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CHAMPAGNE SUBREGIONS OF THE UNIT BALL WITH UNAVOIDABLE BUBBLES

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Abstract. This paper is concerned with the type of region that arises when infinitely many disjoint closed balls, or "bubbles", are removed from the unit ball of Euclidean space. It characterises those configurations of balls which carry full harmonic measure for the resultant region.

1. Main results

Let $B(x,r)$ denote the open ball of centre x and radius r in Euclidean space $\mathbf{R}^n$ $(n \geq 2)$, and let $B = B(0,1)$. This paper is concerned with domains of the form $\Omega = B \backslash (\cup_k \overline{B}(x_k, r_k))$, where the closed balls $\overline{B}(x_k, r_k)$ are pairwise disjoint, $|x_k| \to 1$ and $\sup_k r_k/(1-|x_k|) < 1$. Such domains are known as *champagne regions* and the removed balls are referred to collectively as *the bubbles*. It is convenient to assume that $0 \in \Omega$. The main problem is to determine those configurations of bubbles which cause the unit sphere to carry no harmonic measure for Ω. Since this is equivalent to the bubbles being unavoidable for Brownian motion starting at 0, we will describe such configurations as *unavoidable*.

When $n = 2$ Akeroyd [3] has shown that, for any $\varepsilon > 0$, there are champagne regions for which $\cup_k \overline{B}(x_k, r_k)$ is unavoidable and yet $\sum_k r_k < \varepsilon$. Ortega-Cerdà and Seip [7], also working in the disc, subsequently showed that this phenomenon can occur for any given sequence (x_k) satisfying

$$\inf_{j \neq k} \frac{|x_j - x_k|}{1 - |x_k|} > 0 \tag{1}$$

and

$$B(x, a(1-|x|)) \cap \{x_k \colon k \in \mathbf{N}\} \neq \emptyset \quad (x \in B) \tag{2}$$

for some $a \in (0,1)$. In this case, if $r_k = (1-|x_k|)\phi(|x_k|)$ for some decreasing function $\phi \colon [0,1) \to (0,1)$, it was shown that the bubbles are unavoidable if and only if

$$\int_0^1 \frac{1}{(1-t)\log(1/\phi(t))}\, dt = \infty.$$

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This result was recently extended by O'Donovan [6] to higher dimensions, where the corresponding condition on ϕ is

$$\int_0^1 \frac{\{\phi(t)\}^{n-2}}{1-t}\,dt = \infty.$$

The main purpose of this paper is to obtain results of this nature for more general champagne subregions of the unit ball, where the separation condition (1) is substantially relaxed. From now on we will assume that $n \geq 3$. Normalised surface area measure on ∂B will be denoted by σ.

Theorem 1. *Let Ω be a champagne subregion of the unit ball.*

(a) *If the bubbles are unavoidable, then*

$$\sum_k \frac{(1-|x_k|)^2}{|y-x_k|^n} r_k^{n-2} = \infty \quad \text{for } \sigma\text{-almost every } y \in \partial B. \tag{3}$$

(b) *Conversely, if (3) holds, together with the separation condition*

$$\inf_{j \neq k} \frac{|x_j - x_k|}{r_k^{1-2/n}(1-|x_k|)^{2/n}} > 0, \tag{4}$$

then the bubbles are unavoidable.

We note that condition (4) is strictly weaker than (1) when $n \geq 3$. To see that it cannot be omitted, let K_j denote the closed cube of centre $(1-2^{-j}, 0, ..., 0)$ and sidelength $2^{-j-1}/\sqrt{n}$, with sides parallel to the coordinate hyperplanes. If, for each $j \in \mathbf{N}$, we choose 2^{jn^2} disjoint closed balls of radius $2^{-j-3-jn}/\sqrt{n}$ inside K_j, the resultant configuration of balls is certainly avoidable and yet satisfies (3).

As we will indicate briefly at the end of the paper, our approach to proving Theorem 1 also leads to an improvement of related results for unavoidable configurations of balls in space that have recently been obtained by Carroll and Ortega-Cerdà [5].

Next, following Ortega-Cerdà and Seip [7] and O'Donovan [6], we consider what more can be said when r_k is of the form $(1-|x_k|)\phi(|x_k|)$, where $\phi\colon [0,1) \to (0,1)$ is decreasing. We note that (1) and (2) together imply that the number of points

$$N_a(x) = \#\left[B(x, a(1-|x|)) \cap \{x_k\colon k \in \mathbf{N}\}\right]$$

satisfies $1 \leq N_a(x) \leq b$ for some constants $a \in (0,1)$ and $b > 1$. In the next result we will allow $N_a(x)$ to grow, as $|x| \to 1$, like some increasing function $M\colon [0,1) \to [1,\infty)$ where

$$M(1-t/2) \leq cM(1-t) \quad (0 < t \leq 1),$$

for some $c > 1$.

Theorem 2. *Let ϕ and M be as above, and let Ω be a champagne subregion of the unit ball, where $r_k = (1-|x_k|)\phi(|x_k|)$.*

(a) *If the bubbles are unavoidable and there are constants $a \in (0,1)$ and $b > 0$ such that $N_a(x) \leq bM(|x|)$ for all $x \in B$, then*

$$\int_0^1 \frac{\{\phi(t)\}^{n-2} M(t)}{1-t}\,dt = \infty. \tag{5}$$

(b) *Conversely, if (5) holds, together with the separation condition*

$$\inf_{j \neq k} \frac{|x_j - x_k|}{\{\phi(|x_k|)\}^{1-2/n}(1-|x_k|)} > 0, \tag{6}$$

and there are constants $a \in (0,1)$ and $b > 0$ such that $N_a(x) \geq bM(|x|)$ for all $x \in B$, then the bubbles are unavoidable.

We can now deduce a higher dimensional version of Akeroyd's result.

Corollary 3. *Let $\varepsilon > 0$.*

(a) *There is a champagne subregion of the unit ball satisfying (1), (2) and $\sum_k r_k^{n-1} < \varepsilon$, such that the bubbles are unavoidable.*

(b) *For any $\alpha > n-2$ there is a champagne subregion of the unit ball such that $\sum_k r_k^{\alpha} < \varepsilon$ and the bubbles are unavoidable.*

To see that part (b) of the corollary is sharp, suppose we have a champagne subregion of the unit ball such that $\sum_k r_k^{n-2} < \infty$. By omitting finitely many of the balls we can arrange that $\overline{B}(x_k, r_k) \subset B \backslash B(0, 1/2)$ for all k and $\sum_k r_k^{n-2} < 2^{-n}$. By subadditivity this would imply that the Newtonian capacity of the union of the remaining balls is at most 2^{-n}, whence the associated capacitary potential is valued at most $1/4$ at 0. Thus these balls are avoidable, and it follows that the full collection of balls is also avoidable.

The above results will be proved using Whitney decompositions, two different types of quasiadditivity of Newtonian capacity, and minimal thinness. For potential theoretic background material we refer to the book [4].

2. Proof of Theorem 1

Let $E = \cup_k \overline{B}(x_k, r_k)$, so that $\Omega = B \backslash E$. For a positive superharmonic function u on B we define the usual reduced function

$$R_u^E = \inf\{v \colon v \text{ is positive and superharmonic on } B \text{ and } v \geq u \text{ on } E\}.$$

Then E is unavoidable if and only if $R_1^E(0) = 1$. The Poisson kernel for B with pole at $y \in \partial B$ is given by

$$P(x, y) = \frac{1 - |x|^2}{|x-y|^n} \quad (x \in B).$$

Since $\int P(\cdot, y)\, d\sigma(y) \equiv 1$, we see that

$$R_1^E(0) = R^E_{\int P(\cdot,y)\, d\sigma(y)}(0) = \int_{\partial B} R^E_{P(\cdot,y)}(0)\, d\sigma(y).$$

Hence E is unavoidable if and only if $R^E_{P(\cdot,y)}(0) = 1 = P(0, y)$ for σ-almost every $y \in \partial B$. By the connectedness of Ω and the maximum principle,

$$E \text{ is unavoidable if and only if } R^E_{P(\cdot,y)} \equiv P(\cdot, y) \text{ for } \sigma\text{-a.e. } y \in \partial B. \tag{7}$$

We note, for use below, that the condition $R^A_{P(\cdot,y)} \not\equiv P(\cdot, y)$ characterizes minimal thinness with respect to B of a set $A \subset B$ at a boundary point y (see Chapter 9 of [4]).

Next we choose a Whitney decomposition of B; this is a collection of closed cubes $\{Q_m : m \in \mathbf{N}\}$ with sides parallel to the coordinate hyperplanes such that their union is B, their interiors are pairwise disjoint, and

$$\mathrm{diam}(Q_m) \le \mathrm{dist}(Q_m, \partial B) \le 4\mathrm{diam}(Q_m) \quad (m \in \mathbf{N}) \tag{8}$$

(see Chapter VI of Stein [8]). A Wiener-type criterion for minimal thinness (see Corollary 7.4.4 of Aikawa and Essén [2]), based on the quasiadditivity of Green capacity with respect to Whitney decompositions, tells us that

$$R^E_{P(\cdot,y)} \equiv P(\cdot, y) \text{ if and only if } \sum_m \frac{\{\mathrm{dist}(Q_m, \partial B)\}^2}{\{\mathrm{dist}(y, Q_m)\}^n} \mathscr{C}(E \cap Q_m) = \infty, \tag{9}$$

where $\mathscr{C}(\cdot)$ denotes Newtonian capacity.

We will need the following elementary lemma, the proof of which is left to the reader. It relies on the fact that $\sup_k r_k/(1 - |x_k|) < 1$, and the constant c_1 below depends on the value of this supremum.

Lemma 4. *There is a constant $c_1 > 1$ such that, for any Q_m and any $\overline{B}(x_k, r_k)$ which intersects Q_m:*

$$\frac{1}{c_1} \le \frac{\mathrm{dist}(Q_m, \partial B)}{1 - |x_k|} \le c_1, \quad \textit{and} \quad \frac{1}{c_1} \le \frac{\mathrm{dist}(y, Q_m)}{|y - x_k|} \le c_1 \quad \textit{for all } y \in \partial B.$$

Now suppose that E is unavoidable. By (7) and (9),

$$\sum_m \frac{\{\mathrm{dist}(Q_m, \partial B)\}^2}{\{\mathrm{dist}(y, Q_m)\}^n} \mathscr{C}(E \cap Q_m) = \infty \quad \text{for } \sigma\text{-almost every } y \in \partial B. \tag{10}$$

By the countable subadditivity of Newtonian capacity,

$$\mathscr{C}(E \cap Q_m) = \mathscr{C}\big(\big[\cup_k \overline{B}(x_k, r_k)\big] \cap Q_m\big) \le \sum_k \mathscr{C}(\overline{B}(x_k, r_k) \cap Q_m).$$

Since the number of cubes Q_m which intersect a given ball $\overline{B}(x_k, r_k)$ is bounded above by a constant c_2, independent of k, and since

$$\mathscr{C}(\overline{B}(x_k, r_k) \cap Q_m) \le \mathscr{C}(\overline{B}(x_k, r_k)) = r_k^{n-2},$$

we see from the above lemma that

$$\begin{aligned} \sum_m \frac{\{\mathrm{dist}(Q_m, \partial B)\}^2}{\{\mathrm{dist}(y, Q_m)\}^n} \mathscr{C}(E \cap Q_m) &\le \sum_k \sum_m \frac{\{\mathrm{dist}(Q_m, \partial B)\}^2}{\{\mathrm{dist}(y, Q_m)\}^n} \mathscr{C}(\overline{B}(x_k, r_k) \cap Q_m) \\ &\le c_1^{n+2} c_2 \sum_k \frac{(1 - |x_k|)^2}{|y - x_k|^n} r_k^{n-2}. \end{aligned}$$

Hence (3) follows from (10). This proves part (a) of Theorem 1.

For part (b) we require the following.

Lemma 5. *Suppose that*

$$\frac{|x_j - x_k|}{r_k^{1-2/n} (1 - |x_k|)^{2/n}} \ge 4c_1^{4/n} \quad (j \ne k), \tag{11}$$

where c_1 is as in Lemma 4. Then there is a constant $c_3 > 0$ depending only on n such that, for any Whitney cube Q_m,

$$\mathscr{C}(E \cap Q_m) \geq c_3 \sum_k \mathscr{C}(\overline{B}(x_k, r_k) \cap Q_m). \tag{12}$$

Proof of Lemma. We will establish this by applying a different type of quasiadditivity property of Newtonian capacity to a scaled version of $E \cap Q_m$. Let λ_n denote the Lebesgue measure of B. A result of Aikawa and Borichev [1] tells us that, if F is an analytic subset of $\cup_k B(y_k, \rho_k)$, where $\rho_k \leq \lambda_n^{-1/2} 2^{-n/2}$ for all k, and if the balls $\left\{B(y_k, \lambda_n^{-1/n} \rho_k^{1-2/n}) \colon k \in \mathbf{N}\right\}$ are pairwise disjoint, then

$$\sum_k \mathscr{C}(F \cap B(y_k, \rho_k)) \leq C(n) \mathscr{C}(F),$$

where $C(n)$ is a constant that depends only on n. We will apply it to the set $E^\circ \cap Q_m$ after scaling by the factor

$$\alpha = \frac{\lambda_n^{-1/2} 2^{-n/2}}{c_1 \operatorname{dist}(Q_m, \partial B)}.$$

Thus we define

$$F = (\cup_k B(y_k, \rho_k)) \cap \alpha Q_m,$$

where $y_k = \alpha x_k$, $\rho_k = \alpha r_k$ and $\alpha Q_m = \{\alpha x \colon x \in Q_m\}$. If $B(x_k, r_k) \cap Q_m \neq \emptyset$, then we see from Lemma 4 and (11) that

$$\rho_k = \alpha r_k \leq \alpha(1 - |x_k|) \leq \alpha c_1 \operatorname{dist}(Q_m, \partial B) = \lambda_n^{-1/2} 2^{-n/2}$$

and

$$\begin{aligned} \frac{|y_j - y_k|}{\rho_k^{1-2/n}} = \frac{\alpha\, |x_j - x_k|}{\alpha^{1-2/n} r_k^{1-2/n}} &\geq 4 \left\{\alpha c_1^2 \left(1 - |x_k|\right)\right\}^{2/n} \\ &\geq 4 \left\{\alpha c_1 \operatorname{dist}(Q_m, \partial B)\right\}^{2/n} = 2\lambda_n^{-1/n} \quad (j \neq k). \end{aligned}$$

Thus the hypotheses of the above quasiadditivity theorem are satisfied. The estimate (12) follows, using the facts that $\mathscr{C}(\alpha A) = \alpha^{n-2} \mathscr{C}(A)$ for any analytic set A and that $\mathscr{C}(\overline{B}(x_k, r_k) \cap Q_m) = \mathscr{C}(B(x_k, r_k) \cap Q_m)$ for all m and k. □

Now suppose that (3) and (4) hold. We choose $\delta \in (0, 1)$ small enough so that

$$\frac{|x_j - x_k|}{(\delta r_k)^{1-2/n} \left(1 - |x_k|\right)^{2/n}} \geq 4 c_1^{4/n} \quad (j \neq k)$$

and define $E_\delta = \cup_k \overline{B}(x_k, \delta r_k)$. From Lemmas 4 and 5 we see that

$$\sum_m \frac{\{\operatorname{dist}(Q_m, \partial B)\}^2}{\{\operatorname{dist}(y, Q_m)\}^n} \mathscr{C}(E_\delta \cap Q_m) \geq \frac{c_3}{c_1^{n+2}} \sum_k \sum_m \frac{(1 - |x_k|)^2}{|y - x_k|^n} \mathscr{C}(\overline{B}(x_k, \delta r_k) \cap Q_m).$$

Subadditivity of capacity implies that, for each k, there exists m such that

$$\mathscr{C}(\overline{B}(x_k, \delta r_k) \cap Q_m) \geq c_2^{-1} \mathscr{C}(\overline{B}(x_k, \delta r_k)) = c_2^{-1} \delta^{n-2} r_k^{n-2},$$

where c_2 is as above. Thus

$$\sum_m \frac{\{\operatorname{dist}(Q_m, \partial B)\}^2}{\{\operatorname{dist}(y, Q_m)\}^n} \mathscr{C}(E_\delta \cap Q_m) \geq \frac{c_3 \delta^{n-2}}{c_1^{n+2} c_2} \sum_k \frac{(1 - |x_k|)^2}{|y - x_k|^n} r_k^{n-2},$$

and (3), (7) and (9) together show that E_δ, and hence also E, is unavoidable. This completes the proof of Theorem 1.

3. Proof of Theorem 2

Suppose firstly that E is unavoidable and that $N_a(x) \le bM(|x|)$ for all $x \in B$. Since $r_k = (1-|x_k|)\phi(|x_k|)$, we see from Theorem 1(a) that

$$\sum_k \frac{(1-|x_k|)^n}{|y-x_k|^n}\{\phi(|x_k|)\}^{n-2} = \infty \quad \text{for } \sigma\text{-almost every } y \in \partial B. \tag{13}$$

Any given centre x_k belongs to some Whitney cube Q_m. Clearly

$$1-|x_k| \le \operatorname{dist}(Q_m, \partial B) + \operatorname{diam}(Q_m),$$

so

$$1-|x_k| \le 5\operatorname{diam}(Q_m) \quad \text{and} \quad 1-|x_k| \le 2(1-|x|) \quad (x \in Q_m), \tag{14}$$

by (8). Also, by Lemma 4,

$$|y-x_k| \ge \frac{1}{c_1}\operatorname{dist}(y,Q_m) \ge \frac{1}{2c_1}\{\operatorname{dist}(y,Q_m)+\operatorname{diam}(Q_m)\} \ge \frac{|y-x|}{2c_1} \quad (x \in Q_m),$$

so

$$\frac{(1-|x_k|)^n}{|y-x_k|^n}\{\phi(|x_k|)\}^{n-2} \le \{10c_1\operatorname{diam}(Q_m)\}^n \frac{\{\phi((2|x|-1)^+)\}^{n-2}}{|y-x|^n} \quad (x \in Q_m),$$

in view of the fact that ϕ is decreasing. Since the number of centres x_k that belong to Q_m is bounded above by $C(a,c,n)bM((2|x|-1)^+)$ for all $x \in Q_m$, we see that

$$\sum_k \frac{(1-|x_k|)^n}{|y-x_k|^n}\{\phi(|x_k|)\}^{n-2} \le C(a,c,n)bc_1^n \int_B \frac{\{\phi((2|x|-1)^+)\}^{n-2}M((2|x|-1)^+)}{|y-x|^n}\,dx.$$

Integration with respect to $d\sigma(y)$, together with (13) and the fact that

$$\int_{\partial B} |y-x|^{-n}\,d\sigma(y) = \frac{1}{1-|x|^2} \quad (x \in B)$$

by the harmonicity of the Poisson kernel, yields

$$\int_B \frac{\{\phi((2|x|-1)^+)\}^{n-2}M((2|x|-1)^+)}{1-|x|^2}\,dx = \infty.$$

Hence

$$\int_{1/2}^1 \frac{\{\phi(2t-1)\}^{n-2}M(2t-1)}{1-t}\,dt = \infty,$$

and (5) follows. Thus part (a) of Theorem 2 is established.

To prove part (b), suppose that (5) and (6) hold, and that $N_a(x) \ge bM(|x|)$ for all $x \in B$. Let $y \in \partial B$ and define

$$z_i = \left(1-\frac{\alpha^i}{2}\right)y \quad (i \in \mathbf{N}), \quad \text{where} \quad \alpha = \frac{1-a}{1+a}.$$

The balls $\{B(z_i, a(1-|z_i|))\}$ are then pairwise disjoint. Since

$$1-|x| \ge (1-a)\frac{\alpha^i}{2} \quad \text{and} \quad |y-x| \le (1+a)\frac{\alpha^i}{2} \quad \text{when} \quad x \in B(z_i, a(1-|z_i|)),$$

we see that

$$\begin{aligned}\sum_k \frac{(1-|x_k|)^2}{|y-x_k|^n} r_k^{n-2} &= \sum_k \frac{(1-|x_k|)^n}{|y-x_k|^n}\{\phi(|x_k|)\}^{n-2} \\ &\geq \sum_i \sum_{\{k: x_k \in B(z_i, a(1-|z_i|))\}} \frac{(1-|x_k|)^n}{|y-x_k|^n}\{\phi(|x_k|)\}^{n-2} \\ &\geq \alpha^n b \sum_i \left\{\phi\left(1-(1-a)\frac{\alpha^i}{2}\right)\right\}^{n-2} M(|z_i|) \\ &\geq C(a,b,c,n)\sum_i \left\{\phi\left(1-(1-a)\frac{\alpha^i}{2}\right)\right\}^{n-2} M\left(1-(1-a)\frac{\alpha^{i+1}}{2}\right) \\ &\geq C(a,b,c,n)\int_{1-(1-a)\alpha/2}^1 \frac{\{\phi(t)\}^{n-2} M(t)}{1-t}\,dt = \infty.\end{aligned}$$

Hence E is unavoidable, by Theorem 1(b), using the fact that (6) corresponds to (4) in this case.

4. Proof of the corollary

To prove part (a) of the corollary, let $\{x_k\colon k \in \mathbf{N}\}$ be an enumeration of the centres of the Whitney cubes that do not contain 0, and define

$$\phi(t) = \frac{\{1-\log(1-t)\}^{-1/(n-2)}}{10\sqrt{n}}.$$

Then

$$\int_0^1 \frac{\{\phi(t)\}^{n-2}}{1-t}\,dt = \infty$$

and, in view of (14), the balls $\overline{B}(x_k, (1-|x_k|)\phi(|x_k|))$ lie inside the corresponding Whitney cubes, and so will be disjoint. Thus we can apply Theorem 2(b), with $M(t) \equiv 1$, to see that E is unavoidable. Further,

$$\begin{aligned}\sum_k r_k^{n-1} &= C(n)\sum_k \frac{(1-|x_k|)^{n-1}}{\{1-\log(1-|x_k|)\}^{(n-1)/(n-2)}} \\ &\leq C(n)\int_B \frac{1}{(1-|x|)\{1-\log(1-|x|)\}^{(n-1)/(n-2)}}\,dx < \infty,\end{aligned}$$

by (14). By omitting a finite number of the balls, we can arrange that $\sum_k r_k^{n-1}$ is arbitrarily small.

In proving part (b), we may assume that $\alpha \in (n-2, n-1)$. Let $\phi(t) = c_0(1-t)^\beta$, where $\beta > (n-1-\alpha)/(\alpha-n+2)$ and $c_0 \in (0,1)$. We divide each Whitney cube Q_m that does not contain 0 into p_m^n subcubes of equal size, where p_m is the integer part of $\{\mathrm{dist}(Q_m, \partial B)\}^{\beta(2-n)/n}$. Let $\{x_k\colon k \in \mathbf{N}\}$ be an enumeration of the centres of all such subcubes. Then (6) holds. Also, $N_a(x) \geq \{\phi(|x|)\}^{2-n}$ for all $x \in B$, for a suitable choice of $a \in (0,1)$, and the balls $\overline{B}(x_k, r_k)$, where $r_k = (1-|x_k|)\phi(|x_k|)$, will be pairwise disjoint, provided we choose c_0 to be small enough. Since (5) holds with $M(t) = \{\phi(t)\}^{2-n}$, Theorem 2(b) shows that E is unavoidable. Further,

$$\begin{aligned}\sum_k r_k^\alpha &\leq c_0^\alpha \sum_k (1-|x_k|)^{(1+\beta)\alpha} \\ &\leq c_0^\alpha c_1^{(1+\beta)\alpha} \sum_m \{\operatorname{dist}(Q_m, \partial B)\}^{(1+\beta)\alpha} p_m^n \\ &\leq c_0^\alpha C(n,\alpha,\beta) \int_B (1-|x|)^{\beta(\alpha-n+2)+\alpha-n}\, dx < \infty,\end{aligned}$$

by our choice of β. The result now follows, as before.

5. Unavoidable configurations of balls in space

It is also natural to consider domains of the form $\omega = \mathbf{R}^n \backslash (\cup_k \overline{B}(x_k, r_k))$, where the balls $\overline{B}(x_k, r_k)$ are pairwise disjoint, $0 \in \omega$ and $|x_k| \to \infty$, and to ask when the balls are unavoidable, that is, when they carry full harmonic measure for ω. This is a simpler problem since the underlying domain is the whole of space, rather than B, and the question reduces to asking when the set $F = \cup_k \overline{B}(x_k, r_k)$ is non-thin at infinity (see Theorem 7.6.5 in [4]). For each $j \in \mathbf{N}$ we form the closed cube of centre 0 and sidelength 3^j with sides parallel to the coordinate hyperplanes, divide it into 3^n subcubes of sidelength 3^{j-1}, and discard the central cube. Let $\{\mathscr{R}_m\}$ be an enumeration of the resulting collection of cubes. Wiener's criterion tells us that F is non-thin at infinity if and only if

$$\sum_m \{\operatorname{dist}(0, \mathscr{R}_m)\}^{2-n} \mathscr{C}(F \cap \mathscr{R}_m) = \infty,$$

so this divergence condition characterizes when F is unavoidable. Following the approach of Section 2 we arrive at the following analogue of Theorem 1.

Theorem 6. *Let ω be as above.*

(a) *If F is unavoidable, then*

$$\sum_k \left(\frac{r_k}{|x_k|}\right)^{n-2} = \infty. \tag{15}$$

(b) *Conversely, if (15) holds, together with the separation condition*

$$\inf_{j \neq k} \frac{|x_j - x_k|}{r_k^{1-2/n} |x_k|^{2/n}} > 0, \tag{16}$$

then F is unavoidable.

Part (a) above corresponds to Proposition 1 of Carroll and Ortega-Cerdà [5], and has a straightforward proof. Part (b) improves Theorem 1 of [5] where, in place of (16), there is the stronger pair of assumptions that

$$\inf_{j \neq k} |x_j - x_k| > 0 \quad \text{and} \quad \sup_k r_k^{n-2} |x_k|^2 < \infty.$$

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