $\Delta = \frac{\partial^2}{\partial x_0^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. The solutions of the Laplace–Beltrami equation $\Delta_{lb} f = 0$ are called hyperbolic harmonic functions.

The Clifford algebra $\mathscr{C}\ell_{0,n}$ is the free associative algebra with unit generated by the symbols $e_1, \dots, e_n$ together with the defining relations

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

for $i, j = 1, \dots, n$. As a vector space the dimension of the Clifford algebra $\mathscr{C}\ell_{0,n}$ is 2^n. A canonical basis is given by $e_A = e_{a_1} \cdots e_{a_k}$, where $A = \{a_1, \dots, a_k\} \subset \{1, \dots, n\}$ and $1 \le a_1 < \ldots < a_k \le n$. In particular, we denote $e_\emptyset = e_0 = 1$ and $e_{\{j\}} = e_j$. The

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$(n+1)$-dimensional Euclidean space $\mathbf{R}^{n+1}$ is a subspace of $\mathscr{C}\ell_{0,n}$ under the canonical embedding

$$(x_0, x_1, \dots, x_n) \mapsto \sum_{j=0}^{n} x_j e_j$$

and thus we may assume that $\mathbf{R}^{n+1} \subset \mathscr{C}\ell_{0,n}$. An element $a \in \mathscr{C}\ell_{0,n}$ is called a Clifford number and often the algebra $\mathscr{C}\ell_{0,n}$ is called the algebra of Clifford numbers. Elements $\mathbf{x} = \sum_{j=1}^{n} x_j e_j \in \mathscr{C}\ell_{0,n}$ are called vectors. Thus we see that an element x of $\mathbf{R}^{n+1}$ may be written as

$$x = x_0 + \mathbf{x}$$

with $\mathbf{x} = x_1 e_1 + \cdots + x_n e_n$ and it is called a paravector.

The *conjugation* is the algebra anti-automorphism on the Clifford algebra defined by $\overline{x} = x_0 - \mathbf{x}$, that is, if $a, b \in \mathscr{C}\ell_{0,n}$, then $\overline{ab} = \overline{b}\,\overline{a}$. Also, $\mathbf{x}^2 = \mathbf{x}\mathbf{x} = -x_1^2 - \cdots - x_n^2$. Thus we may compute

$$x\overline{x} = (x_0 + \mathbf{x})(x_0 - \mathbf{x}) = x_0^2 + x_1^2 + \cdots + x_n^2$$

for $x \in \mathbf{R}^{n+1}$. The Euclidean norm is then $|x|^2 = x\overline{x} = \overline{x}x$. The main-involution is the algebra automorphism denoted and defined by $x' = x_0 - \mathbf{x}$, that is, if $a, b \in \mathscr{C}\ell_{0,n}$, then $(ab)' = a'b'$.

Let us consider the Clifford algebra valued functions $f\colon \Omega \to \mathscr{C}\ell_{0,n}$, where $\Omega \subset \mathbf{R}_{+}^{n+1}$ is an open subset. Since the Clifford algebra $\mathscr{C}\ell_{0,n}$ is generated by the symbols $e_1, \dots, e_n$, we obtain that then the Clifford algebra $\mathscr{C}\ell_{0,n-1}$ is generated by the symbols $e_1, \dots, e_{n-1}$. Hence each $a \in \mathscr{C}\ell_{0,n}$ may be represented in the form

$$a = b + c e_n,$$

where $b, c \in \mathscr{C}\ell_{0,n-1}$. We abbreviate $Pa = b$ and $Qa = c$ and $Q'a = (Qa)'$ and $P'a = (Pa)'$. Then we define the *modified Dirac operator* by

$$Mf = Df + \frac{n-1}{x_n} Q'f,$$

where $D = \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}$ is the Dirac operator on $\mathbf{R}^{n+1}$. The theory of null-solutions of the modified Dirac operator is called *hyperbolic function theory*, see, e.g., [4].

The function $f\colon \Omega \to \mathscr{C}\ell_{0,n}$ is called a *hypermonogenic* on Ω if $Mf(x) = 0$ for each $x \in \Omega$. Hypermonogenic functions have many nice function theoretic properties, for example, they have Cauchy-type integral formulas. Also, the function $x \mapsto x^k$, where $k \in \mathbf{Z}$, is hypermonogenic. Many properties and more references can be found from the survey article [4].

The conjugate of the modified Dirac operator is defined by

$$\overline{M}f = \overline{D}f - \frac{n-1}{x_n} Q'f,$$

where $\overline{D} = \frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - \cdots - e_n \frac{\partial}{\partial x_n}$.

In the hyperbolic function theory we define hyperbolic harmonic functions $f\colon \Omega \to \mathscr{C}\ell_{0,n}$ as solutions of the equation

$$\overline{M}Mf(x) = 0$$

for $x \in \Omega$.

The next theorem give us a connection between hypermonogenic functions and hyperbolic harmonic functions. Also, we see that the equation in above is really a good generalization for real-valued hyperbolic harmonic functions.

Theorem 1.1. [2] *Let $\Omega \subset \mathbf{R}^{n+1}_{+}$ be an open subset and let $f\colon \Omega \to \mathcal{C}\ell_{0,n}$ be a twice differentiable function. Then*

$$P(\overline{M}Mf) = \Delta Pf - \frac{n-1}{x_n}\frac{\partial Pf}{\partial x_n}$$

and

$$Q(\overline{M}Mf) = \Delta Qf - \frac{n-1}{x_n}\frac{\partial Qf}{\partial x_n} + (n-1)\frac{Qf}{x_n^2}.$$

If f is hypermonogenic, then Pf satisfies the equation

$$\Delta Pf - \frac{n-1}{x_n}\frac{\partial Pf}{\partial x_n} = 0$$

and Qf satisfies the equation

$$\Delta Qf - \frac{n-1}{x_n}\frac{\partial Qf}{\partial x_n} + (n-1)\frac{Qf}{x_n^2} = 0.$$

Thus we see that the Q-part of a hypermonogenic function is a solution of the following eigenvalue equation

$$\Delta_{lb} h = -(n-1)h.$$

In the next section we shall study more detailed what is the structure of the above eigenfunctions.

Also, we see that the P-part of a hypermonogenic function is a direct generalization of a real-valued hyperbolic harmonic function. For a $\mathcal{C}\ell_{0,n-1}$-valued function, especially for the P-part of a hypermonogenic function, we obtained the following structure theorem.

Theorem 1.2. [5] *Let $\Omega \subset \mathbf{R}^{n+1}_{+}$ be open and $g\colon \Omega \to \mathcal{C}\ell_{0,n-1}$ be a differentiable function. The following properties are equivalent.*

(a) *g is a solution of the equation*

$$\Delta g - \frac{n-1}{x_n}\frac{\partial g}{\partial x_n} = 0.$$

(b) *g is smooth and*

$$g(a) = \frac{1}{\omega_{n+1}\sinh^n R_h}\int_{\partial B_h(a,R_h)} g(x)\, d\sigma_h(x)$$

for all $\overline{B_h(a,R_h)} \subset \Omega$. In the formula ω_{n+1} denotes the surface area of the n-dimensional unit sphere.

(c) *g is smooth and*

$$g(a) = \frac{1}{V(B_h(a,R_h))}\int_{B_h(a,R_h)} g(x)\, dx_h(x)$$

for all $\overline{B_h(a,R_h)} \subset \Omega$, where $V(B_h(a,R_h)) = \sigma_n \int_0^{R_h} \sinh^n t\, dt$ is the volume of the ball $B_h(a,R_h)$.

In the previous theorem $B_h(a, R_h)$ is the hyperbolic ball with the center a and the radius R_h. In the next section we shall give more detailed description for it. Since $\mathbf{R}$ is a canonical subset of $\mathscr{C}\ell_{0,n}$, we obtain the following obvious corollary.

Corollary 1.3. *The preceding theorem is true also for real valued functions.*

In the next section we shall state and prove a similar theorem for the preceding eigenfunctions.

2. A mean-value theorem for some eigenfunctions of the Laplace–Beltrami operator

Our aim is to give a detailed proof for the following structure theorem of the eigenfunctions represented in the previous section. First we recall a few basic facts from the hyperbolic geometry. A more detailed survey to the topic is available in [6]. In [6] it is shown that the hyperbolic ball with the radius R_h and the center a is the Euclidean ball with the center $\tau(a, R_h)$ and the radius $R_e(a, R_h)$,

$$B_h(a, R_n) = \{x \in \mathbf{R}^{n+1}_+ \colon |x - \tau(a, R_h)| < R_e(a, R_h)\},$$

where

$$\tau(a, R_h) = a_0 + a_1 e_1 + \cdots + a_{n-1} e_{n-1} + a_n e_n \cosh R_h$$

and

$$R_e(a, R_h) = a_n \sinh R_h.$$

The n-form

$$d\sigma = \sum_{j=0}^{n} (-1)^j e_j \, d\widehat{x}_j$$

is often very useful vector valued differential form on $\mathbf{R}^{n+1}_+$, where $d\widehat{x}_j = dx_0 \cdots dx_{j-1} \cdot dx_{j+1} \cdots dx_n$.

Let K be an $(n+1)$-dimensional manifold-with-boundary. On the boundary ∂K the form $d\sigma$ admits the representation $d\sigma = \nu dS$, where ν is the outer unit normal vector field and dS a scalar n-form. The corresponding surface form on the hyperbolic space is $d\sigma_h = \frac{d\sigma}{x_n^n}$ and if dx is the volume form on the Euclidean space then the corresponding hyperbolic form is $dx_h = \frac{dx}{x_n^{n+1}}$. More detailed introduction to integration and certain differential forms in the Poincaré upper-half space can be found from [6].

Theorem 2.1. *Let $\Omega \subset \mathbf{R}^{n+1}_+$ be an open subset and let $h \colon \Omega \to \mathscr{C}\ell_{0,n-1}$ be a smooth function. The following properties are equivalent:*

(i) *h is an eigenfunction of the Laplace–Beltrami operator with the eigenvalue $-(n-1)$, i.e, is a solution of*

$$\Delta_{lb} h(x) = -(n-1) h(x)$$

for $x \in \Omega$.

(ii)

$$h(a) = \frac{1}{\omega_{n+1} \psi(R_h)} \int_{\partial B_h(a, R_h)} h(x) \, d\sigma_h(x),$$

where

$$\psi(R_h) = \sinh R_h \int_0^{R_h} \sinh^{n-2}(t) \, dt$$

whenever $\overline{B(a, R_h)} \subset \Omega$.

(iii)

$$h(a) = \frac{n-1}{\omega_{n+1}\phi(R_h)} \int_{B_h(a,R_h)} h(x)\, dx_h,$$

where ω_{n+1} *is the surface area of the* $(n+1)$*-unit sphere and*

$$\phi(R_h) = (n-1)\cosh R_h \int_0^{R_h} \sinh^{n-2}(t)\, dt - \sinh^{n-1} R_h$$

whenever $\overline{B(a, R_h)} \subset \Omega$.

The corresponding result in the case $n = 2$ is already known. Leutwiler proved the theorem in his paper [8] using Green's functions which are simple in the case $n = 2$. Authors wishes to emphasize that the methods of Leutwiler are available only in the special case $n = 2$ since the Green's functions have much more complicated form in higher dimensions.

The first consequence is the following remark.

Corollary 2.2. *The preceding theorem is true also for functions* $h\colon \Omega \to \mathbf{R}$.

The proof of the theorem is based on a sequence of lemmata. First we recall the Cauchy's formula for the Q-part of a hypermonogenic function and other useful results.

Proposition 2.3. [1] *If* f *is a hypermonogenic function on* Ω *and* $K \subset \Omega$ *is an oriented* $(n+1)$*-dimensional manifold-with-boundary, then for each* $a \in K$ *we have*

$$Qf(a) = \frac{2^n a_n^{n-1}}{\omega_{n+1}} \int_{\partial K} Q(q(x,a)\nu(x)f(x))\, dS(x),$$

where dS *is the scalar surface element,* ν *is the outer unit normal vector field, and*

$$q(x,a) = -\frac{1}{2(n-1)}\overline{D}\frac{1}{|x-a|^{n-1}|x-\widehat{a}|^{n-1}} = \frac{1}{2}\frac{(x-a)^{-1} + (x-\widehat{a})^{-1}}{|x-a|^{n-1}|x-\widehat{a}|^{n-1}}.$$

The kernel in the above integral admits the following expression.

Theorem 2.4. [6]

$$q(x,a) = \frac{\overline{(x-\tau(a,x))}\cosh d_h(x,a) - a_n \sinh^2 d_h(x,a) e_n}{(2a_n x_n)^n \sinh^{n+1} d_h(x,a)},$$

where

$$\tau(a,x) = a_0 + a_1 e_1 + \cdots + a_{n-1}e_{n-1} + a_n \cosh d_h(x,a) e_n$$

and d_h *is the distance function with respect to the hyperbolic metric.*

Also we need the following integration result. We define a generalized version of the modified Dirac operator by

$$M_n f = Df - \frac{n}{x_n} Q' f.$$

Theorem 2.5. [3] *Let* Ω *be an open subset of* $\mathbf{R}^{n+1}_+$. *If* $K \subset \Omega$ *is an oriented* $(n+1)$*-dimensional manifold-with-boundary and* g *is a smooth Clifford algebra-valued function on* Ω*, then*

$$\int_{\partial K} P(\nu(x)g(x))\frac{dS(x)}{x_n^n} = \int_K P\big(M_n g(x)\big)\frac{dx}{x_n^n}.$$

Using the preceding resukt we are able to prove the following lemma.

Lemma 2.6. *Assume that f is hypermonogenic on Ω and $\overline{B_h(a,R_h)} \subset \Omega$. Then*

$$\int_{\partial B_h(a,R_h)} Q\Big(\frac{e_n \nu(x) f(x)}{x_n^n}\Big) dS(x) = \int_{B_h(a,R_h)} Qf(x)\, dx_h.$$

Proof. It is easy to see that $Q(e_n \nu(x) f(x)) = P'(\nu(x) f(x))$. Using Theorem 2.5 we have

$$\int_{\partial B_h(a,R_h)} Q\Big(\frac{e_n \nu(x) f(x)}{x_n^n}\Big) dS(x) = \int_{B_h(a,R_h)} P'(M_n f(x)) \frac{dx}{x_n^n}.$$

Since

$$Mf(x) = Df(x) + \frac{n-1}{x_n} Q' f(x) = 0,$$

we obtain

$$M_n f(x) = Mf(x) + \frac{Q' f(x)}{x_n} = \frac{Q' f(x)}{x_n}.$$

The proof is complete. □

Also we shall need the following result.

Lemma 2.7. [5] *If f is a twice continuously differentiable function from $\Omega \subset \mathbf{R}^{n+1}_+$ into $\mathscr{C}\ell_{0,n}$, and $\overline{B_h(e_n,R_h)} \subset \Omega$, we obtain*

$$\frac{d}{dR_h}\Big(\frac{1}{\sinh^n R_h} \int_{\partial B_h(a,R_h)} f\, d\sigma_h\Big) = \frac{1}{\sinh^n R_h} \int_{B_h(a,R_h)} \Delta_h f\, dx_h.$$

Next we deduce that any eigenfunction of the Laplace–Beltrami operator is a Q-part of some hypermonogenic function. The theorem is formulated only for a ball but similar theorem holds also for more general star-shaped domains (cf. [2]).

Theorem 2.8. [2] *Let $h\colon B_h(a,R) \to \mathscr{C}\ell_{0,n-1}$ be a solution of the equation*

$$\Delta_{lb} h(x) = -(n-1) h(x).$$

There exists a hypermonogenic function $f\colon B_h(a,R) \to \mathscr{C}\ell_{0,n}$ satisfying $h = Qf$ on $B_h(a,R)$.

Now we may start to give the proof for the Theorem 2.1. First we show that the statement (1) implies (2).

Lemma 2.9. *Let $h\colon \Omega \to \mathscr{C}\ell_{0,n-1}$ be a solution of*

$$\Delta_{lb} h(x) = -(n-1) h(x)$$

on Ω and let $\overline{B_h(a,R_h)} \subset \Omega$. Then

$$h(a) = \frac{1}{(n-1)\omega_{n+1} \sinh R_h \int_0^{R_h} \sinh^{n-2}(t)\, dt} \int_{\partial B_h(a,R_h)} h(x)\, d\sigma_h(x)$$

Proof. Let f be a hypermonogenic function satisfying $Qf = h$ on $B_h(a,R_h)$. Applying Proposition 2.3 and Theorem 2.4 we obtain

$$\omega_{n+1} Qf(a) = \int_{\partial B_h(a,R_h)} Q\Big(\frac{\overline{(x-\tau(a,x))} \cosh d_h(x,a) - a_n \sinh^2 d_h(x,a) e_n}{a_n x_n^n \sinh^{n+1} d_h(x,a)} \nu(x) f(x)\Big) dS(x).$$

Since on the ball $B_h(a, R_h)$ the unit normal field is given by $\nu(x) = \frac{x-\tau(a,x)}{R_e(a,R_h)}$, we infer

$$\omega_{n+1}Qf(a) = \int_{\partial B_h(a,R_h)} Q\Big(\frac{R_e(a,R_h)\overline{\nu(x)}\cosh d_h(x,a) - a_n \sinh^2 d_h(x,a)e_n}{a_n x_n^n \sinh^{n+1} d_h(x,a)}\nu(x)f(x)\Big)dS(x)$$

Since $\overline{\nu(x)}\nu(x) = 1$, we obtain

$$\omega_{n+1}Qf(a) = \frac{R_e(a,R_h)\cosh R_h}{a_n \sinh^{n+1} R_h}\int_{\partial B_h(a,R_h)} Qf(x)\,d\sigma_h - \frac{1}{\sinh^{n-1} R_h}\int_{\partial B_h(a,R_h)} Q\Big(\frac{e_n\nu(x)f(x)}{x_n^n}\Big)dS(x).$$

Since $R_e(a, R_h) = a_n \sinh R_h$, by virtue of Lemma 2.6 we have

$$\omega_{n+1}Qf(a) = \frac{\cosh R_h}{\sinh^n R_h}\int_{\partial B_h(a,R_h)} Qf(x)d\sigma_h - \frac{1}{\sinh^{n-1} R_h}\int_{B_h(a,R_h)} Qf(x)\,dx_h.$$

Using Lemma 2.7 and the assumption we have

$$\omega_{n+1}Qf(a) = \frac{\cosh R_h}{\sinh^n R_h}\int_{\partial B_h(a,R_h)} Qf(x)\,d\sigma_h - \frac{\sinh R_h}{n-1}\frac{d}{dR_h}\Big(\frac{1}{\sinh^n R_h}\int_{\partial B_h(a,R_h)} Qf(x)\,d\sigma_h\Big).$$

The equation in above give us the differential equation

$$\sinh(R_h)g'(R_h) + (n-1)\cosh(R_h)g(R_h) = C,$$

where $C = (n-1)Qf(a)$ and

$$g(R_h) = \frac{1}{\omega_{n+1}\sinh^n R_h}\int_{\partial B_h(a,R_h)} Qf(x)\,d\sigma_h.$$

The general solution of this equation is

$$g(R_h) = \frac{C\int_0^{R_h}\sinh^{n-2} t\,dt + C_0}{\sinh^{n-1}(R_h)}.$$

Since g is a continuous function, we have

$$\lim_{R_h\to 0+} g(R_h) = Qf(a)$$

and then $C_0 = 0$. The proof is complete. □

We show next that the statement (2) implies (3).

Lemma 2.10. *Assume*

$$h(a) = \frac{1}{\omega_{n+1}\sinh R_h\int_0^{R_h}\sinh^{n-2}(t)\,dt}\int_{\partial B_h(a,R_h)} h(x)\,d\sigma_h(x).$$

Then

$$h(a) = \frac{n-1}{\omega_{n+1}\phi(R_h)}\int_{B_h(a,R_h)} Qf(x)\,dx_h,$$

where

$$\phi(R_h) = (n-1)\cosh R_h\int_0^{R_h}\sinh^{n-2}(t)\,dt - \sinh^{n-1} R_h.$$

Proof. Using Lemma 2.7 we have

$$-\frac{n-1}{\sinh^n R_h}\int_{B_h(a,R_h)} h(x)\,dx_h = \frac{d}{dR_h}\Big(\frac{1}{\sinh^n R_h}\int_{\partial B_h(a,R_h)} h(x)\,d\sigma_h\Big).$$

By the assumptions

$$\begin{aligned}&-\frac{n-1}{\omega_{n+1}\sinh^n R_h}\int_{B_h(a,R_h)} h(x)\,dx_h\\ &=\Big(\frac{1}{\sinh R_h}-(n-1)\frac{\cosh R_h\int_0^{R_h}\sinh^{n-2}t\,dt}{\sinh^n R_h}\Big)h(a).\end{aligned}$$

Then

$$h(a)=\frac{n-1}{\omega_{n+1}(n-1)\cosh R_h\int_0^{R_h}\sinh^{n-2}(t)\,dt-\sinh^{n-1}R_h}\int_{B_h(a,R_h)} h(x)\,dx_h,$$

and the proof is complete. □

We show next that (3) implies (2). First we need the following lemma.

Lemma 2.11. *Let* $T\colon B_h(e_n,R_h)\to B_h(a,R_h)$ *be the mapping*

$$T(x)=a_nx+Pa,$$

where $a\in\mathbf{R}^{n+1}_+$. *Then* T *is diffeomorphism, and the following transformation rules hold:*

(a) $\int_{\partial B_h(a,R_h)} f(y)\,d\sigma_h(y)=\int_{\partial B_h(e_n,R_h)} f\circ T^{-1}(x)\,d\sigma_h(x)$,
(b) $\int_{\partial B_h(a,R_h)} f\circ T(x)\,d\sigma_h(x)=\int_{\partial B_h(e_n,R_h)} f(y)\,d\sigma_h(y)$,
(c) $\int_{B_h(a,R_h)} h(y)\,dy_h=\int_{B_h(e_n,R_h)} h\circ T^{-1}(x)\,dx_h$,
(d) $\int_{B_h(a,R_h)} h\circ T(x)\,dx_h=\int_{B_h(e_n,R_h)} h(y)\,dy_h$.

That allows us to prove the following lemma.

Lemma 2.12. *Assume*

$$h(a)=\frac{n-1}{\omega_{n+1}\phi(R_h)}\int_{B_h(a,R_h)} h(x)\,dx_h.$$

Then

$$h(a)=\frac{1}{\omega_{n+1}\sinh R_h\int_0^{R_h}\sinh^{n-2}(t)\,dt}\int_{\partial B_h(a,R_h)} h(x)\,d\sigma_h(x).$$

Proof. Using the previous proposition we infer

$$h(a)=\frac{n-1}{\omega_{n+1}\phi(R_h)}\int_{B_h(e_n,R_h)} h\circ T^{-1}h(x)\,dx_h.$$

Using the polar coordinates we have

$$h(a)=\frac{n-1}{\omega_{n+1}\phi(R_h)}\int_0^{R_h}\int_{\partial B_h(e_n,t)} h\circ T^{-1}(x)\,d\sigma_h(x)\,dt.$$

Then

$$\omega_{n+1}\phi(R_h)h(a)=(n-1)\int_0^{R_h}\int_{\partial B_h(e_n,t)} h\circ T^{-1}(x)\,d\sigma_h(x)\,dt.$$

Since

$$\phi'(R_h) = (n-1)\sinh R_h \int_0^{R_h} \sinh^{n-2}(t)\,dt,$$

using Lemma 2.7 we have

$$h(a) = \frac{1}{\omega_{n+1}\sinh R_h \int_0^{R_h} \sinh^{n-2}(t)\,dt} \int_{\partial B_h(e_n,R_h)} h\circ T^{-1}(x)\,d\sigma_h(x).$$

Then using the (a)-part of the preceding proposition we have

$$h(a) = \frac{1}{\omega_{n+1}\sinh R_h \int_0^{R_h} \sinh^{n-2}(t)\,dt} \int_{\partial B_h(a,R_h)} h(y)\,d\sigma_h(y).$$

The proof is complete. □

Lastly we deduce that (2) implies (1).

Lemma 2.13. *Assume*

$$h(a) = \frac{1}{\omega_{n+1}\sinh R_h \int_0^{R_h} \sinh^{n-2}(t)\,dt} \int_{\partial B_h(a,R_h)} h(x)\,d\sigma_h(x).$$

Then

$$\Delta_{lb} h(x) = -(n-1)h(x)$$

for $x \in B_h(a,R_h)$.

Proof. Since

$$\frac{d}{dR_h}\frac{\sinh^{n-1} R_h}{\int_0^{R_h} \sinh^{n-2}(t)\,dt} = \frac{\sinh^{n-2} R_h \phi(R_h)}{\left(\int_0^{R_h} \sinh^{n-2}(t)\,dt\right)^2},$$

using Lemma 2.7 we obtain

$$\begin{aligned} 0 = {} & \frac{\sinh^{n-2} R_h \phi(R_h)}{\left(\int_0^{R_h} \sinh^{n-2}(t)\,dt\right)^2}\frac{1}{\sinh^n R_h} \int_{\partial B_h(a,R_h)} h(x)\,d\sigma_h(x) \\ & + \frac{\sinh^{n-1} R_h}{\int_0^{R_h} \sinh^{n-2}(t)\,dt}\frac{1}{\sinh^n R_h} \int_{B_h(a,R_h)} \Delta_{lb} h(x)\,dx_h. \end{aligned}$$

Since (2) and (3) are equivalent, we obtain the formula

$$\int_{\partial B_h(a,R_h)} h(x)\,d\sigma_h(x) = (n-1)\frac{\sinh R_h \int_0^{R_h} \sinh^{n-2}(t)\,dt}{\phi(R_h)} \int_{B_h(a,R_h)} h(x)\,dx_h.$$

Then

$$\begin{aligned} 0 = {} & \frac{(n-1)\phi(R_h)}{\left(\int_0^{R_h} \sinh^{n-2}(t)\,dt\right)^2}\frac{1}{\sinh^2 R_h}\frac{\sinh R_h \int_0^{R_h} \sinh^{n-2}(t)\,dt}{\phi(R_h)} \int_{B_h(a,R_h)} h(x)\,dx_h \\ & + \frac{1}{\int_0^{R_h} \sinh^{n-2}(t)\,dt}\frac{1}{\sinh R_h} \int_{B_h(a,R_h)} \Delta_{lb} h(x)\,dx_h, \end{aligned}$$

that is,

$$\frac{1}{\sinh R_h \int_0^{R_h} \sinh^{n-2}(t)\,dt} \int_{B_h(a,R_h)} (\Delta_{lb} h(x) + (n-1)h(x))\,dx_h = 0.$$

Since R_h is arbitrary, we obtain that

$$\Delta_{lb} h(a) + (n-1)h(a) = 0.$$

The proof is complete. □

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