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BOUNDED DISTORTION HOMEOMORPHISMS ON ULTRAMETRIC SPACES

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Dedicated to José María Montesinos on the occasion of his 65th birthday.

Abstract. It is well-known that quasi-isometries between **R**-trees induce power quasi-symmetric homeomorphisms between their ultrametric end spaces. This paper investigates power quasi-symmetric homeomorphisms between bounded, complete, uniformly perfect, ultrametric spaces (i.e., those ultrametric spaces arising up to similarity as the end spaces of bushy trees). A bounded distortion property is found that characterizes power quasi-symmetric homeomorphisms between such ultrametric spaces that are also pseudo-doubling. Moreover, examples are given showing the extent to which the power quasi-symmetry of homeomorphisms is not captured by the quasiconformal and bi-Hölder conditions for this class of ultrametric spaces.

1. Introduction

A theme in the study of noncompact spaces is the extent to which the geometry and topology of such a space is reflected in a natural boundary at infinity.

By choosing a root v for an **R**-tree T, the boundary at infinity naturally becomes a complete, ultrametric space $\mathrm{end}(T,v)$ of diameter ≤ 1, called the *end space* of (T,v). It is known that a quasi-isometry between **R**-trees induces a bi-Hölder, quasiconformal homeomorphism between their ultrametric end spaces. In fact, the induced homeomorphism has the stronger power quasi-symmetric, or PQ-symmetric, property.

This paper is concerned with the natural question, How close do the bi-Hölder and quasiconformal conditions come to characterizing PQ-symmetric homeomorphisms on bounded, complete ultrametric spaces? What if one restricts to bounded, complete, uniformly perfect, ultrametric spaces? The significance of restricting to this class of ultrametric spaces is that they are (up to similarity) exactly the ones that arise as end spaces of bushy **R**-trees. We introduce a notion of bounded distortion for homeomorphisms and a pseudo-doubling property of metric spaces. We show that the bounded distortion property characterizes PQ-symmetric homeomorphisms on

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bounded, complete, uniformly perfect, ultrametric spaces that also have the pseudo-doubling property. The following is a statement of our main positive result.

Theorem 1.1. *A homeomorphism $h\colon X \to Y$ between bounded, complete, uniformly perfect, pseudo-doubling, ultrametric spaces is PQ-symmetric if and only if h is a bounded distortion equivalence.*

The class of ultrametric spaces to which Theorem 1.1 applies includes end spaces of rooted, geodesically complete, bushy, simplicial **R**-trees—see Remark 5.10.

The more difficult part of the proof of Theorem 1.1 is in showing that bounded distortion equivalences on the given class of ultrametric spaces are PQ-symmetric. This is accomplished in Corollary 5.11. The proof of that corollary relies on Theorem 5.5, which establishes Theorem1.1 for the case of end spaces of rooted, geodesically complete, simplicial, bushy **R**-trees. The converse of Theorem 1.1 is Proposition 3.6.

In Example 6.8 it is shown that this theorem does not hold for compact, uniformly perfect, ultrametric spaces that are not pseudo-doubling. Moreover, we give several examples in Section 6 illuminating the difference between bi-Hölder, quasiconformal homeomorphisms on one hand, and PQ-symmetric homeomorphisms on the other hand for this class of ultrametric spaces. In fact, the examples are defined on end spaces of locally finite, simplicial trees of minimal vertex degree three and answer several questions of Mirani [11, 12].

The study of quasi-isometries between trees and the induced maps on their end spaces has a voluminous literature. This is often set in the more general context of hyperbolic metric spaces and their boundaries. See Bonk and Schramm [1], Buyalo and Schroeder [3], Ghys and de la Harpe [5], Martínez-Pérez [9], Mirani [11, 12], and Paulin [14] to name a few. For homeomorphisms induced by **R**-tree morphisms that are less restrictive than quasi-isometries, see Martínez-Pérez and Morón [10]. For **R**-tree morphisms more restrictive than quasi-isometries, see Hughes [7].

2. Preliminaries on trees, end spaces, and ultrametrics

In this section, we recall the definitions of the trees and their end spaces that are relevant to this paper. We also describe a well-known correspondence between trees and ultrametric spaces. See Feĭnberg [4] for an early result along these lines and Hughes [7] for additional background.

Definition 2.1. Let (T, d) be a metric space.

(1) (T, d) is an ***R**-tree* if T is uniquely arcwise connected and for all $x, y \in T$, the unique arc from x to y, denoted $[x, y]$, is isometric to the subinterval $[0, d(x, y)]$ of **R**.
(2) A *rooted **R**-tree* (T, v) consists of an **R**-tree (T, d) and a point $v \in T$, called the *root*.
(3) A rooted **R**-tree (T, v) is *geodesically complete* if every isometric embedding $f\colon [0, t] \to T$ with $t > 0$ and $f(0) = v$ extends to an isometric embedding $F\colon [0, \infty) \to T$.
(4) A *simplicial **R**-tree* is an **R**-tree (T, d) such that T is the (geometric realization of) a simplicial complex and every edge of T is isometric to the closed unit interval $[0, 1]$.

Definition 2.2. An *ultrametric space* is a metric space (X, d) such that $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

Definition 2.3. The *end space* of a rooted $\mathbf{R}$-tree (T, v) is given by:

$$\text{end}(T, v) = \{F : [0, \infty) \to T \mid F(0) = v \text{ and } F \text{ is an isometric embedding}\}.$$

Let $F, G \in \text{end}(T, v)$.

(1) The *Gromov product at infinity* is $(F|G)_v := \sup\{t \geq 0 \mid F(t) = G(t)\}$.
(2) The *end space metric* is $d_v(F, G) := \mathrm{e}^{-(F|G)_v}$.
(3) The arc $F([0, (F|G)_v])$ is denoted $[F|G]$.
(4) The *bifurcation point* of F and G is $F((F|G)_v) \in T$.

Proposition 2.4. *If (T, v) is a rooted $\mathbf{R}$-tree, then $(\text{end}(T, v), d_v)$ is a complete ultrametric space of diameter ≤ 1.* □

Definition 2.5. Let (U, d) be a complete ultrametric space with diameter ≤ 1. Define an equivalence relation $\sim$ on $U \times [0, \infty)$ by:

$$(x, t) \sim (y, t') \iff t = t' \text{ and } d(x, y) \leq \mathrm{e}^{-t}.$$

Then $T_U := U \times [0, \infty)/\sim$ is the *tree associated to* (U, d). Define a metric D on T_U by:

$$D([x, t], [y, s]) = \begin{cases} |t - s| & \text{if } x = y, \\ t + s - 2\min\{-\ln(d(x, y)), t, s\} & \text{if } x \neq y. \end{cases}$$

A proof of the first item in the following proposition can be found in [7, Theorem 6.3] and a proof of the second item can be found in [7, Proposition 6.4] and [10, Proposition 5.5].

Proposition 2.6. *Let (U, d) be a complete, ultrametric space of diameter ≤ 1 and let (T_U, D) be its associated tree.*

(1) *(T_U, D) is a rooted, geodesically complete $\mathbf{R}$-tree with root $v = [x, 0]$ for any $x \in U$.*
(2) *U is isometric to* $\text{end}(T_U, v)$. □

In this article a *map* is a function that needs not be continuous.

Definition 2.7. A map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is a *quasi-isometric map* if there are constants $\lambda \geq 1$ and $A > 0$ such that for all $x, x' \in X$,

$$\frac{1}{\lambda} d_X(x, x') - A \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + A.$$

If $f(X)$ is a net in Y (i.e., there exists $\epsilon > 0$ such that for each $y \in Y$ there exists $x \in X$ such that $d_Y(f(x), y) < \epsilon$), then f is a *quasi-isometry* and X, Y are *quasi-isometric.*

Remark 2.8. It is well-known that a quasi-isometry $f : T \to T'$ between rooted $\mathbf{R}$-trees (T, v) and (T', w) induces a homeomorphism $\tilde{f} : \text{end}(T, v) \to \text{end}(T', w)$. See, for example, Bridson and Haefliger [2, Chapter I.8], where they work in the more general setting of proper, geodesic, metric spaces.

For a quasi-isometry $f : X \to Y$ between Gromov hyperbolic, almost geodesic metric spaces, Bonk and Schramm define [1, Proposition 6.3] the induced map $\partial f : \partial X \to \partial Y$ between the boundaries at infinity and prove [1, Theorem 6.5] that ∂f is PQ-symmetric (see Definition 3.4 below) with respect to any metrics on ∂X and ∂Y in their canonical gauges. In the special case that X and Y are $\mathbf{R}$-trees,

$\partial X = \text{end}(X, v)$, $\partial Y = \text{end}(Y, w)$ and the end space metrics are in the canonical gauges for any choice of roots.

Another source for the result that quasi-isometries between **R**-trees induce PQ-symmetric homeomorphisms on their ultrametric end spaces, is Buyalo and Schroeder [3, Theorem 5.2.17]. They work with Gromov hyperbolic, geodesic metric spaces and with visual boundaries on their boundaries. When specialized to **R**-trees, these boundaries are the ultrametric end spaces.

3. Homeomorphisms on metric spaces

In this section we begin our discussion of various geometric properties that may be satisfied by homeomorphisms between metric spaces. Homeomorphisms between end spaces of rooted, geodesically complete **R**-trees induced by quasi-isometries of the trees are examples where these properties are encountered. With the possible exceptions of homeomorphisms of bounded distortion and bounded distortion equivalences, defined in Definition 3.2 below, all of these concepts are well-known. In addition to Buyalo and Schroeder [3], other sources for background include Bonk and Schramm [1], Heinonen [6], Roe [15], Semmes [16], and Tukia and Väisälä [17].

Definition 3.1. Let $f\colon X \to Y$ be a homeomorphism between metric spaces (X, d_X) and (Y, d_Y). If $x_0 \in X$ and $\epsilon > 0$, then the *distortion by f of the ϵ-sphere $S(x_0, \epsilon) := \{x \in X \mid d_X(x_0, x) = \epsilon\}$ at x_0* is

$$D_f(x_0, \epsilon) := \begin{cases} \frac{\sup\{d_Y(f(x_0), f(x)) \mid d_X(x_0, x) = \epsilon\}}{\inf\{d_Y(f(x_0), f(x)) \mid d_X(x_0, x) = \epsilon\}} & \text{if } S(x_0, \epsilon) \neq \emptyset, \\ 1 & \text{if } S(x_0, \epsilon) = \emptyset. \end{cases}$$

Definition 3.2. Let $f\colon X \to Y$ be a homeomorphism between metric spaces.

(1) f is *conformal* if $\limsup_{\epsilon \to 0} D_f(x_0, \epsilon) = 1$ for all $x_0 \in X$.

(2) f is *K-quasiconformal*, where $K > 0$, if $\limsup_{\epsilon \to 0} D_f(x, \epsilon) \leq K$ for all $x \in X$.

(3) f is *quasiconformal* if f is K-quasiconformal for some $K > 0$.

(4) f has *bounded distortion* if there exists $K > 0$ such that

$$\sup_{x \in X} \sup_{\epsilon > 0} D_f(x, \epsilon) \leq K.$$

(5) f is a *bounded distortion equivalence* if f and $f^{-1}\colon Y \to X$ have bounded distortion.

Quasiconformal maps are those maps with control on the distortion of sufficiently small spheres. Homeomorphisms of bounded distortion have control on the distortion of every sphere. An example of a quasiconformal (in fact, conformal) homeomorphism that is not of bounded distortion is provided by Example 6.2 below.

Definition 3.3. A map $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is *quasi-symmetric* if f is not constant and if there is a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$, called the *control function for f*, such that whenever $x, a, b \in X$, $t \geq 0$, and $d_X(x, a) \leq t\, d_X(x, b)$ it follows that $d_Y(f(x), f(a)) \leq \eta(t)\, d_Y(f(x), f(b))$.

Thus, a quasi-symmetric map controls distortion of annuli; more precisely, it controls the distortion of ratios between inner and outer radii of annuli.

Definition 3.4. A quasi-symmetric map is said to be *power quasi-symmetric*, or *PQ-symmetric*, if it has a control function of the form

$$\eta(t) = q \max\{t^p, t^{1/p}\}$$

for some $p, q \geq 1$.

Remark 3.5. It follows from Tukia and Väisälä [17, Theorem 2.2] that inverses and compositions of quasi-symmetric homeomorphisms are quasi-symmetric; moreover, inverses and compositions of PQ-symmetric homeomorphisms are PQ-symmetric.

Proposition 3.6. *If a homeomorphism $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is quasi-symmetric, then f is a bounded distortion equivalence.*

Proof. Let η be a control function for f and let $x \in X$ and $\epsilon > 0$ be given. If $a, b \in X$ with $d_X(x,a) = \epsilon = d_X(x,b)$, then $d_Y(f(x), f(a)) \leq \eta(1)\, d_Y(f(x), f(b))$. Hence, $D_f(x, \epsilon) \leq \eta(1)$ and f has bounded distortion with constant $K = \eta(1)$. It follows from Remark 3.5 that f^{-1} is also quasi-symmetric (hence, of bounded distortion) and f is a bounded distortion equivalence. □

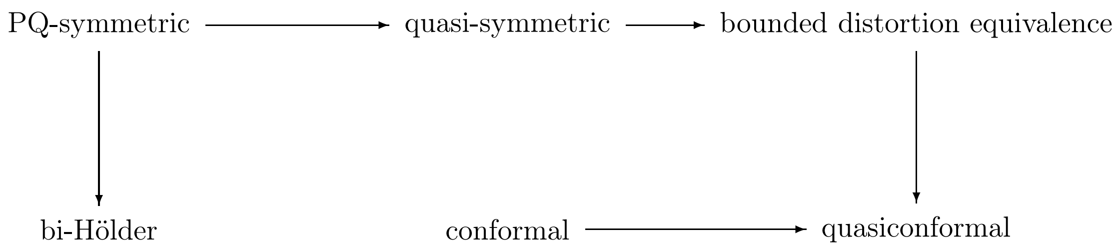
Definition 3.7. A homeomorphism $f\colon X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is *bi-Hölder* if there exists constants $\alpha > 0$ and $c > 0$ such that

$$\frac{1}{c} d_X(x,y)^{1/\alpha} \leq d_Y(f(x), f(y)) \leq c\, d_X(x,y)^{\alpha}$$

for all $x, y \in X$. If $\alpha = 1$, then f is *bi-Lipschitz.*

Remark 3.8. Note that a bi-Lipschitz homeomorphism is PQ-symmetric with $p = 1$, $q = c^2$, and $\eta(t) = qt$. It follows from the proof of Tukia and Väisälä [17, Theorem 3.14] that a PQ-symmetric homeomorphism $f\colon X \to Y$ between bounded metric spaces is bi-Hölder. A quasiconformal homeomorphism need not be bi-Hölder as shown in Example 6.3 below.

Remark 3.9. For homeomorphisms between arbitrary bounded metric spaces, there are the following implications:



In fact, the implication that PQ-symmetric homeomorphisms are bi-Hölder is the only one that requires that the metric spaces be bounded. No other implications hold in general.

4. Homeomorphisms on ultrametric spaces

In this section, the Gromov product at infinity is used to characterize homeomorphisms of bounded distortion, quasi-symmetries, and PQ-symmetries in the case of interest to us, namely, maps between end spaces of trees. It is also shown that local similarity equivalences between compact, ultrametric spaces are PQ-symmetric.

We begin with the characterization of bounded distortion homeomorphisms in terms of the Gromov product at infinity.

Proposition 4.1. *If (T, v) and (T', w) are rooted, geodesically complete* **R**-*trees, then a homeomorphism $h\colon \operatorname{end}(T, v) \to \operatorname{end}(T', w)$ has bounded distortion if and only if there exists a constant $A \geq 0$ such that whenever $F, G, H \in \operatorname{end}(T, v)$ and $(F|G)_v = (F|H)_v$ it follows that $|(h(F)|h(G))_w - (h(F)|h(H))_w| \leq A$.*

Proof. Using the relation between the end space metric and the Gromov product at infinity, it follows that for $F \in \operatorname{end}(T, v)$, $\epsilon > 0$, and $K \geq 1$, we have $D_h(F, \epsilon) \leq K$ if and only if

$$\frac{\mathrm{e}^{-(h(F)|h(G))_w}}{\mathrm{e}^{-(h(F)|h(H))_w}} \leq K,$$

whenever $G, H \in \operatorname{end}(T, v)$ and $(F|G)_v = (F|H)_v = -\ln \epsilon$. Thus, if $F \in \operatorname{end}(T, v)$ and $K \geq 1$, then $\sup_{\epsilon>0} D_h(F, \epsilon) \leq K$ if and only if

$$\mathrm{e}^{(h(F)|h(H))_w - (h(F)|h(G))_w} \leq K,$$

whenever $G, H \in \operatorname{end}(T, v)$ and $(F|G)_v = (F|H)_v$; in turn, this holds if and only if

$$|(h(F)|h(H))_w - (h(F)|h(G))_w| \leq \ln K,$$

whenever $G, H \in \operatorname{end}(T, v)$ and $(F|G)_v = (F|H)_v$. The result follows; furthermore, the relationship between the bounded distortion constant K and the constant A is given by $A = \ln K$. □

The following result gives the characterization of quasi-symmetric maps between end spaces of trees in terms of the Gromov product at infinity.

Proposition 4.2. *If (T, v) and (T', w) are rooted, geodesically complete* **R**-*trees, then a map $f\colon \operatorname{end}(T, v) \to \operatorname{end}(T', w)$ is quasi-symmetric if and only if there exists an orientation-preserving homeomorphism $\gamma\colon \mathbf{R} \to \mathbf{R}$ such that whenever $F, G, H \in \operatorname{end}(T, v)$, it follows that*

$$-\gamma((F|G)_v - (F|H)_v) \leq (F'|H')_w - (F'|G')_w \leq \gamma((F|H)_v - (F|G)_v),$$

where $F' = f(F), G' = f(G)$, and $H' = f(H)$.

Proof. According to Tukia and Väisälä [17, page 99], f is quasi-symmetric if and only if there exists a homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ such that

$$\eta(\rho^{-1})^{-1} \leq \frac{d_w(F', G')}{d_w(F', H')} \leq \eta(\rho)$$

whenever $F, G, H \in \operatorname{end}(T, v)$ and $\rho = \frac{d_v(F,G)}{d_v(F,H)}$. The result now follows easily via the correspondence of η and γ implied by the relation $\gamma(t) = \ln \eta(\mathrm{e}^t)$ and the relation between the end space metric and the Gromov product at infinity. □

The following result gives the characterization of PQ-symmetric maps between end spaces of trees in terms of the Gromov product at infinity.

Proposition 4.3. *If (T, v) and (T', w) are rooted, geodesically complete* **R**-*trees, then a map $f\colon \operatorname{end}(T, v) \to \operatorname{end}(T', w)$ is PQ-symmetric if and only if there exist constants $\lambda \geq 1$ and $A \geq 0$ such that whenever $F, G, H \in \operatorname{end}(T, v)$ and*

$(F|G)_v \le (F|H)_v < \infty$, it follows that

$$\frac{1}{\lambda}((F|H)_v - (F|G)_v) - A \le (F'|H')_w - (F'|G')_w \le \lambda((F|H)_v - (F|G)_v) + A,$$

where $F' = f(F), G' = f(G)$ and $H' = f(H)$.

Proof. Suppose first that f is PQ-symmetric with constants $p, q \ge 1$ as in Definition 3.4 and let $\lambda = p$ and $A = \ln q$. If $F, G, H \in \operatorname{end}(T, v)$ and $(F|G)_v \le (F|H)_v < \infty$, then it follows from the relation between the end space metric and the Gromov product at infinity that

$$\frac{d_v(F,G)}{d_v(F,H)} = \mathrm{e}^{(F|H)_v - (F|G)_v} \ge 1.$$

Thus, $d_v(F,G) = t \cdot d_v(F,H)$, where $t \ge 1$ and $\ln t = (F|H)_v - (F|G)_v$. The PQ-symmetric property implies

$$d_w(F', G') \le qt^p d_w(F', H')$$

and

$$d_w(F', H') \le q(1/t)^{1/p} d_w(F', G').$$

These two inequalities are equivalent to

$$(F'|G')_w \ge (F'|H')_w - \lambda((F|H)_v - (F|G)_v) - A$$

and

$$(F'|H')_w \ge (F'|G')_w + \frac{1}{\lambda}((F|H)_v - (F|G)_v) - A,$$

which are in turn equivalent to the desired inequalities.

Conversely, given $\lambda \ge 1$ and $A \ge 0$ satisfying the given conditions, let $p = \lambda$ and $q = \mathrm{e}^A$. Suppose $F, G, H \in \operatorname{end}(T, v)$, $t \ge 0$, and $d_v(F,G) \le t\, d_v(F,H)$. It must be shown that

$$d_w(F', G') \le \begin{cases} qt^p\, d_w(F', H') & \text{if } t \ge 1, \\ qt^{1/p}\, d_w(F', H') & \text{if } t \le 1 \end{cases}$$

(where F', G', H' continue to denote the images of F, G, H, respectively, under f). We may assume $0 < d_v(F,G)$ and $t > 0$, for otherwise the result is trivial.

We first consider the case $(F|G)_v \le (F|H)_v$. In particular, $d_v(F,H) \le d_v(F,G)$ and $t \ge 1$. The right-hand part of the assumed inequalities takes the form

$$\ln \frac{d_w(F',G')}{d_w(F',H')} \le p \ln \frac{d_v(F,G)}{d_v(F,H)} + \ln q = \ln \left[q \left(\frac{d_v(F,G)}{d_v(F,H)} \right)^p \right].$$

Therefore,

$$\frac{d_w(F',G')}{d_w(F',H')} \le q \left(\frac{d_v(F,G)}{d_v(F,H)} \right)^p \le qt^p,$$

as required.

Finally, consider the case $(F|H)_v \le (F|G)_v$. The left-hand part of the assumed inequalities takes the form

$$\ln \left[\frac{1}{q} \left(\frac{d_v(F,H)}{d_v(F,G)} \right)^{1/p} \right] = \frac{1}{p} \ln \frac{d_v(F,H)}{d_v(F,G)} - \ln q \le \ln \frac{d_w(F',H')}{d_w(F',G')}.$$

Therefore,

$$\frac{1}{q}\left(\frac{1}{t}\right)^{1/p} \leq \frac{1}{q}\left(\frac{d_v(F,H)}{d_v(F,G)}\right)^{1/p} \leq \frac{d_w(F',H')}{d_w(F',G')}$$

and

$$d_w(F',G') \leq qt^{1/p}\, d_w(F',H').$$

This is exactly what is required if $t \leq 1$; if $t \geq 1$, what is required follows by using $t^{1/p} \leq t^p$. □

Remark 4.4. The following characterization of bi-Hölderness in terms of the Gromov product at infinity is straightforward to verify. If (T,v) and (T',w) are rooted, geodesically complete **R**-trees, then a homeomorphism $f\colon \operatorname{end}(T,v) \to \operatorname{end}(T',w)$ is bi-Hölder if and only if there exist constants $\lambda \geq 1$ and $A \geq 0$ such that whenever $F, G \in \operatorname{end}(T,v)$, it follows that

$$\frac{1}{\lambda}(F|G)_v - A \leq (F'|G')_w \leq \lambda(F|G)_v + A,$$

where $F' = f(F)$ and $G' = f(G)$.

Remark 4.5. Suppose (T,v) and (T',w) are rooted, geodesically complete, **R**-trees and $h\colon \operatorname{end}(T,v) \to \operatorname{end}(T',w)$ is a homeomorphism induced by a rooted homeomorphism $\hat{h}\colon (T,v) \to (T',w)$; i.e., $h(F) = \hat{h}\circ F$ for all $F \in \operatorname{end}(T,v)$. It follows that if $F, G, H \in \operatorname{end}(T,v)$ and $(F|G)_v = (F|H)_v$, then $(h(F)|h(G))_w = (h(F)|h(H))_w$. Thus, h is conformal and a bounded distortion equivalence (with constant $K = 1$).

Finally, we show that the local similarity equivalences between compact ultrametric spaces studied in [7, 8] are PQ-symmetric. In fact, we show that they are bi-Lipschitz. In particular, this affirms a conjecture of Mirani [11, 12].

Definition 4.6. A function $f\colon X \to Y$ between metric spaces (X, d_X), (Y, d_Y) is a *similarity* if there exists $\lambda > 0$ such that $d_Y(f(x), f(y)) = \lambda d_X(x,y)$ for all $x, y \in X$. In this case, f is a *λ-similarity.*

Definition 4.7. A homeomorphism $h\colon X \to Y$ between metric spaces is a *local similarity equivalence* if for every $x \in X$ there exist $\varepsilon > 0$ and $\lambda > 0$ such that the restriction $h|_{B(x,\varepsilon)}\colon B(x,\varepsilon) \to B(h(x), \lambda\varepsilon)$ is a surjective λ-similarity.

Proposition 4.8. *If $f : U \to V$ is a local similarity equivalence between compact, ultrametric spaces, then f is bi-Lipschitz. In particular, f is PQ-symmetric.*

Proof. Up to similarity homeomorphism, we may assume that the diameters of U and V are ≤ 1. Proposition 2.6 then implies that $U = \operatorname{end}(T,v)$ and $V = \operatorname{end}(T',w)$, where (T,v) and (T',w) are rooted, geodesically complete **R**-trees. There is a local similarity equivalence $h\colon \operatorname{end}(T,v) \to \operatorname{end}(T',w)$ induced by conjugating f by similarities. Clearly, it suffices to show that h is bi-Lipschitz. To this end, we will show that there exists some constant $K > 0$ such that if $F, G \in \operatorname{end}(T,v)$, then $|(F|G)_v - (h(F)|h(G))_w| \leq K$. If there is such a constant K, then it is clear that: if $F, G \in \operatorname{end}(T,v)$, then $(F|G)_v - K \leq (h(F)|h(G))_w \leq (F|G)_v + K$ and $e^{-K}d(F,G) \leq d(h(F), h(G)) \leq e^K d(F,G)$. That is, h is bi-Lipschitz.

To complete the proof, we will establish the existence of K. For every $F \in \operatorname{end}(T,v)$ there exist $\varepsilon > 0$ and $\lambda > 0$ such that the restriction $h|_{B(x,\varepsilon)}\colon B(x,\varepsilon) \to B(h(x), \lambda\varepsilon)$ is a surjective λ-similarity. Since $\operatorname{end}(T,v)$ is compact, there is a finite, open covering $\{B(x_i, \varepsilon_i)\}_{i=1}^n$ and associated similarity constants $\{\lambda_i\}_{i=1}^n$. Since h is a homeomorphism, $\{B(h(x_i), \lambda_i\varepsilon_i))\}_{i=1}^n$ is also an open covering of $\text{end}(T', w)$. Let δ_1 a Lebesgue number for the covering $\{B(x_i, \varepsilon_i)\}_{i=1}^n$ and δ_2 a Lebesgue number for the covering $\{B(h(x_i), \lambda_i\varepsilon_i)\}_{i=1}^n$. Define

$$K := \max\{\max\{|\ln(\lambda_i)| \mid i = 1, \ldots, n\}, -2\ln(\delta_1), -2\ln(\delta_2)\}.$$

If $F, G \in B(x_i, \varepsilon_i)$ for some $i = 1, \ldots, n$, then $d(h(F), h(G)) = \lambda_i d(F, G)$ and $|(F|G)_v - (h(F)|h(G))_w| = |\ln(\lambda_i)| \leq K$, as desired. On the other hand, if F, G are not in any such ball, then $h(F)$ and $h(G)$ are not in the same ball $B(h(x_i), \lambda_i\varepsilon_i))$ for any i; therefore, $d(h(F), h(G)) > \delta_2$. Thus, $|(F|G)_v - (h(F)|h(G))_w| \leq (F|G)_v + (h(F)|h(G))_w \leq -\ln(\delta_1) - \ln(\delta_2) \leq K$, which establishes the desired property of K. □

Remark 4.9. An alternative proof that a local similarity equivalence between compact, ultrametric spaces is PQ-symmetric can be obtained as follows. First, it is possible to modify the proof of Tukia and Väisälä [17, Theorem 2.23] to show that a local PQ-symmetric embedding $f : X \to Y$ between metric spaces, where X is compact, is PQ-symmetric. Then observe that, since a similarity is PQ-symmetric (with $p = 1 = q$), a local similarity equivalence is locally PQ-symmetric.

5. Homeomorphisms on uniformly perfect, ultrametric spaces

In this section, it is shown that a bounded distortion homeomorphism between bounded, complete, uniformly perfect, pseudo-doubling, ultrametric spaces is PQ-symmetric—see Corollary 5.11. This is the remaining part of the main result Theorem 1.1, namely, the sufficiency of bounded distortion for PQ-symmetry. The proof of Corollary 5.11 consists of a reduction to the end spaces of simplicial, bushy **R**-trees. That reduction is contained in Theorem 5.5. This section also contains the definition of pseudo-doubling metric spaces and a brief discussion of its relationship to the well-known notion of doubling metric spaces.

We begin by recalling the following definition of uniformly perfect for metric spaces and its relation to bushy **R**-trees.

Definition 5.1. A metric space X is *uniformly perfect* if there is a constant $\mu \in (0, 1)$ such that for every $x \in X$ and every $r > 0$, it follows that $B_r(x) \backslash B_{\mu r}(x) \neq \emptyset$ unless $X = B_r(x)$.

Tukia and Väisälä [17] proved that a quasi-symmetric homeomorphism between uniformly perfect metric spaces is PQ-symmetric (see also [6, Theorem 11.3, page 89]).

Now recall the following definition from Mosher, Sageev, and Whyte [13].

Definition 5.2. An **R**-tree T is *bushy* if there is a constant $K > 0$, called a *bushy constant*, such that for any point $x \in T$ there is a point $y \in T$ such that $d(x, y) < K$ and $T \backslash \{y\}$ has at least 3 unbounded components.

Remark 5.3. Note that a rooted, geodesically complete **R**-tree T is bushy if and only if $\text{end}(T, v)$ is uniformly perfect for some (respectively, for every) $v \in T$.

Remark 5.4. As Mosher, Sageev, and Whyte [13, page 118] point out, any two bounded valence, locally finite, simplicial, bushy trees are quasi-isometric. Therefore, any such tree is quasi-isometric to the infinite binary tree. See also Bridson and

Haefliger [2, page 141, Exercise 8.20(2)] for the special case of regular, simplicial trees.

Theorem 5.5. *If a map $h\colon \operatorname{end}(T,v) \to \operatorname{end}(T',w)$ between the end spaces of rooted, geodesically complete, simplicial, bushy $\mathbf{R}$-trees is a bounded distortion equivalence, then h is PQ-symmetric.*

Proof. Let $K > 0$ be a bushy constant for both T and T', and let e^A be the constant for the bounded distortion equivalence, where $A \geq 0$. Let $p = A + 2K$, $q_1 = 4K + 3A$, $q_2 = \max\{q_1^{1/p}, q_1\mathrm{e}^{2A}\}$, and $q = \max\{q_1, q_2\}$. We will show that h is PQ-symmetric with constants p and q.

Consider any three points $G_0, G_1, G_2 \in \operatorname{end}(T,v)$ and suppose $d(G_0,G_1) = d(G_1,G_2) \geq d(G_0,G_2)$. Let $t_0 = (G_0|G_1)_v$ and $t_1 = (G_0|G_2)_v$. Then, $d(G_0,G_1) = t \cdot d(G_0,G_2)$ with $t = \mathrm{e}^{t_1 - t_0} \geq 1$. Since T is a simplicial $\mathbf{R}$-tree, there exists $k \in \mathbf{N}$ and vertices $x_0, \dots, x_{k+1} \in T$ as shown in Figure 1 such that

(1) $[G_0|G_1] = [v, x_0]$ and $[G_0|G_2] = [v, x_{k+1}]$
(2) $d(x_0, x_{k+1}) = k + 1$
(3) $d(v, x_i) = d(v, x_0) + i \in \mathbf{N}$ for each $i = 0, \dots, k+1$
(4) $x_i \in [v, x_{k+1}]$ for each $i = 0, \dots, k+1$

For each $i = 0, \dots, k+1$, let $\mathscr{F}_i = \{F \in \operatorname{end}(T,v) \mid [F|G_0] = [v, x_i]\}$. Note that $\mathscr{F}_i$ may be empty.

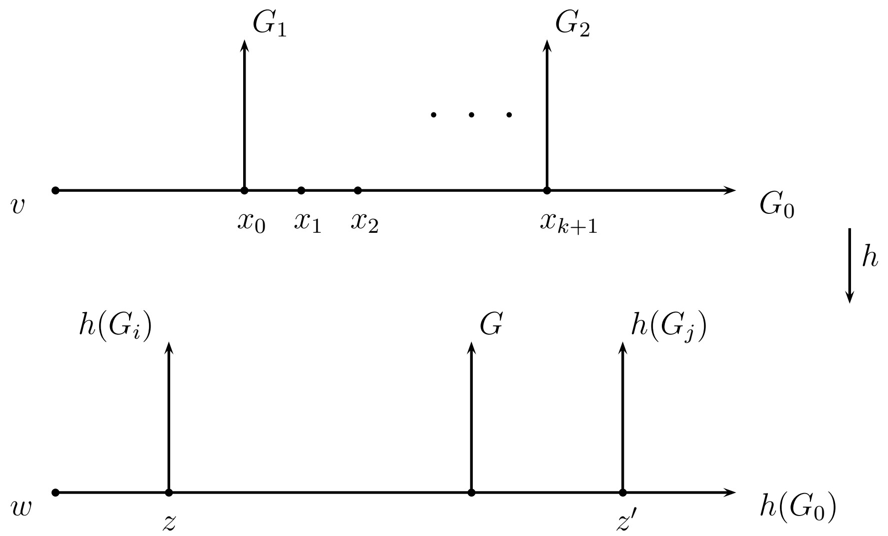


Figure 1. $h\colon \operatorname{end}(T,v) \to \operatorname{end}(T',w)$.

The first step is to show that

$$d(h(G_0), h(G_1) \leq q_1 \cdot t^p d(h(G_0), h(G_2)).$$

Suppose on the contrary that

$$d(h(G_0), h(G_1) > (4K + 3A) \cdot t^{A+2K} d(h(G_0), h(G_2)),$$

which is to say,

$$(h(G_0)|h(G_1))_w - (h(G_0)|h(G_2))_w > (k+1)(A + 2K) + 4K + 3A.$$

For any two $F_1, F_2 \in \mathscr{F}_i$, $0 \leq i \leq k+1$, the bounded distortion condition with respect to G_0 implies that

$$|(h(G_0)|h(F_1))_w - (h(G_0)|h(F_1))_w| \leq A.$$

It follows that for each $i = 0, \ldots, k+1$, there exists a subinterval I_i of $h(G_0)$ of length A such that the bifurcation points of $h(G_0)$ and elements of $h(\mathscr{F}_i)$ and are all contained in the subinterval I_i. Since T' is bushy with constant K, there exists $G \in \mathrm{end}(T', w) \setminus \bigcup_{i=0}^{k+1} h(\mathscr{F}_i)$ such that

$$(h(G_0)|h(G_1))_w + A < (G|h(G_0))_w < (h(G_0)|h(G_2))_w - A.$$

Suppose z is the bifurcation point of $h(G_0)$ and $h(G_1)$, and z' the bifurcation point of $h(G_0)$ and $h(G_2)$. The interval $I := [G_0(\|z\| + A), G_0(\|z'\| - A)]$ has length at least $(k+1)(A+2K) + 4K + A$. Thus, I contains an open interval of length $2K$ disjoint from each of the $k+2$ subintervals I_i, for $0 \leq i \leq k+1$. Since K is the bushy constant, there exists $H \in \mathrm{end}(T', w)$ such that the bifurcation point of $h(G_0)$ and H is in I. The surjectivity of h implies there exists $G \in \mathrm{end}(T, v) \setminus \bigcup_{i=0}^{k+1} \mathscr{F}_i$ such that $h(G) = H$. There are two cases:

(a) $(G|G_0) < \|x_0\|$. This leads to a contradiction by applying the bounded distortion condition on G with respect to G_0, G_1 in conjunction with the fact that $|(h(G)|h(G_0))_w - (h(G)|h(G_1))_w| > A$.

(b) $(G|G_0) > \|x_{k+1}\|$. This leads to a contradiction by applying the bounded distortion condition on G_2 with respect to G, G_0 in conjunction with the fact that $(H|h(G_0))_w < (h(G_0)|h(G_2))_w - A$.

In either case, there is a contradiction. The conclusion is that $d(h(G_0), h(G_1)) \leq q_1 t^p \cdot d(h(G_0), h(G_2))$ and the first step is complete.

The second and final step is to show that

$$d(h(G_0), h(G_2)) \leq q_2 \left(\frac{1}{t}\right)^{\frac{1}{p}} \cdot d(h(G_0), h(G_1)),$$

which is to say,

$$d(h(G_0), h(G_1)) \geq \frac{1}{q_2} t^{\frac{1}{p}} \cdot d(h(G_0), h(G_2)).$$

To this end note that the bounded distortion assumption together with the fact that $d(G_0, G_1) = d(G_1, G_2) \geq d(G_0, G_2)$, implies that

$$d(h(G_0), h(G_1)) \geq \mathrm{e}^{-A} \cdot d(h(G_0), h(G_2)).$$

Let $t_0' = (h(G_0)|h(G_1))_v$, $t_1' = (h(G_0)|h(G_2))_v$ and $t' = \mathrm{e}^{|t_1 - t_0|}$. By the same argument as before, $(4K+3A)t'^p \geq t$. Therefore, $t' \geq (\frac{t}{4K+3A})^{\frac{1}{p}}$. In other words, either

$$d(h(G_0), h(G_1)) \geq \left(\frac{1}{4K+3A}\right)^{\frac{1}{p}} t^{\frac{1}{p}} d(h(G_0), h(G_2))$$

or

$$\mathrm{e}^A d(h(G_0), h(G_1)) \geq d(h(G_0), h(G_2)) \geq \left(\frac{1}{4K+3A}\right)^{\frac{1}{p}} t^{\frac{1}{p}} d(h(G_0), h(G_1)),$$

where the second is only possible if $t^{\frac{1}{p}} \leq (4K+3A)\mathrm{e}^A$. In the first case, the desired result follows from the fact that $q_2 \geq (4K+3A)^{\frac{1}{p}}$. In the second case,

$$d(h(G_0),h(G_1)) \geq \mathrm{e}^{-A} d(h(G_0),h(G_2)) \geq \mathrm{e}^{-A}\frac{t^{\frac{1}{p}}}{(4K+3A)\mathrm{e}^A} d(h(G_0),h(G_2)).$$

Thus, the desired result follows from the fact that $q_2 \geq (4K+3A)\mathrm{e}^{2A}$. In both cases, it is readily seen that $d(h(G_0),h(G_1)) \geq \frac{1}{q_2} t^{\frac{1}{p}} d(h(G_0),h(G_2))$, completing the second step. Thus, h is PQ-symmetric. □

Definition 5.6. A map between metric spaces $f\colon X \to Y$ is *roughly isometric* if there is a constant $a > 0$ such that $d(x,x') - a \leq d(f(x)f(x')) \leq d(x,x') + a$ for all $x, x' \in X$. If, in addition, $f(X)$ is a net in Y, then f is a *rough isometry* and X and Y are *roughly isometric.*

Obviously, a rough isometry is a quasi-isometry.

Proposition 5.7. *If T is an* $\mathbf{R}$*-tree T, then there exists a simplicial* $\mathbf{R}$*-tree S such that T and S are (continuously) rough isometric. In particular, T and S are (continuously) quasi-isometric.*

Proof. Fix a root $v \in T$ and consider (T,v) the rooted $\mathbf{R}$-tree. For each $n = 1,2,3,\ldots$, let $S_n = \partial B(v,n)$. Let $S_0 = \{w\}$, where w is a point not in $\bigcup_{i=1}^{\infty} S_n$. Define the rooted, simplicial $\mathbf{R}$-tree (S,w) by joining with edges of length 1 all the points in S_1 to w and each point $s \in S_k$ to the unique point $t \in S_{k-1}$ such that $t \in [v,s]$. Thus, $\bigcup_{i=0}^{\infty} S_i$ is the vertex set of S.

Any isometric embedding of the form $F\colon [0,\infty) \to T$ or $F\colon [0,N] \to T$, where $F(0) = v$ and $N \in \mathbf{N}$, is completely determined by the sequence $\{F(k)\}_{k=1}^{\infty}$ or $\{F(k)\}_{k=1}^{N}$, respectively. Since these sequences are in the vertex set of S, there is a corresponding isometric embedding of the form $F'\colon [0,\infty) \to S$ or $F'\colon [0,N] \to S$, where $F'(0) = w$. Extend this construction to isometric embeddings of the form $F\colon [0,t] \to T$, where $F(0) = v$ and $t > 0$, by letting F' denote $(F|[0,\lfloor t\rfloor])'$, where $\lfloor t\rfloor$ denotes the greatest integer less than or equal to t. Denote this association by $p\colon F \mapsto F'$. (Note that p is surjective between the sets of isometric embeddings. However, p need not be injective because of the existence of domains of the form $[0,t]$ for non-integral t.) Define the map $h\colon (T,v) \to (S,w)$ as follows: First, $h(x) = w$ for all $x \in B(v,1)$. Second, $h(F(t)) = F'(t-1)$ for any isometric embedding of the form $F\colon [0,\infty) \to T$ or $F\colon [0,t] \to T$, where $F(0) = v$ and $t \geq 1$. In particular, for every integer $k > 0$ in the domain of F, $h(F(k)) = F'(k-1)$ which, by the definition of S, is the point $F(k-1)$. In order to show that h is well-defined, let F and G be two isometric embeddings into T of the type considered above. If t is in the domains of F and G, then $F(t) = G(t)$ implies that $F(\lfloor t\rfloor) = G(\lfloor t\rfloor)$. In particular, $F'(t-1) = G'(t-1)$. Therefore, $h(F(t)) = F'(t-1) = G'(t-1) = h(G(t))$.

Finally, define $j\colon (S,w) \to (T,v)$ by $j(H(t)) = F(t)$, where $H\colon [0,\infty) \to S$ or $H\colon [0,N] \to S$ is an isometric embedding such that $H(0) = w$ and $N \in \mathbf{N}$, and F is an isometric embedding in T such that $p(F) = H$. Note that $j(H(t))$ does not depend on the choice of $F \in p^{-1}(H)$. For $p(F) = p(G)$ and $F \neq G$ if and only if F and G have finite domains and $F(k) = H(k) = G(k)$ and for all integers k in their domains. Clearly, h is a roughly isometric map with with constant $a = 1$. Moreover, j shows that h is a rough isometry between T and S . □

The following definition is well-known. The name "doubling" comes from the fact that usually $C = 2$. The version given here appears in Bonk and Schramm [1, Section 9].

Definition 5.8. A metric space is *doubling* if for every $C > 1$ there exists $N \in \mathbf{N}$ such that: if $0 < r < R$ with $R/r = C$, then every open ball of radius R in X can be covered by N open balls of radius r.

Definition 5.9. A metric space is *pseudo-doubling* if for every $C > 1$ there exists $N \in \mathbf{N}$ such that: if $0 < r < R$ with $R/r = C$ and $x \in X$, then there are at most N balls B such that $B(x, r) \subseteq B \subseteq B(x, R)$.

Remark 5.10. The following facts are easy to verify.

(1) Every doubling ultrametric space is pseudo-doubling.
(2) Let $X = \mathrm{end}(T, v)$, where (T, v) is a rooted, geodesically complete, simplicial $\mathbf{R}$-tree. Then X is pseudo-doubling. However, if the set of valences of the vertices of T is unbounded, then X is not doubling.
(3) The closed unit interval $[0, 1]$ with its standard metric is a doubling metric space that is not pseudo-doubling.

Corollary 5.11. *If $f\colon X \to Y$ is a bounded distortion equivalence between bounded, complete, uniformly perfect, pseudo-doubling ultrametric spaces, then f is a PQ-symmetric homeomorphism.*

Proof. Let $X = \mathrm{end}(T_X, v), Y = \mathrm{end}(T_Y, w)$, where T_X, T_Y are geodesically complete, bushy $\mathbf{R}$-trees (see Remark 5.3). Since X is pseudo-doubling, for $R/r = \mathrm{e}$ there exists a constant N such that for any $x \in X$ there are at most N balls B such that $B(x, r) \subseteq B \subseteq B(x, R)$. It follows that for each $k \in \mathbf{N}$ there are at most N different real numbers $\alpha_1, \ldots, \alpha_n$ with $k < \alpha_1 < \alpha_2 < \cdots < \alpha_n < k + 1$ such that whenever $F, G \in \mathrm{end}(T_X, v)$ and $k < (F|G)_v < k + 1$, then $(F|G)_v = \alpha_i$ for some $i = 1, \ldots, n$.

We will now distort the tree T_X up to rooted homeomorphism so that Remark 4.5 applies. Divide each interval $[k, k + 1]$ into $N + 1$ subintervals of length $1/(N + 1)$. For any pair of points $F, G \in \mathrm{end}(T_X, v)$, if $(F|G)_v = \alpha_i$, change it to $(F|G)_v = k + i/(N + 1)$. Denote this new tree by $[T_X]$. It is immediate to see that the natural rooted homeomorphism $T_X \to [T_X]$ induces a bi-Lipschitz equivalence $\mathrm{end}(T_X, v) \to \mathrm{end}([T_X], v)$ with constant e.

Now consider S_X, a barycentric subdivision of $[T_X]$ obtained by dividing each edge into $N + 1$ subintervals of length $1/(N + 1)$ and adding the corresponding N new vertices. Consider each new edge to be of length 1. Then S_X is similar to $[T_X]$ (the distance is multiplied by $N + 1$) and, by construction of $[T_X]$, S_X is simplicial. Also, the canonical map $h_X\colon \mathrm{end}(T_X, v) \to \mathrm{end}(S_X, v)$ is such that $(\frac{1}{\mathrm{e}} d(F, G))^{N+1} \leq d(h_X(F), h_X(Y)) \leq (\mathrm{e} \cdot d(F, G))^{N+1}$. Thus, h_X is PQ-symmetric.

Likewise distort T_Y to $[T_Y]$ and construct S_Y and $h_Y\colon \mathrm{end}(T_Y, w) \to \mathrm{end}(S_Y, w)$.

It follows that f induces a bounded distortion equivalence

$$\tilde{f} := h_Y \circ f \circ h_X^{-1}\colon \mathrm{end}(S_X, v) \to \mathrm{end}(S_Y, w)$$

between ends of rooted, geodesically complete, simplicial, bushy $\mathbf{R}$-trees. Theorem 5.5 implies that $\tilde{f}\colon \mathrm{end}(S_X, v) \to \mathrm{end}(S_Y, w)$ is PQ-symmetric. Since $f = h_Y^{-1} \circ \tilde{f} \circ h_X$, Remark 3.5 implies that f is PQ-symmetric. $\square$

6. The examples

This section contains several examples that illustrate the sharpness of the results in the previous sections. In addition, some of the examples answer several questions raised by Mirani [11, 12]. The basic building block for our examples is the infinite binary tree. We begin by introducing notation that will help us describe the examples.

Notation 6.1. Let (T_2, v) denote the rooted, infinite binary tree, also known as the Cantor tree. Thus, T_2 is a locally finite, simplicial $\mathbf{R}$-tree, the root has valency two, and all other roots have valency three. All edges are labeled 0 or 1; every vertex is incident to at least one edge labeled 0 and to at least one edge labeled 1. Let $X_2 = \text{end}(T, v)$. Thus,

$$X_2 = \{x = (x_1, x_2, \dots) \mid x_i \in \{0,1\}, i = 1, 2, \dots\} = \prod_1^\infty \{0,1\}$$

with Gromov product at infinity given by $(x|y)_v = i$ if and only if $x_j = y_j$ for $j \leq i$ and $x_{i+1} \neq y_{i+1}$.

If $a = (a_1, \dots, a_n) \in \prod_1^n \{0,1\}$, then

$$aX_2 := \{y \in X_2 \mid y_i = a_i \text{ for } 1 \leq i \leq n\}.$$

Thus, each $y \in aX_2$ can be written uniquely as $y = ax$, where $x \in X_2$. Note that aX_2 is a closed ball in X_2.

We denote certain infinite and finite sequences as follows:

$$\bar{0} = (0,0,0,\dots) \in X_2, \qquad \bar{1} = (1,1,1,\dots) \in X_2,$$
$$\bar{0}_n = (0,\dots,0) \in \prod_1^n \{0,1\}, \qquad \bar{1}_n = (1,\dots,1) \in \prod_1^n \{0,1\}.$$

If $n = 0$, then $\bar{0}_n$ and $\bar{1}_n$ denote the empty sequences.

Since much of Ghys and de la Harpe [5] and Mirani [11, 12] is in the setting of simplicial $\mathbf{R}$-trees with the property that each vertex has valency at least 3, we want to provide examples in that class of trees whenever possible. A simple way to achieve this is to let $(T_3, v))$ be the unique rooted, simplicial $\mathbf{R}$-tree containing (T_2, v) as a rooted subtree such that every vertex of T_3 has valency exactly 3. In other words, T_3 is the infinite 3-regular tree.

Let $X_3 = \text{end}(T_3, v)$. Note that X_2 and X_3 are compact, uniformly perfect, doubling ultrametric spaces. Moreover, X_2 is a closed and open subspace of X_3.

The first example shows that the hypothesis of Theorems 1.1 and 5.5 can not be changed from "bounded distortion equivalence" to "bi-Hölder and conformal homeomorphism".

Example 6.2. *There exist a bi-Hölder, conformal homeomorphism $h\colon X_3 \to X_3$ that does not have bounded distortion. In particular, h is not PQ-symmetric.*

Proof. For each $i \geq 1$, define the following sequences:

$$\begin{aligned} a_i &= \bar{1}_{i-1}011, \\ b_i &= \bar{1}_{i-1}\bar{0}_{i+1}11, \\ c_i &= \bar{1}_{i-1}010, \\ F_i &= \bar{1}_{i-1}\bar{0}. \end{aligned}$$

Define a homeomorphism $h\colon X_3 \to X_3$ by

$$\begin{cases} h(a_i x) = b_i x & \text{for all } x \in X_2 \text{ and } i = 1, 2, \dots, \\ h(b_i x) = a_i x & \text{for all } x \in X_2 \text{ and } i = 1, 2, \dots, \\ h(x) = x & \text{for all } x \in X_3 \setminus \bigcup_{i=1}^{\infty} (a_i X_2 \cup b_i X_2). \end{cases}$$

To see that h is conformal, note that if $\bar{1} \neq F \in X_2$, then there exists $n_F \geq 1$ such that the $n_F{}^{\text{th}}$ term of F is 0. It follows that the image under h of any sphere centered at F of radius $r \leq \mathrm{e}^{-1-n_F}$ is a sphere; thus, $D_h(F, r) = 1$. Clearly, for all $r > 0$, $D_h(\bar{1}, r) = 1$ and $D_h(F, r) = 1$ whenever $F \in X_3 \setminus X_2$. Thus, h is conformal.

To see that h is bi-Hölder, it suffices to check that for any pair of points $F, G \in X_3$, $\frac{1}{2}(F|G)_v - 1 \leq (h(F)|h(G))_v \leq 2(F|G)_v + 1$. It only happens that $(h(F)|h(G))_v \neq (F|G)_v$ when

(1) one of F or G is in $a_i X_2$ or $b_i X_2$ and the other is in $X_3 \setminus \bigcup_{i=1}^{\infty} (a_i X_2 \cup b_i X_2)$, or
(2) $F, G \in a_i X_2$ for some $i \geq 1$, or
(3) $F, G \in b_i X_2$ for some $i \geq 1$.

In each case, the required inequalities are easy to check.

To see that h does not have bounded distortion, choose $G_i \in a_i X_2$ and $H_i \in c_i X_2$ for each $i \geq 1$. Note that $h(F_i) = F_i$, $h(G_i) \in b_i X_2$, and $h(H_i) = H_i$ for all $i \geq 1$. Moreover, $(F_i|G_i)_v = (F_i|H_i)_v = i$ and $(h(F_i)|h(G_i))_v = 2i$ for all $i \geq 1$. Thus, $|(h(F_i|h(G_i))_v - (h(F_i|h(H_i))_v| = i$ and Proposition 4.1 shows that h does not have bounded distortion.

Finally, it follows from Proposition 3.6 that h is not PQ-symmetric. □

The following example illustrates that the result of Tukia and Väisälä [17, Theorem 3.14] mentioned in Remark 3.8 above, that a PQ-symmetric homeomorphism between bounded metric spaces is bi-Hölder, does not hold if the hypothesis "PQ-symmetric" is weakened to "quasiconformal", even for ultrametric spaces as nice as X_3.

Example 6.3. *There exists a conformal homeomorphism $h\colon X_3 \to X_3$ that is not bi-Hölder.*

Proof. For each $i \geq 1$, define the following sequences:

$$\begin{aligned} g_i &= \bar{1}_{i-1}01, \\ h_i &= \bar{1}_{i-1}\bar{0}_{i^2-i}1. \end{aligned}$$

Define a homeomorphism $h\colon X_3 \to X_3$ by

$$\begin{cases} h(g_i x) = h_i x & \text{for all } x \in X_2 \text{ and } i = 1, 2, \dots, \\ h(h_i x) = g_i x & \text{for all } x \in X_2 \text{ and } i = 1, 2, \dots, \\ h(x) = x & \text{for all } x \in X_3 \setminus \bigcup_{i=1}^{\infty} (g_i X_2 \cup h_i X_2). \end{cases}$$

The argument to prove that h is conformal is similar to the one in Example 6.2. To see that h is not bi-Hölder, for each $i = 1, 2, 3, \dots$ choose $G_i \in g_i X_2$ and let $F_i \in X_2$ be as in Example 6.2. Note that $h(F_i) = F_i$ and $h(G_i) \in h_i X_2$ for all $i \geq 1$. Thus, $d(h(F_i), h(G_i)) = \mathrm{e}^{-i^2}$ while $d(F_i, G_i) = \mathrm{e}^{-i}$ for all $i \geq 1$. It follows that h is not bi-Hölder. □

In the converse direction, Mirani [11, 12] speculated that bi-Hölder homeomorphisms on spaces such as X_3 might be quasi-conformal. The following example shows this is not the case.

Example 6.4. *There exists a bi-Hölder homeomorphism $h\colon X_3 \to X_3$ that is not not quasiconformal.*

Proof. For each $i \geq 1$, define the following sequences:

$$g_i = \bar{0}_{4^i} 11,$$
$$h_i = \bar{0}_{2 \cdot 4^i} 11.$$

Define a homeomorphism $h\colon X_3 \to X_3$ by

$$\begin{cases} h(g_i x) = h_i x & \text{for all } x \in X_2 \text{ and } i = 1, 2, \dots, \\ h(h_i x) = g_i x & \text{for all } x \in X_2 \text{ and } i = 1, 2, \dots, \\ h(x) = x & \text{for all } x \in X_3 \setminus \bigcup_{i=1}^{\infty} (g_i X_2 \cup h_i X_2). \end{cases}$$

The argument to prove that h is bi-Hölder is similar to the one in Example 6.2: one checks that $F, G \in \mathrm{end}(T, v)$, $\frac{1}{2}(F|G)_v - 1 \leq (h(F)|h(G))_v \leq 2(F|G)_v + 1$ for every $F, G \in X_3$.

To see that h is not quasiconformal, it suffices to show that $\limsup_{\epsilon \to 0} D_h(\bar{0}, \epsilon) = \infty$. To this end, let $K > 0$ and $N \in \mathbf{N}$ be given. There exists $i_0 > N$ such $4^{i_0} > K$. Choose $G_1 \in g_{i_0} X_2$ and $G_2 \in \bar{0}_{4^{i_0}} 10 X_2$. It follows that $h(\bar{0}) = \bar{0}$, $h(G_1) \in h_{i_0} X_2$, and $h(G_2) = G_2$. Therefore, $(h(F_0)|h(G_2))_v = 4^{i_0}$ and $(h(F_0)|h(G_1))_v = 2 \cdot 4^{i_0}$. Hence, $D_h\left(\bar{0}, \mathrm{e}^{-4^{i_0}}\right) \geq \frac{\mathrm{e}^{-4^{i_0}}}{\mathrm{e}^{-2 \cdot 4^{i_0}}} = \mathrm{e}^{4^{i_0}}$. □

For functorial reasons, Mirani [11, 12] was interested in compositions of bi-Hölder, quasiconformal homeomorphisms between spaces such as X_3. The following example illustrates that these compositions need not be well-behaved. This is in contrast to the situation for PQ-symmetric homeomorphisms as observed by Tukia and Väisälä [17, Theorem 2.2] (see Remark 3.5).

Example 6.5. *There exist two bi-Hölder, quasiconformal homeomorphisms*

$$h_1, h_2\colon X_3 \to X_3$$

such that the composition $h_2 \circ h_1$ is not quasiconformal. Moreover, h_1 and h_2 are bounded distortion equivalences and h_2 is conformal.

Proof. For each $i \geq 1$, define the following sequences:

$$g_i^a = \bar{0}_{2i} 11,$$
$$h_i^a = \bar{0}_{2i+1} 11,$$
$$g_i^b = \bar{0}_{4i} 1,$$
$$h_i^b = \bar{0}_{8i} 1.$$

Define a homeomorphism $h_1\colon X_3 \to X_3$ by

$$\begin{cases} h_1(g_i^a x) = h_i^a x & \text{for all } x \in X_2 \text{ and } i = 1, 2, \dots, \\ h_1(h_i^a x) = g_i^a x & \text{for all } x \in X_2 \text{ and } i = 1, 2, \dots, \\ h_1(x) = x & \text{for all } x \in X_3 \setminus \bigcup_{i=1}^{\infty}(g_i^a X_2 \cup h_i^a X_2). \end{cases}$$

Define a homeomorphism $h_2\colon X_3 \to X_3$ by

$$\begin{cases} h_2(g_i^b x) = h_i^b x & \text{for all } x \in X_2 \text{ and } i = 1, 2, \dots, \\ h_2(h_i^b x) = g_i^b x & \text{for all } x \in X_2 \text{ and } i = 1, 2, \dots, \\ h_2(x) = x & \text{for all } x \in X_3 \setminus \bigcup_{i=1}^{\infty}(g_i^b X_2 \cup h_i^b X_2). \end{cases}$$

It is readily seen that h_1 is e-quasiconformal. Indeed, $d_{h_1}(F, \epsilon) \le \mathrm{e}$ for all $F \in X_3$ and for all $\epsilon > 0$, from which it also follows that h_1 is a bounded distortion equivalence. Likewise, $d_{h_2}(F, \epsilon) = 1$ for all $F \in X_3$ and for all $\epsilon > 0$, from which it also follows that h_2 is conformal and a bounded distortion equivalence. □

The next two examples show that “uniformly perfect” can not be weakened to “perfect” in Theorem 1.1 and Corollary 5.11, and that the “bushy” condition is needed in Theorem 5.5.

Example 6.6. *There exist a compact, perfect, pseudo-doubling, ultrametric space Z and a conformal homeomorphism $h\colon Z \to Z$ such that h is a bounded distortion equivalence, but h is not quasi-symmetric. Moreover, h can be chosen to be bi-Hölder or not bi-Hölder.*

Proof. We begin by constructing compact, perfect, pseudo-doubling, ultrametric spaces X and Y and a conformal, bounded distortion equivalence $f\colon X \to Y$ such that f is neither quasi-symmetric nor bi-Hölder. Let (T_2^i, v_i) denote a copy of the rooted, infinite binary tree (T, v) for each $i \ge 1$. Form the tree $T = T_2 \bigvee_{i=1}^{\infty} T_2^i$ by attaching, for each $i \ge 1$, T_2^i to T_2 by identifying v_i with the vertex on $\bar{1} \subseteq T_2$ that is a distance i from v. Note that (T, v) is a rooted, geodesically complete, simplicial $\mathbf{R}$-tree.

For each $i \ge 1$, let $X_2^i = \mathrm{end}(T_2^i, v_i)$. Then

$$X := \mathrm{end}(T, v) = X_2 \bigcup_{i=1}^{\infty} \bar{1}_i X_2^i.$$

As before, the points of X_2 are infinite sequences of 0’s and 1’s. For each $i \ge 1$, the points of X_2^i are denoted by infinite sequences of 0_i’s and 1_i’s. A typical point of X is either a point of X_2 or of the form $\bar{1}_i x = (\underbrace{1, \dots, 1}_{i})x$, where $x \in X_2^i$. Note that X is a compact, uniformly perfect, pseudo-doubling, ultrametric space.

For each $i \ge 1$ define

$$Y_2^i = \{y \in X_2^i \mid y = (\underbrace{x_1, \dots, x_1}_{i}, \underbrace{x_2, \dots, x_2}_{i}, \underbrace{x_3, \dots, x_3}_{i}, \dots), x_j \in \{0_i, 1_i\} \text{ for each } j \ge 1\}.$$

Note that for each $i \ge 1$, Y_2^i is a closed, uniformly perfect subset of X_2^i. There is an evident subtree S_2^i of T_2^i such that $\mathrm{end}(S_2^i, v_i) = Y_2^i$. The vertices of S_2^i of valency 3 are those vertices of T_2^i that are a distance ni from v_i for some $n = 1, 2, 3, \dots$.

Let $T' = T_2 \bigvee_{i=1}^{\infty} S_2^i$ so that (T', v) is a rooted, geodesically complete subtree of (T, v). For each $i \geq 1$, let $Y_2^i = \text{end}(S_2^i, v_i)$. Then

$$Y := \text{end}(T', v) = X_2 \bigcup_{i=1}^{\infty} \bar{1}_i Y_2^i.$$

Note that Y is a compact, perfect, pseudo-doubling, ultrametric space. The simplicial $\mathbf{R}$-tree T' is not bushy because for any $K > 0$, if $i \geq 2K$, then there are points x in S_2^i such that if y is within K of x, then $T' \setminus \{y\}$ has exactly 2 components. In particular, Y is not uniformly perfect.

Define $f \colon X \to Y$ by

$$f(\bar{1}_i x) = \begin{cases} x & \text{if } i = 0 \text{ and } x \in X_2, \\ \bar{1}_i(\underbrace{x_1, \dots, x_1}_{i}, \underbrace{x_2, \dots, x_2}_{i}, \underbrace{x_3, \dots, x_3}_{i}, \dots) & \text{if } i \geq 1 \text{ and } x \in X_2^i. \end{cases}$$

Note that $f(X_2) = Y_2$ and $f(X_2^i) = Y_2^i$ for all $i \geq 1$.

Clearly, f is induced by a rooted homeomorphism $\hat{f} \colon (T, v) \to (T', v)$; i.e., $f(F) = \hat{f} \circ F$ for each $F \in \text{end}(T, v)$. It follows from Remark 4.5 that f is conformal and f is a bounded distortion equivalence.

To see that f is not quasi-symmetric, choose $G \in X_2$ and $F_i, H_i \in X_2^i$ such that $(F_i|H_i)_v = i + 1$ for each $i \geq 1$. Then $|(F_i|H_i)_v - (F_i|G)_v| = 1$ and

$$|(f(F_i)|f(H_i))_v - (f(F_i)|f(G))_v| = i$$

for each $i \geq 1$. Proposition 4.2 implies that f is not quasi-symmetric.

To verify that f is not bi-Hölder, choose for each $i \geq 1$, $F_i, G_i \in X_2^i$ such that $(F_i|G_i)_v = 2i$. Then $(f(F_i)|f(G_i))_v = i + i^2$. It follows that the criterion in Remark 4.4 is violated.

We have now constructed a conformal, bounded distortion equivalence $f \colon X \to Y$ that is neither quasi-symmetric nor bi-Hölder. In order, to get an example in which the domain and range are the same, let $Z = \text{end}(T \vee T', v)$ so that $Z = X \cup Y$ and define $h \colon Z \to Z$ by $h_X = f$ and $h|Y = f^{-1}$.

Finally, we briefly indicate the modifications that need to be made in order to get a bi-Hölder example. For each $i \geq 1$ replace Y_2^i by

$$\hat{Y}_2^i = \{y \in X_2^i \mid y = (\underbrace{x_1, \dots, x_1}_{i}, x_2, x_3, \dots), x_j \in \{0_i, 1_i\} \text{ for each } j \geq 1\}$$

and replace f by

$$\hat{f}(\bar{1}_i x) = \begin{cases} x & \text{if } i = 0 \text{ and } x \in X_2, \\ \bar{1}_i(\underbrace{x_1, \dots, x_1}_{i}, x_2, x_3, \dots) & \text{if } i \geq 1 \text{ and } x \in X_2^i. \end{cases}$$

These changes are enough to construct a bi-Hölder, conformal, bounded distortion equivalence that is not quasi-symmetric. □

Example 6.7. *There exist a compact, perfect, pseudo-doubling, ultrametric space Z and a conformal homeomorphism $h \colon Z \to Z$ such that h is quasi-symmetric, but h is not PQ-symmetric. Moreover, h can be chosen to be bi-Hölder or not bi-Hölder.*

Proof. These examples are modifications of Example 6.6. Therefore, we will just briefly indicate the changes that need to be made.

We begin by showing how to produce the example that is not bi-Hölder. Let Y be as in Example 6.6. Construct a space W as Y is constructed above—except that Y_2^i is replaced by

$$\widetilde{Y}_2^i = \{y \in X_2^i \mid y = (\underbrace{x_1, \dots, x_1}_{i^2}, \underbrace{x_2, \dots, x_2}_{i^2}, \underbrace{x_3, \dots, x_3}_{i^2}, \dots), x_j \in \{0_i, 1_i\} \text{ for each } j \geq 1\}.$$

Let

$$\widetilde{Y} = X_2 \bigcup_{i=1}^{\infty} \bar{1}_i \widetilde{Y}_2^i$$

and define $\tilde{f}\colon Y \to \widetilde{Y}$ by $\tilde{f}(x) = x$ if $x \in X_2$ and

$$\tilde{f}(\bar{1}_i(\underbrace{x_1, \dots, x_1}_{i}, \underbrace{x_2, \dots, x_2}_{i}, \underbrace{x_3, \dots, x_3}_{i}, \dots)) = \bar{1}_i(\underbrace{x_1, \dots, x_1}_{i^2}, \underbrace{x_2, \dots, x_2}_{i^2}, \underbrace{x_3, \dots, x_3}_{i^2}, \dots)$$

if $i \geq 1$ and $(x_1, x_2, x_3, \dots) \in X_2^i$. It follows from Remark 4.5 that $\tilde{f}$ is conformal. One can verify that $\tilde{f}$ is quasi-symmetric by invoking Proposition 4.2 with

$$\gamma(t) = \begin{cases} t^2 & \text{if } t \geq 0, \\ t & \text{if } t < 0. \end{cases}$$

To verify that f is not bi-Hölder, choose for each $i \geq 1$, $F_i, G_i \in Y_2^i$ such that $(F_i|G_i)_v = 2i$. Then $(\tilde{f}(F_i)|\tilde{f}(G_i))_v = i + i^2$. It follows that the criterion in Remark 4.4 is violated. To see that $\tilde{f}$ is not PQ-symmetric, choose $G \in X_2$ and $F_i, H_i \in Y_2^i$ such that $(F_i|H_i)_v = 2i$ for each $i \geq 1$. Then $|(F_i|H_i)_v - (F_i|G)_v| = i$ and

$$|(\tilde{f}(F_i)|\tilde{f}(H_i))_v - (\tilde{f}(F_i)|\tilde{f}(G))_v| = i^2$$

for each $i \geq 1$. Proposition 4.3 implies that f is not PQ-symmetric. The passage from $\tilde{f}\colon Y \to \tilde{Y}$ to $h\colon Z \to Z$ takes place as in Example 6.6.

To construct the bi-Hölder example, replace $\tilde{f}\colon Y \to \tilde{Y}$ by $\tilde{f}'\colon Y' \to \widetilde{Y}'$ with the following explanations. First, just as Y arises by attaching trees S_2^i to points on $\bar{1}$ a distance i from v, the space Y' arises by attaching those trees to points a distance i^2 from v. Second, just as $\widetilde{Y}$ arises from a modification of Y by replacing each Y_2^i by $\widetilde{Y}_2^i$, the space $\widetilde{Y}'$ arises by replacing $Y_2^i \subseteq Y'$ by

$$\left(\widetilde{Y}_2^i\right)' = \{y \in X_2^i \mid y = (\underbrace{x_1, \dots, x_1}_{i^2}, \underbrace{x_2, \dots, x_2}_{i}, \underbrace{x_3, \dots, x_3}_{i}, \dots), \\ x_j \in \{0_i, 1_i\} \text{ for each } j \geq 1\}.$$

The map $\tilde{f}'$ is obtained by modifying $\tilde{f}$ in the obvious way. □

The final example shows the necessity of the "pseudo-doubling" in Theorem 1.1 and Corollary 5.11.

Example 6.8. *There exist a compact, uniformly perfect, ultrametric space X that is not pseudo-doubling and a bounded distortion equivalence $f\colon X \to X_3$ that is not PQ-symmetric. Moreover, f is not bi-Hölder.*

Proof. For each $i \in \mathbf{N}$ consider the edge e_i on the tree $T_2 \subseteq T_3$ joining $\bar{1}_i$ to $\bar{1}_{i+1}$. Choose points $\bar{1}_i = x_i^0, x_i^1, \dots, x_i^i, x_i^{i+1} = \bar{1}_{i+1}$ dividing e_i into $i+1$ subintervals of length $1/(i+1)$. For each $i \in \mathbf{N}$ and to each point x_i^j, $1 \le j \le i$, attach a copy T_i^j of $T_2 \subseteq T_3$ to T_3 by identifying the root v_i^j of T_i^j to x_i^j. Let $T = T_3 \bigvee_{i=1}^{\infty} \bigvee_{j=1}^{i} T_i^j$ be the resulting tree and let $X = \mathrm{end}(T, v)$. Define $f\colon X \to X_3$ as follows. If $x \in (X \setminus \mathrm{end}(T_2, v)) \cup 0X_2$, define $f(x) = x$. If $x \in 1X_2$ is of the form $x = \bar{1}_i$ for some $i \in \mathbf{N}$, define $f(x) = \bar{1}_{i^2} y$. Any other point $x \in X$ may be written (using self-explanatory notation) as $x = \bar{1}_i x_i^1 \dots x_i^j z$, where $i \in \mathbf{N}$, $1 \le j \le i$, and $z = z_1 z_2 z_3 \cdots \in X_2$. For such an x, define $f(x) = \bar{1}_{i+2j+z_1-1} z_2 z_3 z_4 \dots$. It can be verified that f is a bounded distortion equivalence (with constant e) and that f is neither PQ-symmetric nor bi-Hölder. □

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