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UNIONS OF JOHN DOMAINS AND UNIFORM DOMAINS IN REAL NORMED VECTOR SPACES

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Abstract. Let E be real normed vector spaces with the dimension at least 2. In this paper we study the following questions: When is the union of two John domains in E a John domain and when is the union of two uniform domains in E a uniform domain?

1. Introduction and main results

Throughout the paper, we always assume that E denotes a real normed vector space with $\dim E \geq 2$ and that D is a proper subdomain in E. The norm of a vector z in E is written as $|z|$, and for any two points z_1, z_2 in E, the distance between them is denoted by $|z_1 - z_2|$, and the closed line segment with endpoints z_1 and z_2 by $[z_1, z_2]$. For $x \in E$ and $r > 0$, we let $\mathbf{B}(x, r)$ denote the open ball in E with center x and radius r. For real numbers r and s, we use the notation: $r \wedge s = \min\{r, s\}$.

John domains in Euclidean spaces $\mathbf{R}^n$ were introduced by John [1] in connection with his work on elasticity. The term is due to Martio and Sarvas [3]. Roughly speaking, a domain is a John domain if it is possible to travel from one point of the domain to another without going too close to the boundary. The precise definition is as follows.

Definition 1.1. D is called a *c-John domain* if for every pair of points $x_1, x_2 \in D$ there is a rectifiable arc γ joining them with

$$\ell(\gamma[x_1, x]) \wedge \ell(\gamma[x_2, x]) \leq c\, d(x)$$

for all $x \in \gamma$, where c is a positive constant, $\gamma[x_j, x]$ denotes the closed subarc of γ with endpoints x_j and x $(j = 1, 2)$, $\ell(\gamma[x_j, x])$ the arclength of $\gamma[x_j, x]$. γ is called a *c-John arc* joining x_1 and x_2.

See [4] for several characterizations of John domains. In the study of John domains, the following question is natural:

Question 1.1. Is the union of two John domains in E still a John domain when their intersection is not empty?

Väisälä considered this question when $E = \mathbf{R}^n$. In [7], Väisälä constructed an example to show that, in general, the answer to this question is negative. Note

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that the definition of John domains used in [7] is based on diameter cigar, which is quantitatively equivalent to Definition 1.1 when $E = \mathbf{R}^n$. In the same paper, Väisälä proved that if the intersection of two John domains is not too thin then their union is a John domain as the following result shows.

Theorem A. [7, Theorem 3.1] *Suppose that D_1 and D_2 are c-John domains in $\mathbf{R}^n$. If there exist $z_0 \in D_1 \cap D_2$ and $r > 0$ such that $\mathbf{B}(z_0, r) \subset D_1 \cap D_2$ and $d(D_1) \wedge d(D_2) \le c_0 r$, where $d(D_i)$ denotes the diameter of D_i $(i = 1, 2)$, then $D_1 \cup D_2$ is a c'-John domain with $c' = 2c(c_0 + 1)$.*

As the first aim of this paper, we study Question 1.1 further. Our result is as follows.

Theorem 1.1. *Suppose that both D_1 and D_2 are c-John domains in E, and that there are $z_0 \in D_1 \cap D_2$ and $r > 0$ such that $\mathbf{B}(z_0, r) \subset D_1 \cap D_2$. If there exists some $r_1 > 0$ such that $r_1 \le c_0 r$ and $D_1 \subset \mathbf{B}(z_0, r_1)$, where $c_0 > 1$ is a constant, then $D_1 \cup D_2$ is a c'-John domain with $c' = c(4cc_0 + 1)$.*

The proof of Theorem 1.1 will be presented in Section 2. Our proof method is different from that in [7]. Hence when $E = \mathbf{R}^n$, we also give a different proof for Theorem A.

We remark that the assumption "$d(D_1) \wedge d(D_2) \le c_0 r$" in Theorem A is equivalent to the statement "at least one of D_1 and D_2 is bounded", and the assumption "there exists some $r_1 > 0$ such that $r_1 \le c_0 r$ and $D_1 \subset \mathbf{B}(z_0, r_1)$" in Theorem 1.1 is equivalent to the statement "D_1 is bounded". The following example shows that the requirement that "at least one of D_1 and D_2 must be bounded" in Theorem 1.1 is necessary.

Example 1.1. Let $D_1 = \{(x, y) \in \mathbf{R}^2 \colon x < 0\}$ and $D_2 = \{(x, y) \in \mathbf{R}^2 \colon |y| < x + 1\}$. Then both D_1 and D_2 are John domains, but $D = D_1 \cup D_2$ is not a John domain.

The proof of Example 1.1 will be given in Section 3.

Definition 1.2. D is called *c-uniform* in the norm metric in E provided there exists a positive constant c with the property that each pair of points z_1, z_2 in D can be joined by a rectifiable arc γ in D satisfying (cf. [5, Section 6.3])

(1) $\ell(\gamma[z_1, z]) \wedge \ell(\gamma[z_2, z]) \le c\, d(z)$ for all $z \in \gamma$, and
(2) $\ell(\gamma) \le c\, |z_1 - z_2|$.

D is called *uniform* if it is c-uniform for some $c > 0$, and γ is called a *c-uniform arc* if it satisfies (1) and (2) (cf. [6, Section 2.16]). See [2, 9] for the generalization of this definition.

As the second aim of this paper, we consider the following question:

Question 1.2. Does Theorem 1.1 hold for uniform domains in E?

The following example shows that even when both D_1 and D_2 are bounded uniform domains their union may not be uniform.

Example 1.2. Let $D_1 = \{(x, y) \in \mathbf{R}^2 \colon -2 < x < 1,\ 0 < y < 2\}$ and $D_2 = D_3 \cup D_4$, where $D_3 = \{(x, y) \in \mathbf{R}^2 \colon 0 < x < 1,\ -1 < y < 1\}$ and $D_4 = \{(x, y) \in \mathbf{R}^2 \colon -2 < x < 1,\ -1 < y < 0\}$. Then both D_1 and D_2 are uniform domains, but $D = D_1 \cup D_2$ is not a uniform domain.

The proof of Example 1.2 will be given in Section 3. For convex and bounded domains in E, the following result due to Väisälä, which is from [6].

Theorem B. [6, Theorem 2.19] *Suppose that G is a convex domain and that $\mathbf{B}(x_0, r') \subset G \subset \mathbf{B}(x_0, R')$. Then G is c_1-uniform with $c_1 = 2\frac{R'}{r'}$.*

In the following we consider Question 1.2 for convex domains and we get

Theorem 1.2. *Suppose that D_1 and D_2 are convex domains in E, where D_1 is bounded and D_2 is c-uniform for some $c > 1$, and that there exist $z_0 \in D_1 \cap D_2$ and $r > 0$ such that $\mathbf{B}(z_0, r) \subset D_1 \cap D_2$. If there exist constants $R_1 > 0$ and $c_0 > 1$ such that $R_1 \le c_0 r$ and $D_1 \subset \mathbf{B}(z_0, R_1)$, then $D_1 \cup D_2$ is a c'-uniform domain with $c' = (c+1)(2c_0+1) + c$.*

The proof of Theorem 1.2 will be presented in Section 4.

Remark 1.1. Example 1.2 shows that the hypothesis "D_2 being convex" in Theorem 1.2 is necessary.

2. The proof of Theorem 1.1

We show that the theorem holds with $c' = c(4cc_0 + 1)$. Set $D = D_1 \cup D_2$. Let $x \in D \setminus D_2$, $y \in D \setminus D_1$, $d_j(x) = d(x, \partial D_j)$ for $j = 1, 2$. Then there are John arcs $\alpha \subset D_1$ from x to z_0 and $\beta \subset D_2$ from z_0 to y and an arc $\gamma_1 \subset \alpha \cup \beta$ from x and y. To prove that γ_1 is a c'-John arc in D it suffices to show that

(1) $\ell(\alpha[x, w]) \le c'd(w)$ for all $w \in \alpha$,

(2) $\zeta(z) := \big(\ell(\alpha) + \ell(\beta[z_0, z])\big) \wedge \ell(\beta[z, y]) \le c'd(z)$ for all $z \in \beta$.

We let $x_0 \in \alpha$ be the point bisecting the length of α and choose x_1 such that $\ell(\alpha[x_1, z_0]) = \frac{r}{2}$. For any $w \in \alpha$, if $w \in \alpha[x, x_0]$, then we have

$$\ell(\alpha[x, w]) = \ell(\alpha[x, w]) \wedge \ell(\alpha[w, z_0]) \le cd_1(w) \le cd(w),$$

and (1) is proved.

If $w \in \alpha[x_1, z_0]$, then

$$d(w) \ge \frac{r}{2} \ge \frac{r_1}{2c_0}$$

and

$$\ell(\alpha) = 2\ell(\alpha[x, x_0]) \le 2cd_1(x_0) \le 2cr_1, \tag{2.1}$$

which show that

$$\ell(\alpha[x, w]) \le \ell(\alpha) \le 2cr_1 \le 4cc_0 d(w),$$

and we obtain (1).

Let $w \in \alpha[x_0, x_1]$. Obviously, $\ell(\alpha[x, w]) \wedge \ell(\alpha[z_0, w]) = \ell(\alpha[z_0, w]) \ge \frac{r}{2}$, which together with (2.1) show that

$$\ell(\alpha[x, w]) \le 2cr_1 \le 4c\frac{r_1}{r}\ell(\alpha[z_0, w]) \le 4c^2c_0 d(w),$$

which is (1).

The proof of (1) is complete. In the following, we come to prove (2). We let $y_0 \in \beta$ be the point bisecting the length of β and choose y_1 such that $\ell(\alpha[z_0, y_1]) = \frac{r}{2}$. For any $z \in \beta$, if $z \in \beta[y, y_0]$, then (2) easily follows because β is c-John in D_2.

If $z \in \beta[y_1, z_0]$, then (2.1) implies that

$$\zeta(z) \le \ell(\alpha) + \ell(\beta[z_0, z]) \le 2cr_1 + cd_2(z) \le 4cc_0 d(z) + cd(z) = c(4c_0 + 1)d(z),$$

since $r_1 \le c_0 r \le 2c_0 d(z)$. If $z \in \beta[y_0, y_1]$, then we have

$$\ell(\beta[y,z]) \wedge \ell(\beta[z,z_0]) = \ell(\beta[z,z_0]) \ge \frac{r}{2} \ge \frac{r_1}{2c_0}$$

and

$$\ell(\beta[y,z]) \wedge \ell(\beta[z,z_0]) \le cd_2(z) \le cd(z),$$

which together with (2.1) imply

$$\zeta(z) \le \ell(\alpha) + \ell(\beta[z_0,z]) \le 2cr_1 + cd(z) \le 4c^2c_0 d(z) + cd(z) = c(4cc_0+1)d(z).$$

The arbitrariness of x and y shows that $D = D_1 \cup D_2$ is a c'-John domain with $c' = c(4cc_0+1)$.

3. The proof of Examples 1.1 and 1.2

3.1. Proof of Example 1.1. The proof of both D_1 and D_2 being John domains easily follows from the fact that an L-bilipschitz image of a c-John domain is c'-John with $c' = L^2 c$. Obviously, $z_0 = (-\frac{1}{2}, 0) \in D_1 \cap D_2$ and $\mathbf{B}(z_0, \frac{1}{3}) \subset D_1 \cap D_2$. Let $D = D_1 \cup D_2$. Then for any positive integer n, $w_n = (-n, 0)$ and $z_n = (n, 0) \in D$. For any γ_n joining w_n and z_n, there must exist a point $u_n \in \gamma_n \cap (D_1 \cap D_2)$ such that $d(u_n) < 2$ and $\ell(\gamma_n[w_n,u_n]) \wedge \ell(\gamma_n[z_n,u_n]) \to \infty$ as $n \to \infty$. This implies that $D_1 \cup D_2$ is not a John domain.

In order to prove Example 1.2, we introduce the following definition.

Definition 3.1. A domain $D \subset E$ is *c-quasiconvex* if each pair of points $a, b \in D$ can be joined with an arc $\gamma \subset D$ with

$$\ell(\gamma) \le c\,|a-b|,$$

where $c > 1$ is a constant.

Proposition 3.1. *If $D \subset E$ is uniform, then it must be quasiconvex.*

3.2. Proof of Example 1.2. Theorem B implies that all domains D_1, D_3 and D_4 are uniform, and Theorem 1.2 shows that D_2 is also uniform. Obviously, $D = D_1 \cup D_2$ is not quasiconvex, hence by Proposition 3.1, D is not uniform.

4. The proof of Theorem 1.2

Before the proof of Theorem 1.2, we introduce two lemmas.

Lemma C. [8, Lemma 3.4] *Suppose that $D \subset E$ is a convex domain. The function $d\colon D \to \mathbf{R}$ is concave, that is,*

$$d\big(ta + (1-t)b\big) \ge td(a) + (1-t)d(b)$$

whenever $a, b \in D$ and $t \in [0,1]$.

Lemma 4.1. *Suppose that X is a vector space with $\dim X \ge 2$, the vectors x_i, y_i, $z_i \in X$ are linearly independent for each $i \in \{1,2\}$, and that*

$$x_1 - y_1 = \lambda(x_2 - y_2),\ y_1 - z_1 = \mu(y_2 - z_2) \text{ and } z_1 - x_1 = \tau(z_2 - x_2)$$

for constants λ, μ, τ. Then $\lambda = \mu = \tau$.

Proof. Since

$$z_1 - x_1 = (z_1 - y_1) + (y_1 - x_1) = \mu(z_2 - y_2) + \lambda(y_2 - x_2),$$

we know

$$(\tau - \lambda)x_2 + (\lambda - \mu)y_2 + (\mu - \tau)z_2 = 0.$$

By the linear independence of $\{x_2, y_2, z_2\}$ we get $\lambda = \mu = \tau$. $\square$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We show that the theorem holds with $c' = (c+1)(2c_0 + 1) + c$. Set $D = D_1 \cup D_2$ and $D_0 = D_1 \cap D_2$. Let $a \in D \setminus D_2$, $b \in D \setminus D_1$, $z_1 = \frac{a+b}{2}$ and $s = \frac{|a-b|}{2}$.

Case 4.1. $|z_1 - z_0| \le 2c_0 s$. There is a c-uniform arc $\beta \subset D_2$ from z_0 to b and an arc $\gamma \subset |a - z_0| \cup \beta$ from a to b. To prove that γ is c'-uniform in D it suffices to show that

(1) $|a - z_0| + \ell(\beta) \le c'|a - b|$,
(2) $|a - x| \le c_0 d(x)$ for all $x \in [a, z_0]$,
(3) $\zeta(y) := \big(|a - z_0| + \ell(\beta[z_0, y])\big) \wedge \ell(\beta[b, y]) \le c' d(y)$ for all $y \in \beta$.

Since $|a - z_0| \le |a - z_1| + |z_1 - z_0| \le s + 2c_0 s$ and similarly $|b - z_0| \le s + 2c_0 s$, we have

$$|a - z_0| + \ell(\beta) \le (2c_0 + 1)s + c|b - z_0| \le (c+1)(2c_0 + 1)s \le c'|a - b|,$$

and (1) is proved.

If $x \in [a, z_0]$, then $x = (1-t)a + tz_0$ for some $t \in [0, 1]$, and we have $|a - x| = t|a - z_0| \le tc_0 r$. Lemma C implies that

$$d(x) \ge d_1(x) \ge (1-t)d_1(a) + td_1(z_0) \ge td_1(z_0) \ge tr.$$

As $|a - x| = t|a - z_0| \le tc_0 r$, this yields (2).

Let $y \in \beta$ and let $y_0 \in \beta$ be the point bisecting the length of β. If $y \in \beta[y_0, b]$, then the c-uniformity of β gives $\zeta(y) = \ell(\beta[y, b]) \le cd_2(y)$. If $y \in \beta[z_0, y_0]$, then

$$\zeta(y) \le |a - z_0| + \ell(\beta[z_0, y]) \le c_0 r + cd_2(y) \le c_0 d(z_0) + cd(y).$$

Here $d(z_0) \le d(y) + |z_0 - y| \le d(y) + \ell(\beta[z_0, y]) \le (1 + c)d(y)$, and we obtain (3).

Case 4.2. $|z_1 - z_0| > 2c_0 s$. Set $e = (b-a)/|b-a|$ and $a_0 = z_0 + re$. As $a \notin D_2$ and $b \notin D_1$, these points do not lie on the line through z_0 and z_1. Hence there is a unique point $w \in [z_0, z_1] \cap [a_0, a]$. Applying Lemma 4.1 to the triples (w, z_0, a_0) and (w, z_1, a) we get $|w - z_1| = s|w - z_0|/r$. Replacing a and a_0 by b and $b_0 = z_0 - re$, respectively, we see that $w \in [z_0, z_1] \cap [b_0, b]$. Hence $w \in D_0$, which implies that $[z_0, w] \subset D_0$. Since $w \in D_1 \subset \mathbf{B}(z_0, c_0 r)$, we have $|w - z_0| \le c_0 r$, whence $|w - z_1| \le c_0 s$.

Set $u = (z_1 - z_0)/|z_1 - z_0|$ and $y_1 = w - c_0 s u$. Then $|y_1 - z_1| = |y_1 - w| + |w - z_1| \le 2c_0 s$, whence $y_1 \in [z_0, w]$. There is a c-uniform arc $\beta_1 \subset D_2$ from y_1 to b and an arc $\gamma \subset [a, y_1] \cup \beta_1$ from a to b. To prove that γ is c'-uniform in D it suffices to show that

(1) $|a - y| + \ell(\beta_1) \le c'|a - b|$,
(2) $|a - x| \le (2c_0 + 1)d(x)$ for all $x \in [a, y_1]$,
(3) $\zeta_1(y) := (|a - y_1| + \ell(\beta_1[y_1, y])) \wedge \ell(\beta_1[b, y]) \le c' d(y)$ for all $y \in \beta_1$.

We have

$$|a-y_1| \le |a-z_1| + |z_1-y_1| \le s + 2c_0 s, \quad \ell(\beta_1) \le c|b-y_1| \le c(2c_0+1)s,$$

and (1) follows.

If $x \in [a, y_1]$, then $x = (1-t)a + ty_1$ for some $t \in [0,1]$. It follows from Lemma C that

$$d(x) \ge d_1(x) \ge (1-t)d_1(a) + td_1(y_1) \ge td_1(y_1).$$

As $|w-z_0| \le c_0 r$, we similarly obtain

$$d_1(y_1) \ge \frac{c_0 s}{|w-z_0|} d_1(z_0) \ge \frac{c_0 sr}{|w-z_0|} \ge s.$$

Hence

$$|a-x| = t|a-y_1| \le t(2c_0+1)s \le (2c_0+1)d(x),$$

which is (2).

Let $y \in \beta_1$ and $y_0 \in \beta_1$ be the point bisecting the length of β_1. If $y \in \beta_1[y_0, b]$, then (3) follows from the c-uniformity of β_1 in D_2. Let $y \in \beta_1[y_1, y_0]$. Now (2) and the c-uniformity of β_1 imply that

$$\zeta_1(y) \le |a-y_1| + \ell(\beta_1[y_1, y]) \le (2c_0+1)d(y_1) + cd_2(y).$$

Here $d(y_1) \le d(y) + |y-y_1| \le d(y) + \ell(\beta_1[y_1, y]) \le (c+1)d(y)$, and we obtain (3).

The proof of Theorem 1.2 is complete. □

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