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APPROXIMATE IDENTITIES AND YOUNG TYPE INEQUALITIES IN VARIABLE LEBESGUE–ORLICZ SPACES $L^{p(\cdot)}(\log L)^{q(\cdot)}$

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Abstract. Our aim in this paper is to deal with approximate identities in generalized Lebesgue spaces $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$. As a related topic, we also study Young type inequalities for convolution with respect to norms in such spaces.

1. Introduction

Following Cruz-Uribe and Fiorenza [2], we consider two variable exponents $p(\cdot)\colon \mathbf{R}^n \to [1,\infty)$ and $q(\cdot)\colon \mathbf{R}^n \to \mathbf{R}$, which are continuous functions. Letting $\Phi_{p(\cdot),q(\cdot)}(x,t) = t^{p(x)}(\log(c_0+t))^{q(x)}$, we define the space $L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$ of all measurable functions f on an open set Ω such that

$$\int_\Omega \Phi_{p(\cdot),q(\cdot)}\left(y, \frac{|f(y)|}{\lambda}\right) dy < \infty$$

for some $\lambda > 0$; here we assume

(Φ) $\Phi_{p(\cdot),q(\cdot)}(x,\cdot)$ is convex on $[0,\infty)$ for every fixed $x \in \mathbf{R}^n$.

Note that (Φ) holds for some $c_0 \geq e$ if and only if there is a positive constant K such that

$$K(p(x)-1) + q(x) \geq 0 \quad \text{for all } x \in \mathbf{R}^n \tag{1.1}$$

(see Appendix). Further, we see from (Φ) that $t^{-1}\Phi_{p(\cdot),q(\cdot)}(x,t)$ is nondecreasing in t.

We define the norm

$$\|f\|_{\Phi_{p(\cdot),q(\cdot)},\Omega} = \inf\left\{\lambda > 0\colon \int_\Omega \Phi_{p(\cdot),q(\cdot)}\left(y, \frac{|f(y)|}{\lambda}\right) dy \leq 1\right\}$$

for $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$. Note that $L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$ is a Musielak–Orlicz space [9]. Such spaces have been studied in [2, 8, 10]. In case $q(\cdot) = 0$ on $\mathbf{R}^n$, $L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$ is denoted by $L^{p(\cdot)}(\Omega)$ ([7]).

We assume throughout the article that our variable exponents $p(\cdot)$ and $q(\cdot)$ are continuous functions on $\mathbf{R}^n$ satisfying:

(p1) $1 \leq p_- := \inf_{x\in\mathbf{R}^n} p(x) \leq \sup_{x\in\mathbf{R}^n} p(x) =: p_+ < \infty$;

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(p2) $|p(x) - p(y)| \le \dfrac{C}{\log(e + 1/|x-y|)}$ whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$;

(p3) $|p(x) - p(y)| \le \dfrac{C}{\log(e + |x|)}$ whenever $|y| \ge |x|/2$;

(q1) $-\infty < q_- := \inf_{x\in\mathbf{R}^n} q(x) \le \sup_{x\in\mathbf{R}^n} q(x) =: q_+ < \infty$;

(q2) $|q(x) - q(y)| \le \dfrac{C}{\log(e + \log(e + 1/|x-y|))}$ whenever $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^n$

for a positive constant C.

We choose $p_0 \ge 1$ as follows: we take $p_0 = p_-$ if $t^{-p_-}\Phi_{p(\cdot),q(\cdot)}(x,t)$ is uniformly almost increasing in t; more precisely, if there exists $C > 0$ such that $s^{-p_-}\Phi_{p(\cdot),q(\cdot)}(x,s) \le Ct^{-p_-}\Phi_{p(\cdot),q(\cdot)}(x,t)$ whenever $0 < s < t$ and $x \in \mathbf{R}^n$. Otherwise we choose $1 \le p_0 < p_-$. Then note that $t^{-p_0}\Phi_{p(\cdot),q(\cdot)}(x,t)$ is uniformly almost increasing in t in any case.

Let ϕ be an integrable function on $\mathbf{R}^n$. For each $t > 0$, define the function ϕ_t by $\phi_t(x) = t^{-n}\phi(x/t)$. Note that by a change of variables, $\|\phi_t\|_{L^1,\mathbf{R}^n} = \|\phi\|_{L^1,\mathbf{R}^n}$. We say that the family $\{\phi_t\}$ is an approximate identity if $\int_{\mathbf{R}^n} \phi(x)\,dx = 1$. Define the radial majorant of ϕ to be the function

$$\hat{\phi}(x) = \sup_{|y|\ge|x|} |\phi(y)|.$$

If $\hat{\phi}$ is integrable, we say that the family $\{\phi_t\}$ is of potential-type.

Cruz-Uribe and Fiorenza [1] proved the following result:

Theorem A. *Let $\{\phi_t\}$ be an approximate identity. Suppose that either*

(1) *$\{\phi_t\}$ is of potential-type, or*

(2) *$\phi \in L^{(p_-)'}(\mathbf{R}^n)$ and has compact support.*

Then

$$\sup_{0<t\le 1} \|\phi_t * f\|_{L^{p(\cdot)},\mathbf{R}^n} \le C\|f\|_{L^{p(\cdot)},\mathbf{R}^n}$$

and

$$\lim_{t\to+0} \|\phi_t * f - f\|_{L^{p(\cdot)},\mathbf{R}^n} = 0$$

for all $f \in L^{p(\cdot)}(\mathbf{R}^n)$.

Our aim in this note is to extend their result to the space $L^{p(\cdot)}(\log L)^{q(\cdot)}(\Omega)$ of two variable exponents.

Theorem 1.1. *Let $\{\phi_t\}$ be a potential-type approximate identity. If $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$, then $\{\phi_t * f\}$ converges to f in $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$:*

$$\lim_{t\to 0} \|\phi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} = 0.$$

Theorem 1.2. *Let $\{\phi_t\}$ be an approximate identity. Suppose that $\phi \in L^{(p_0)'}(\mathbf{R}^n)$ and has compact support. If $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$, then $\{\phi_t * f\}$ converges to f in $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$:*

$$\lim_{t\to 0} \|\phi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} = 0.$$

We show by an example that the conditions on ϕ are necessary; see Remarks 3.5 and 3.6 below.

Finally, in Section 4, we give some Young type inequalities for convolution with respect to the norms in $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$.

2. The case of potential-type

Throughout this paper, let C denote various positive constants independent of the variables in question.

Let us begin with the following result due to Stein [11].

Lemma 2.1. *Let $1 \le p < \infty$ and $\{\phi_t\}$ be a potential-type approximate identity. Then for every $f \in L^p(\mathbf{R}^n)$, $\{\phi_t * f\}$ converges to f in $L^p(\mathbf{R}^n)$.*

We denote by $B(x,r)$ the open ball centered at $x \in \mathbf{R}^n$ and with radius $r > 0$. For a measurable set E, we denote by $|E|$ the Lebesgue measure of E.

The following is due to Lemma 2.6 in [8].

Lemma 2.2. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le 1$ such that $f(x) \ge 1$ or $f(x) = 0$ for each $x \in \mathbf{R}^n$. Set*

$$J = J(x,r,f) = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy$$

and

$$L = L(x,r,f) = \frac{1}{|B(x,r)|}\int_{B(x,r)} \Phi_{p(\cdot),q(\cdot)}(y, f(y))\,dy.$$

Then

$$J \le C L^{1/p(x)} (\log(c_0 + L))^{-q(x)/p(x)},$$

where $C > 0$ does not depend on x, r, f.

Further we need the following result.

Lemma 2.3. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $(1+|y|)^{-n-1} \le f(y) \le 1$ or $f(y) = 0$ for each $y \in \mathbf{R}^n$. Set*

$$J = J(x,r,f) = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy$$

and

$$L = L(x,r,f) = \frac{1}{|B(x,r)|}\int_{B(x,r)} \Phi_{p(\cdot),q(\cdot)}(y, f(y))\,dy.$$

Then

$$J \le C\left\{L^{1/p(x)} + (1+|x|)^{-n-1}\right\},$$

where $C > 0$ does not depend on x, r, f.

Proof. We have by Jensen's inequality

$$\begin{aligned}
J &\le \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)^{p(x)} dy\right)^{1/p(x)} \\
&\le \left(\frac{1}{|B(x,r)|}\int_{B(x,r)\cap B(0,|x|/2)} f(y)^{p(x)} dy\right)^{1/p(x)} \\
&\quad + \left(\frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|/2)} f(y)^{p(x)} dy\right)^{1/p(x)} \\
&= J_1 + J_2.
\end{aligned}$$

We see from (p3) that

$$J_1 \le C \left(\frac{1}{|B(x,r)|} \int_{B(x,r)\cap B(0,|x|/2)} f(y)^{p(y)} dy \right)^{1/p(x)}.$$

Similarly, setting $E_2 = \{y \in \mathbf{R}^n \colon f(y) \ge (1+|x|)^{-n-1}\}$, we see from (p3) that

$$\begin{aligned} J_2 &\le C \left(\frac{1}{|B(x,r)|} \int_{\{B(x,r)\setminus B(0,|x|/2)\}\cap E_2} f(y)^{p(y)} dy \right)^{1/p(x)} \\ &\quad + \left(\frac{1}{|B(x,r)|} \int_{\{B(x,r)\setminus B(0,|x|/2)\}\setminus E_2} (1+|x|)^{-p(x)(n+1)} dy \right)^{1/p(x)} \\ &\le C \left\{ \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)^{p(y)} dy \right)^{1/p(x)} + (1+|x|)^{-(n+1)} \right\}. \end{aligned}$$

Since $f(y) \le 1$, $f(y)^{p(y)} \le C\Phi_{p(\cdot),q(\cdot)}(y, f(y))$. Hence, we have the required estimate. $\square$

By using Lemmas 2.2 and 2.3, we show the following theorem.

Theorem 2.4. *If $\{\phi_t\}$ is of potential-type, then*

$$\|\phi_t * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le C\|\hat{\phi}\|_{L^1,\mathbf{R}^n} \|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n}$$

for all $t > 0$ and $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$.

Proof. Suppose $\|\hat{\phi}\|_{L^1,\mathbf{R}^n} = 1$ and take a nonnegative measurable function f on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le 1$. Write

$$\begin{aligned} f &= f\chi_{\{y\in\mathbf{R}^n : f(y)\ge 1\}} + f\chi_{\{y\in\mathbf{R}^n : (1+|y|)^{-n-1}\le f(y)<1\}} + f\chi_{\{y\in\mathbf{R}^n : f(y)\le(1+|y|)^{-n-1}\}} \\ &= f_1 + f_2 + f_3, \end{aligned}$$

where χ_E denotes the characteristic function of a measurable set $E \subset \mathbf{R}^n$.

Since $\hat{\phi}_t$ is a radial function, we write $\hat{\phi}_t(r)$ for $\hat{\phi}_t(x)$ when $|x| = r$. First note that

$$\begin{aligned} |\phi_t * f(x)| &\le \int_{\mathbf{R}^n} \hat{\phi}_t(|x-y|) f_1(y)\, dy \\ &= \int_0^\infty \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} f_1(y)\, dy \right) |B(x,r)|\, d(-\hat{\phi}_t(r)), \end{aligned}$$

so that Jensen's inequality and Lemma 2.2 yield

$$\begin{aligned} &\Phi_{p(\cdot),q(\cdot)}(x, |\phi_t * f_1(x)|) \\ &\le \int_0^\infty \Phi_{p(\cdot),q(\cdot)} \left(x, \frac{1}{|B(x,r)|} \int_{B(x,r)} f_1(y)\, dy \right) |B(x,r)|\, d(-\hat{\phi}_t(r)) \\ &\le C \int_0^\infty \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} \Phi_{p(\cdot),q(\cdot)}(y, f_1(y))\, dy \right) |B(x,r)|\, d(-\hat{\phi}_t(r)) \\ &= C(\hat{\phi}_t * g)(x), \end{aligned}$$

where $g(y) = \Phi_{p(\cdot),q(\cdot)}(y, f(y))$. The usual Young inequality for convolution gives

$$\int_{\mathbf{R}^n} \Phi_{p(\cdot),q(\cdot)}(x, |\phi_t * f_1(x)|)\, dx \le C \int_{\mathbf{R}^n} (\hat{\phi}_t * g)(x)\, dx \le C\|\hat{\phi}_t\|_{L^1,\mathbf{R}^n} \|g\|_{L^1,\mathbf{R}^n} \le C.$$

Similarly, noting that $\frac{1}{|B(x,r)|}\int_{B(x,r)} f_2(y)\, dy \le 1$ and applying Lemma 2.3, we derive the same result for f_2.

Finally, noting that $|\phi_t * f_3| \le C\|\phi_t\|_{L^1,\mathbf{R}^n} \le C$, we obtain

$$\int_{\mathbf{R}^n} \Phi_{p(\cdot),q(\cdot)}(x, |\phi_t * f_3(x)|)\, dx \le C \int_{\mathbf{R}^n} |\phi_t * f_3(x)|\, dx \le C\|\phi_t\|_{L^1,\mathbf{R}^n} \|f_3\|_{L^1,\mathbf{R}^n} \le C,$$

as required. □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Given $\varepsilon > 0$, we find a bounded function g in $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$ with compact support such that $\|f - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} < \varepsilon$. By Theorem 2.4 we have

$$\begin{aligned} &\|\phi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \\ &\le \|\phi_t * (f-g)\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} + \|\phi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} + \|f-g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \\ &\le C\varepsilon + \|\phi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n}. \end{aligned}$$

Since $|\phi_t * g| \le \|g\|_{L^\infty,\mathbf{R}^n}$,

$$\|\phi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le C'\|\phi_t * g - g\|_{L^1,\mathbf{R}^n} \to 0$$

by Lemma 2.1. (Here C' depends on $\|g\|_{L^\infty,\mathbf{R}^n}$.) Hence

$$\limsup_{t\to 0} \|\phi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le C\varepsilon,$$

which completes the proof. □

As another application of Lemmas 2.2 and 2.3, we can prove the following result, which is an extension of [4, Theorem 1.5] and [8, Theorem 2.7] (see also [6]).

Let Mf be the Hardy–Littlewood maximal function of f.

Proposition 2.5. *Suppose $p_- > 1$. Then the operator M is bounded from $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$ to $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$.*

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le 1$ and write $f = f_1 + f_2 + f_3$ as in the proof of Theorem 2.4. Take $1 < p_1 < p_-$ and apply Lemmas 2.2 and 2.3 with $p(\cdot)$ and $q(\cdot)$ replaced by $p(\cdot)/p_1$ and $q(\cdot)/p_1$, respectively. Then

$$\Phi_{p(\cdot),q(\cdot)}(x, Mf_1(x)) \le C[Mg_1(x)]^{p_1}$$

and

$$\Phi_{p(\cdot),q(\cdot)}(x, Mf_2(x)) \le C\left\{[Mg_1(x)]^{p_1} + (1+|x|)^{-n-1}\right\},$$

where $g_1(y) = \Phi_{p(\cdot)/p_1,q(\cdot)/p_1}(y, f(y))$. As to f_3, we have

$$\Phi_{p(\cdot),q(\cdot)}(x, Mf_3(x)) \le C[Mf_3(x)]^{p_1}.$$

Then the boundedness of the maximal operator in $L^{p_1}(\mathbf{R}^n)$ proves the proposition. □

Remark 2.6. If $p_- > 1$, then the function $\Phi_{p(\cdot),q(\cdot)}$ is a proper N-function and our Proposition 2.5 implies that this function is of class $\mathscr{A}$ in the sense of Diening [5] (see [5, Lemma 3.2]). It would be an interesting problem to see whether "class $\mathscr{A}$" is also a sufficient condition or not for the boundedness of M on $L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$.

3. The case of compact support

We know the following result due to Zo [12]; see also [1, Theorem 2.2].

Lemma 3.1. *Let $1 \le p < \infty$, $1/p + 1/p' = 1$ and $\{\phi_t\}$ be an approximate identity. Suppose that $\phi \in L^{p'}(\mathbf{R}^n)$ has compact support. Then for every $f \in L^p(\mathbf{R}^n)$, $\{\phi_t * f\}$ converges to f pointwise almost everywhere.*

Set

$$\overline{p}(x) = p(x)/p_0 \quad \text{and} \quad \overline{q}(x) = q(x)/p_0;$$

recall that $p_0 \in [1, p_-]$ is chosen such that $t^{-p_0}\Phi_{p(\cdot),q(\cdot)}(x,t)$ is uniformly almost increasing in t.

For a proof of Theorem 1.2, the following is a key lemma.

Lemma 3.2. *Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le 1$ such that $f(x) \ge 1$ or $f(x) = 0$ for each $x \in \mathbf{R}^n$ and let ϕ have compact support in $B(0,R)$ with $\|\phi\|_{L^{(p_0)'},\mathbf{R}^n} \le 1$. Set*

$$F = F(x,t,f) = |\phi_t * f(x)|$$

and

$$G = G(x,t,f) = \int_{\mathbf{R}^n} |\phi_t(x-y)|\Phi_{\overline{p}(\cdot),\overline{q}(\cdot)}(y, f(y))\,dy.$$

Then

$$F \le CG^{1/\overline{p}(x)}(\log(c_0 + G))^{-\overline{q}(x)/\overline{p}(x)}$$

for all $0 < t \le 1$, where $C > 0$ depends on R.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le 1$ such that $f(x) \ge 1$ or $f(x) = 0$ for each $x \in \mathbf{R}^n$ and let ϕ have compact support in $B(0,R)$ with $\|\phi\|_{L^{(p_0)'},\mathbf{R}^n} \le 1$. By Hölder's inequality, we have

$$G \le \|\phi_t\|_{L^{(p_0)'},\mathbf{R}^n}\left(\int_{\mathbf{R}^n} \Phi_{p(\cdot),q(\cdot)}(y, f(y))\,dy\right)^{1/p_0} \le t^{-n/p_0}.$$

First consider the case when $G \ge 1$. Since $G \le t^{-n/p_0}$, for $y \in B(x,tR)$ we have by (p2)

$$G^{-p(y)} \le G^{-p(x)+C/\log(e+(tR)^{-1})} \le CG^{-p(x)}$$

and by (q2)

$$(\log(c_0 + G))^{q(y)} \le C(\log(c_0 + G))^{q(x)}.$$

Hence it follows from the choice of p_0 that

$$
\begin{aligned}
F &\le G^{1/\overline{p}(x)}(\log(c_0+G))^{-\overline{q}(x)/\overline{p}(x)}\int_{\mathbf{R}^n}|\phi_t(x-y)|\,dy \\
&\quad + C\int_{\mathbf{R}^n}|\phi_t(x-y)|f(y)\left\{\frac{f(y)}{G^{1/\overline{p}(x)}(\log(c_0+G))^{-\overline{q}(x)/\overline{p}(x)}}\right\}^{\overline{p}(y)-1} \\
&\quad \cdot\left\{\frac{\log(c_0+f(y))}{\log(c_0+G^{1/\overline{p}(x)}(\log(c_0+G))^{-\overline{q}(x)/\overline{p}(x)})}\right\}^{\overline{q}(y)} dy \\
&\le CG^{1/\overline{p}(x)}(\log(c_0+G))^{-\overline{q}(x)/\overline{p}(x)}
\end{aligned}
$$

(cf. the proof of [8, Lemma 2.6]).

In the case $G<1$, noting from the choice of p_0 that $f(y)\le C\Phi_{\overline{p}(\cdot),\overline{q}(\cdot)}(y,f(y))$ for $y\in\mathbf{R}^n$, we find

$$F\le CG\le CG^{1/\overline{p}(x)}\le CG^{1/\overline{p}(x)}(\log(c_0+G))^{-\overline{q}(x)/\overline{p}(x)}.$$

Now the result follows. □

Lemma 3.3. *Suppose that $\|\phi\|_{L^1,\mathbf{R}^n}\le 1$. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n}\le 1$. Set*

$$I=I(x,t,f)=\int_{\{y\in\mathbf{R}^n:\,|y|>|x|/2\}}|\phi_t(x-y)|f(y)\,dy$$

and

$$H=H(x,t,f)=\int_{\mathbf{R}^n}|\phi_t(x-y)|\Phi_{p(\cdot),q(\cdot)}(y,f(y))\,dy.$$

If $A>0$ and $H\le H_0$, then

$$I\le C(H^{1/p(x)}+|x|^{-A/p(x)})$$

for $|x|>1$ and $0<t\le 1$, where $C>0$ depends on A and H_0.

Proof. Suppose that $\|\phi\|_{L^1,\mathbf{R}^n}\le 1$. Let f be a nonnegative measurable function on $\mathbf{R}^n$ with $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n}\le 1$.

Let $|x|>1$. In the case $H_0\ge H\ge |x|^{-A}$ with $A>0$, we have by (p3)

$$H^{-p(y)}\le CH^{-p(x)-C/\log(e+|x|)}\le CH^{-p(x)}$$

for $|y|\ge |x|/2$. Hence we find by (Φ)

$$
\begin{aligned}
I &\le C\bigg\{H^{1/p(x)}+\int_{\{y\in\mathbf{R}^n:\,|y|>|x|/2\}}|\phi_t(x-y)|f(y) \\
&\qquad \cdot\left\{\frac{f(y)}{H^{1/p(x)}}\right\}^{p(y)-1}\left\{\frac{\log(c_0+f(y))}{\log(c_0+H^{1/p(x)})}\right\}^{q(y)}dy\bigg\} \\
&\le CH^{1/p(x)}.
\end{aligned}
$$

Next note from (p3) that

$$|x|^{p(y)}\le |x|^{p(x)+C/\log(e+|x|)}\le C|x|^{p(x)}$$

for $|y| \geq |x|/2$. Hence, when $H \leq |x|^{-A}$, we obtain by (Φ)

$$\begin{aligned} I &\leq C\bigg\{|x|^{-A/p(x)} + \int_{\{y\in\mathbf{R}^n:|y|>|x|/2\}} |\phi_t(x-y)|f(y) \\ &\quad \cdot \left\{\frac{f(y)}{|x|^{-A/p(x)}}\right\}^{p(y)-1} \left\{\frac{\log(c_0+f(y))}{\log(c_0+|x|^{-A/p(x)})}\right\}^{q(y)} dy\bigg\} \\ &\leq C|x|^{-A/p(x)}, \end{aligned}$$

which completes the proof. □

Theorem 3.4. *Suppose that* $\phi \in L^{(p_0)'}(\mathbf{R}^n)$ *has compact support in* $B(0,R)$. *Then*

$$\|\phi_t * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \leq C\|\phi\|_{L^{(p_0)'},\mathbf{R}^n}\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n}$$

for all $0 < t \leq 1$ *and* $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$, *where* $C > 0$ *depends on* R.

Proof. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \leq 1$ and let ϕ have compact support in $B(0,R)$ with $\|\phi\|_{L^{(p_0)'},\mathbf{R}^n} \leq 1$. Write

$$f = f\chi_{\{y\in\mathbf{R}^n:f(y)\geq 1\}} + f\chi_{\{y\in\mathbf{R}^n:f(y)<1\}} = f_1 + f_2.$$

We have by Lemma 3.2,

$$|\phi_t * f_1(x)| \leq C(|\phi_t| * g(x))^{p_0/p(x)}(\log(c_0 + |\phi_t| * g(x)))^{-q(x)/p(x)},$$

where $g(y) = \Phi_{\bar{p}(\cdot),\bar{q}(\cdot)}(y, f(y)) = \Phi_{p(\cdot),q(\cdot)}(y, f(y))^{1/p_0}$, so that

$$\Phi_{p(\cdot),q(\cdot)}(x, |\phi_t * f_1(x)|) \leq C(|\phi_t| * g(x))^{p_0}. \tag{3.1}$$

Hence, since $g \in L^{p_0}(\mathbf{R}^n)$, the usual Young inequality for convolution gives

$$\begin{aligned} \int_{\mathbf{R}^n} \Phi_{p(\cdot),q(\cdot)}(x, |\phi_t * f_1(x)|)\,dx &\leq C\int_{\mathbf{R}^n} (|\phi_t| * g(x))^{p_0}dx \\ &\leq C\left(\|\phi_t\|_{L^1,\mathbf{R}^n}\|g\|_{L^{p_0},\mathbf{R}^n}\right)^{p_0} \leq C. \end{aligned}$$

Next we are concerned with f_2. Write

$$f_2 = f_2\chi_{B(0,R)} + f_2\chi_{\mathbf{R}^n\setminus B(0,R)} = f_2' + f_2''.$$

Since $|\phi_t * f_2(x)| \leq C$ on $\mathbf{R}^n$, we have

$$\int_{B(0,2R)} \Phi_{p(\cdot),q(\cdot)}(x, |\phi_t * f_2(x)|)\,dx \leq C.$$

Further, noting that $\phi_t * f_2' = 0$ outside $B(0,2R)$, we find

$$\int_{\mathbf{R}^n} \Phi_{p(\cdot),q(\cdot)}(x, |\phi_t * f_2'(x)|)\,dx \leq C.$$

Therefore it suffices to prove

$$\int_{\mathbf{R}^n\setminus B(0,2R)} \Phi_{p(\cdot),q(\cdot)}(x, |\phi_t * f_2''(x)|)\,dx \leq C.$$

Thus, in the rest of the proof, we may assume that $0 \leq f < 1$ on $\mathbf{R}^n$ and $f = 0$ on $B(0,R)$. Note that

$$\int_{B(0,|x|/2)} \phi_t(x-y)f(y)\,dy = 0$$

for $|x| > 2R$. Hence, applying Lemma 3.3, we have

$$|\phi_t * f(x)|^{p(x)} \le C(|\phi_t| * h(x) + |x|^{-A})$$

for $|x| > 2R$, where $h(y) = \Phi_{p(\cdot),q(\cdot)}(y, f(y))$. Thus the integration yields

$$\int_{\mathbf{R}^n \setminus B(0,2R)} |\phi_t * f(x)|^{p(x)} dx \le C,$$

which completes the proof. $\square$

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. Given $\varepsilon > 0$, choose a bounded function g with compact support such that $\|f - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} < \varepsilon$. As in the proof of Theorem 1.1, using Theorem 3.4 this time, we have

$$\|\phi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le C\varepsilon + \|\phi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n}.$$

Obviously, $g \in L^{p_0}(\mathbf{R}^n)$. Hence by Lemma 3.1, $\phi_t * g \to g$ almost everywhere in $\mathbf{R}^n$. Since there is a compact set S containing all the supports of $\phi_t * g$,

$$\|\phi_t * g - g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le C'\|\phi_t * g - g\|_{L^{p_+ +1},\mathbf{R}^n}$$

with C' depending on $|S|$, and the Lebesgue convergence theorem implies $\|\phi_t * g - g\|_{L^{p_+ +1},\mathbf{R}^n} \to 0$ as $t \to \infty$. Hence

$$\limsup_{t \to 0} \|\phi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le C\varepsilon,$$

which completes the proof. $\square$

Remark 3.5. In Theorem 1.2 (and in Theorem A), the condition $\phi \in L^{(p_-)'}(\mathbf{R}^n)$ cannot be weakened to $\phi \in L^q(\mathbf{R}^n)$ for $1 \le q < (p_-)'$. In fact, for given $p_1 > 1$ and $1 \le q < (p_1)'$, we can find a smooth exponent $p(\cdot)$ on $\mathbf{R}^n$ such that $p_- = p_1$, $f \in L^{p(\cdot)}(\mathbf{R}^n)$ and $\phi \in L^q(\mathbf{R}^n)$ having compact support for which

$$\|\phi * f\|_{L^{p(\cdot)},\mathbf{R}^n} = \infty.$$

For this, let $a \in \mathbf{R}^n$ be a fixed point with $|a| > 1$ and let p_2 satisfy

$$\frac{1}{(p_1)'} + \frac{1}{p_2} < \frac{1}{q}.$$

Then choose a smooth exponent $p(\cdot)$ on $\mathbf{R}^n$ such that

$$p(x) = p_1 \text{ for } x \in B(0, 1/2), \quad p(x) = p_2 \text{ for } x \in B(a, 1/2),$$

$p_- = p_1$ and $p(x) = \text{const.}$ outside $B(0, |a| + 1)$. Take

$$\phi_j = j^{n/q}\chi_{B(a,j^{-1})} \quad \text{and} \quad f_j = j^{n/p_1}\chi_{B(0,j^{-1})}, \quad j = 2, 3, \ldots.$$

Then

$$\|\phi_j\|_{L^q,\mathbf{R}^n} = C < \infty \quad \text{and} \quad \|f_j\|_{L^{p(\cdot)},\mathbf{R}^n} = \|f_j\|_{L^{p_1},B(0,1/2)} = C < \infty.$$

Note that if $x \in B(a, j^{-1})$, then

$$\phi_j * f_j(x) = j^{n/q+n/p_1}|B(a, j^{-1}) \cap B(x, j^{-1})| \ge Cj^{n/q+n/p_1}j^{-n},$$

so that

$$\begin{aligned}\int_{\mathbf{R}^n}\{\phi_j * f_j(x)\}^{p(x)}dx &\geq \int_{B(a,j^{-1})}\{\phi_j * f_j(x)\}^{p(x)}dx\\ &\geq Cj^{p_2(n/q+n/p_1-n)}j^{-n}\\ &= Cj^{p_2 n(1/q-1/(p_1)'-1/p_2)}.\end{aligned}$$

Now consider

$$\phi = \sum_{j=2}^{\infty} j^{-2}\phi_{2^j} \quad \text{and} \quad f = \sum_{j=2}^{\infty} j^{-2} f_{2^j}.$$

Then $\phi \in L^q(\mathbf{R}^n)$ and $f \in L^{p(\cdot)}(\mathbf{R}^n)$. On the other hand,

$$\begin{aligned}\int_{\mathbf{R}^n}\{\phi * f(x)\}^{p(x)}dx &\geq j^{-4}\int_{\mathbf{R}^n}\{\phi_{2^j} * f_{2^j}(x)\}^{p(x)}dx\\ &\geq Cj^{-4}2^{p_2 nj(1/q-1/(p_1)'-1/p_2)} \to \infty\end{aligned}$$

as $j \to \infty$. Hence, $\|\phi * f\|_{L^{p(\cdot)},\mathbf{R}^n} = \infty$.

Remark 3.6. Cruz-Uribe and Fiorenza [1] gave an example showing that it can occur

$$\limsup_{t\to 0}\|\phi_t * f\|_{L^{p(\cdot)},\mathbf{R}} = \infty$$

for $f \in L^{p(\cdot)}(\mathbf{R})$ when ϕ does not have compact support.

By modifying their example, we can also find $p(\cdot)$ and $\phi \in L^{(p_-)'}(\mathbf{R})$, whose support is not compact, such that

$$\|\phi * f\|_{L^{p(\cdot)},\mathbf{R}} \leq C\|f\|_{L^{p(\cdot)},\mathbf{R}}$$

does not hold, namely there exists f_N $(N = 1, 2, \dots)$ such that $\|f_N\|_{L^{p(\cdot)},\mathbf{R}} \leq 1$ and

$$\lim_{N\to\infty}\|\phi * f_N\|_{L^{p(\cdot)},\mathbf{R}} = \infty.$$

For this purpose, choose $p_1 > 1$, $p_2 > p_1$ and $a > 1$ such that

$$-p_1/p_2 - ap_1 + 2 > 0,$$

and let $p(\cdot)$ be a smooth variable exponent on $\mathbf{R}$ such that

$$p(x) = p_1 \text{ for } x \leq 0, \quad p(x) = p_2 \text{ for } x \geq 1$$

and $p_1 \leq p(x) \leq p_2$ for $0 < x < 1$. Set $\phi = \sum_{j=1}^{\infty}\chi_j$, where $\chi_j = \chi_{[-j,-j+j^{-a})}$. Then

$$\int_{\mathbf{R}}\phi(x)^q\,dx = \sum_{j=1}^{\infty}\int_{-j}^{-j+j^{-a}}\chi_j(x)^q\,dx = \sum_j j^{-a} \leq C(a) < \infty$$

for any $q > 0$. Further set $f_N = N^{-1/p_2}\chi_{[1,N+1]}$. Note that for $1 - j + j^{-a} < x < 0$ and $j \leq N$

$$\chi_j * f_N(x) \geq \int_{x+j-j^{-a}}^{x+j}\chi_j(x-y)f_N(y)\,dy = N^{-1/p_2}j^{-a},$$

so that

$$\int_{\mathbf{R}} \{\phi * f_N(x)\}^{p(x)} dx \geq \int_{-\infty}^{0} \left\{ \sum_{j=1}^{\infty} \chi_j * f_N(x) \right\}^{p_1} dx$$
$$\geq \sum_{j=2}^{N} \int_{1-j-j^{-a}}^{0} \{\chi_j * f_N(x)\}^{p_1} dx$$
$$\geq N^{-p_1/p_2} \sum_{j=2}^{N} j^{-ap_1}(j - j^{-a} - 1)$$
$$\geq CN^{-p_1/p_2 - ap_1 + 2} \to \infty \quad (N \to \infty).$$

4. Young type inequalities

Cruz-Uribe and Fiorenza [1] conjectured that Theorem A remains true if ϕ satisfies the additional condition

$$|\phi(x-y) - \phi(x)| \leq \frac{|y|}{|x|^{n+1}} \quad \text{when } |x| > 2|y|. \tag{4.1}$$

Noting that this condition implies

$$\sup_{x,z \in B(0,2^{j+1}) \setminus B(0,2^j)} |\phi(x) - \phi(z)| \leq C2^{-nj},$$

we see that $\lim_{|x|\to\infty} \phi(x) = 0$ since $\phi \in L^1(\mathbf{R}^n)$ and

$$|\phi(x)| \leq C|x|^{-n}. \tag{4.2}$$

if ϕ satisfies (4.1). In this connection we show

Theorem 4.1. *Let $p_- > 1$. Suppose that $\phi \in L^1(\mathbf{R}^n) \cap L^{(p_0)'}(B(0,R))$ and ϕ satisfies* (4.2) *for $|x| \geq R$. Then*

$$\|\phi * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \leq C(\|\phi\|_{L^1,\mathbf{R}^n} + \|\phi\|_{L^{(p_0)'},B(0,R)})\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n}$$

for all $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$.

Remark 4.2. Theorem 4.1 does not imply an inequality

$$\|\phi_t * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \leq C\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n}$$

with a constant C independent of $t \in (0,1]$ even if ϕ satisfies (4.2) for all x, because $\{\|\phi_t\|_{L^{(p_0)'},B(0,R)}\}_{0<t\leq 1}$ is not bounded.

Proof of Theorem 4.1. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \leq 1$. Suppose that ϕ satisfies (4.2) for $|x| \geq R$ and $\|\phi\|_{L^1,\mathbf{R}^n} + \|\phi\|_{L^{(p_0)'},B(0,R)} \leq 1$. Decompose $\phi = \phi' + \phi''$, where $\phi' = \phi\chi_{B(0,R)}$. We first note by Theorem 1.2 that

$$\|\phi' * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \leq C.$$

Hence it suffices to show that

$$\|\phi'' * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \leq C.$$

For this purpose, write

$$f = f\chi_{\{y\in\mathbf{R}^n : f(y)\geq 1\}} + f\chi_{\{y\in\mathbf{R}^n : f(y)<1\}} = f_1 + f_2,$$

as before. Then we have by (4.2) and (Φ)

$$\begin{aligned}|\phi'' * f_1(x)| &\le C \int_{\mathbf{R}^n \setminus B(x,R)} |x-y|^{-n} f_1(y)\,dy \\ &\le CR^{-n} \int_{\mathbf{R}^n} f_1(y)\,dy \\ &\le CR^{-n} \int_{\mathbf{R}^n} \Phi_{p(\cdot),q(\cdot)}(y, f(y))\,dy \le C.\end{aligned}$$

Noting that $|\phi'' * f_2| \le 1$, we obtain

$$\int_{B(0,R)} \Phi_{p(\cdot),q(\cdot)}(x, \phi'' * f(x))\,dx \le C.$$

Next, let $h(y) = \Phi_{p(\cdot),q(\cdot)}(y, f(y))$. Then

$$|\phi''| * h(x) \le CR^{-n} \int_{\mathbf{R}^n} h(y)\,dy \le CR^{-n}.$$

If $x \in \mathbf{R}^n \setminus B(0,R)$, then we have by (4.2) and Lemma 3.3

$$\begin{aligned}|\phi'' * f(x)| &\le \int_{B(0,|x|/2)} |\phi''(x-y)| f(y)\,dy + \int_{\mathbf{R}^n \setminus B(0,|x|/2)} |\phi''(x-y)| f(y)\,dy \\ &\le C\left\{ |x|^{-n} \int_{B(x,3|x|/2)} f(y)\,dy + \left(|\phi''| * h(x)\right)^{1/p(x)} + |x|^{-A/p(x)} \right\} \\ &\le C\left\{ Mf(x) + \left(|\phi''| * h(x)\right)^{1/p(x)} + |x|^{-A/p(x)} \right\}\end{aligned}$$

with $A > n$. Now it follows from Proposition 2.5 that

$$\begin{aligned}\int_{\mathbf{R}^n \setminus B(0,R)} \Phi_{p(\cdot),q(\cdot)}(x, |\phi'' * f(x)|)\,dx &\le C\left\{ \int_{\mathbf{R}^n \setminus B(0,R)} \Phi_{p(\cdot),q(\cdot)}(x, Mf(x))\,dx \right. \\ &\qquad \left. + \int_{\mathbf{R}^n} |\phi| * h(x)\,dx + \int_{\mathbf{R}^n \setminus B(0,R)} |x|^{-A}\,dx \right\} \\ &\le C,\end{aligned}$$

as required. □

Theorem 4.3. *Let $1 - p_-/p_+ \le \theta < 1$, $1 < \tilde{p} < p_-$,*

$$\frac{1}{s} = 1 - \frac{\theta}{\tilde{p}} \quad \text{and} \quad \frac{1}{r(x)} = \frac{1-\theta}{p(x)}.$$

Take $\nu = p_-/\tilde{p}$, if $t^{-p_-/\tilde{p}} \Phi_{p(\cdot)/\tilde{p},q(\cdot)}(x,t)$ is uniformly almost increasing in t; otherwise choose $1 \le \nu < p_-/\tilde{p}$. Suppose that $\phi \in L^1(\mathbf{R}^n) \cap L^s(\mathbf{R}^n) \cap L^{s\nu'}(B(0,R))$ and ϕ satisfies

$$|\phi(x)| \le C|x|^{-n/s}$$

for $|x| \ge R$. Then

$$\|\phi * f\|_{\Phi_{r(\cdot),q(\cdot)},\mathbf{R}^n} \le C(\|\phi\|_{L^1,\mathbf{R}^n} + \|\phi\|_{L^s,\mathbf{R}^n} + \|\phi\|_{L^{s\nu'},B(0,R)}) \|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n}$$

for all $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$.

Proof. Suppose that $\|\phi\|_{L^1,\mathbf{R}^n} + \|\phi\|_{L^s,\mathbf{R}^n} + \|\phi\|_{L^{s\nu'},B(0,R)} \le 1$ and ϕ satisfies

$$|\phi(x)| \le C|x|^{-n/s}$$

for $|x| \ge R$. Let f be a nonnegative measurable function on $\mathbf{R}^n$ such that $\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le 1$, and decompose

$$f = f_1 + f_2,$$

where $f_1 = f\chi_{\{x\in\mathbf{R}^n:\, f(x)\ge 1\}}$. Let

$$\frac{1}{r} = \frac{1-\theta}{p_-} \quad \text{and} \quad \frac{1}{s_1} = 1 + \frac{1}{r} - \frac{1}{p_+}.$$

By our assumption, $s_1 \ge 1$. It follows from Young's inequality for convolution that

$$\|\phi * f_2\|_{L^r,\mathbf{R}^n} \le \|\phi\|_{L^{s_1},\mathbf{R}^n}\|f_2\|_{L^{p_1},\mathbf{R}^n}.$$

Here note that $1 \le s_1 < s$, so that $\|\phi\|_{L^{s_1},\mathbf{R}^n} \le \|\phi\|_{L^1,\mathbf{R}^n} + \|\phi\|_{L^s,\mathbf{R}^n} \le 1$. Since $0 \le f_2 < 1$, $\|f_2\|_{L^{p_+},\mathbf{R}^n} \le C\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le C$. Thus, noting that $|\phi * f_2| \le 1$ and

$$\frac{1}{r(x)} - \frac{1}{r} = \frac{1-\theta}{p(x)} - \frac{1-\theta}{p_-} \le 0,$$

we see that

$$\|\phi * f_2\|_{\Phi_{r(\cdot),q(\cdot)},\mathbf{R}^n} \le C\|\phi * f_2\|_{L^r,\mathbf{R}^n} \le C. \tag{4.3}$$

On the other hand, we have by Hölder's inequality

$$\begin{aligned} |\phi * f_1(x)| &\le \left(\int_{\mathbf{R}^n} |\phi(x-y)|^s f_1(y)^{\tilde{p}}\,dy\right)^{(1-\theta)/\tilde{p}} \left(\int_{\mathbf{R}^n} |\phi(x-y)|^s\,dy\right)^{1-1/\tilde{p}} \\ &\quad\cdot \left(\int_{\mathbf{R}^n} |f_1(y)|^{\tilde{p}}\,dy\right)^{\theta/\tilde{p}} \\ &\le C\left(|\phi|^s * f_1^{\tilde{p}}(x)\right)^{(1-\theta)/\tilde{p}}. \end{aligned} \tag{4.4}$$

Noting that $|\phi|^s \in L^1(\mathbf{R}^n) \cap L^{\nu'}(B(0,R))$, $|\phi|^s$ satisfies (4.2) for $|x| \ge R$ and $\|f_1^{\tilde{p}}\|_{\Phi_{p(\cdot)/\tilde{p},q(\cdot)},\mathbf{R}^n} \le C$, we find by Theorem 4.1

$$\|\phi^s * f_1^{\tilde{p}}\|_{\Phi_{p(\cdot)/\tilde{p},q(\cdot)},\mathbf{R}^n} \le C.$$

Since (4.4) implies

$$\Phi_{r(\cdot),q(\cdot)}(x, \phi * f_1(x)) \le C\Phi_{p(\cdot)/\tilde{p},q(\cdot)}(x, |\phi|^s * f_1^{p_1}(x)),$$

it follows that

$$\|\phi * f_1\|_{\Phi_{r(\cdot),q(\cdot)},\mathbf{R}^n} \le C.$$

Thus, together with (4.3), we obtain

$$\|\phi * f\|_{\Phi_{r(\cdot),q(\cdot)},\mathbf{R}^n} \le C,$$

as required. □

Remark 4.4. Cruz-Uribe and Fiorenza [1] conjectured that Theorem A remains true if ϕ satisfies the additional condition (4.1).

If $p_- > 1$, this conjecture was shown to be true by Cruz-Uribe, Fiorenza, Martell and Pérez in [3], using an extrapolation theorem ([3, Theorem 1.3 or Corollary 1.11]). Using our Proposition 2.5, we can prove the following extension of [3, Theorem 1.3]:

Proposition 4.5. *Let $\mathscr{F}$ be a family of ordered pairs (f,g) of nonnegative measurable functions on $\mathbf{R}^n$. Suppose that for some $0<p_0<p^-$,*

$$\int_{\mathbf{R}^n} f(x)^{p_0} w(x)\,dx \le C_0 \int_{\mathbf{R}^n} g(x)^{p_0} w(x)\,dx$$

for all $(f,g)\in\mathscr{F}$ and for all A_1-weights w, where C_0 depends only on p_0 and the A_1-constant of w. Then

$$\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le C\|g\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n}$$

for all $(f,g)\in\mathscr{F}$ such that $g\in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$.

Then, as in [3, p. 249], we can prove:

Theorem 4.6. *Assume that $p_->1$. If ϕ is an integrable function on $\mathbf{R}^n$ satisfying* (4.1)*, then*

$$\|\phi_t * f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n} \le C\|f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n}$$

for all $t>0$ and $f\in L^{p(\cdot)}(\log L)^{q(\cdot)}(\mathbf{R}^n)$. If, in addition, $\int\phi(x)\,dx=1$, then

$$\lim_{t\to 0}\|\phi_t * f - f\|_{\Phi_{p(\cdot),q(\cdot)},\mathbf{R}^n}=0.$$

5. Appendix

For $p\ge 1$, $q\in\mathbf{R}$ and $c\ge e$, we consider the function

$$\Phi(t)=\Phi(p,q,c;t)=t^p\big(\log(c+t)\big)^q,\quad t\in[0,\infty).$$

In this appendix, we give a proof of the following elementary result:

Theorem 5.1. *Let X be a non-empty set and let $p(\cdot)$ and $q(\cdot)$ be real valued functions on X such that $1\le p(x)\le p_0<\infty$ for all $x\in X$. Then, the following* (1) *and* (2) *are equivalent to each other:*

(1) *There exists $c_0\ge e$ such that $\Phi(p(x),q(x),c_0;\,\cdot\,)$ is convex on $[0,\infty)$ for every $x\in X$;*
(2) *There exists $K>0$ such that $K(p(x)-1)+q(x)\ge 0$ for all $x\in X$.*

This theorem may be well known; however, the authors fail to find any literature containing this result.

This theorem is a corollary to the following

Proposition 5.2. (1) *If*

$$(1+\log c)(p-1)+q\ge 0,$$

then Φ is convex on $[0,\infty)$.

(2) *Given $p_0>1$ and $c\ge e$, there exists $K=K(p_0,c)>0$ such that Φ is not convex on $[0,\infty)$ whenever $1\le p\le p_0$ and $q<-K(p-1)$.*

Proof. By elementary calculation we have

$$\Phi''(t)=t^{p-2}(c+t)^{-2}\big(\log(c+t)\big)^{q-2}G(t)$$

with

$$G(t)=p(p-1)(c+t)^2\big(\log(c+t)\big)^2+2pqt(c+t)\log(c+t)-qt^2\log(c+t)+q(q-1)t^2$$

for $t>0$. $\Phi(t)$ is convex on $[0,\infty)$ if and only if $G(t)\ge 0$ for all $t\in(0,\infty)$.

(1) If $q \geq 0$, then

$$G(t) \geq qt\big(2p(c+t)-t\big)\log(c+t) - qt^2 \geq qt\big(2pc+2(p-1)t\big) \geq 0$$

for all $t \in (0,\infty)$, so that Φ is convex on $[0,\infty)$.

If $-(1+\log c)(p-1) \leq q < 0$, then

$$\begin{aligned} G(t) &= p\left\{\sqrt{p-1}(c+t)\log(c+t) + \frac{q}{\sqrt{p-1}}t\right\}^2 \\ &\quad - \frac{pq^2}{p-1}t^2 - qt^2\log(c+t) + q(q-1)t^2 \\ &\geq (-q)t^2\left(\frac{pq}{p-1} + \log c - (q-1)\right) \\ &= (-q)t^2\left(\frac{q}{p-1} + \log c + 1\right) \geq 0 \end{aligned}$$

for all $t \in (0,\infty)$, so that Φ is convex on $[0,\infty)$.

(2) If $p = 1$ and $q < 0$, then

$$G(t) = qt\big((t+2c)\log(c+t) + (q-1)t\big) \to -\infty$$

as $t \to \infty$. Hence Φ is not convex on $[0,\infty)$.

Next, let $1 < p \leq p_0$ and $q = -k(p-1)$ with $k > 0$. Then

$$\begin{aligned} \frac{G(t)}{p-1} &= p\big((c+t)\log(c+t) - kt\big)^2 + k\big(\log(c+t) - k + 1\big)t^2 \\ &\leq p_0\big((c+t)\log(c+t) - kt\big)^2 + k\big(\log(c+t) - k + 1\big)t^2. \end{aligned}$$

Let $\lambda = 1 - 1/(2p_0)$. Then $0 < \lambda < 1$. If $k > (\log c)/\lambda$, there is (unique) $t_k > 0$ such that $\log(c+t_k) = \lambda k$. Note that $t_k/k \to \infty$ as $k \to \infty$. We have

$$\begin{aligned} \frac{G(t_k)}{p-1} &\leq p_0\big((c+t_k)\lambda k - kt_k\big)^2 + k(\lambda k - k + 1)t_k^2 \\ &= kt_k^2\left\{\big(p_0(1-\lambda)-1\big)(1-\lambda)k + 1 - 2p_0 c\lambda(1-\lambda)\frac{k}{t_k} + p_0 c^2\lambda^2\frac{k}{t_k^2}\right\}. \end{aligned}$$

Since $p_0(1-\lambda) - 1 = -1/2$, it follows that there is $K = K(c,p_0) > (\log c)/\lambda$ such that $G(t_k) < 0$ whenever $k \geq K$. Hence Φ is not convex if $1 < p \leq p_0$ and $q \leq -K(p-1)$. $\square$

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