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A SPACE OF PROJECTIONS ON THE BERGMAN SPACE

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Abstract. We define a set of projections on the Bergman space A^2, which is parameterized by an affine subset of a Banach space of holomorphic functions in the disk and which includes the classical Forelli–Rudin projections.

1. Introduction

Recall that the Bergman projection of $L^2(\mathbf{D})$ onto the holomorphic Bergman space $A^2 = L^2(\mathbf{D}) \cap \mathscr{H}(\mathbf{D})$, where $\mathscr{H}(\mathbf{D})$ denotes the space of holomorphic functions in the unit disk, is given by

$$P\varphi(z) = \int_{\mathbf{D}} \frac{\varphi(w)}{(1-z\overline{w})^2}\, dA(w),$$

where dA is the normalized Lebesgue measure in the disk. Recall also the family of Forelli–Rudin projections parameterized by $\alpha > -1$

$$P_\alpha\varphi(z) = \int_{\mathbf{D}} (\alpha+1) \left(\frac{1-|w|^2}{1-z\overline{w}}\right)^{\alpha} \frac{\varphi(w)}{(1-z\overline{w})^2}\, dA(w).$$

These are the orthogonal projections of the weighted $L^2(\mathbf{D},(1-|w|)^\alpha\, dA(w))$ onto $\mathscr{H}(\mathbf{D}) \cap L^2(\mathbf{D},(1-|w|)^\alpha dA(w))$. It is well known (see [6, Th. 7.1.4]) that P_α is a continuous projection of $L^2(\mathbf{D})$ onto A^2, for each $\alpha > -1/2$.

Since

$$\left\{\frac{1-|w|^2}{1-z\overline{w}}, z, w \in \mathbf{D}\right\} \subset \mathbf{D}_1$$

where $\mathbf{D}_1 = \{z : |z-1| < 1\}$, we may replace the function $g_\alpha(\zeta) = (\alpha+1)\zeta^\alpha$ in the definition of P_α by any holomorphic function g on $\mathbf{D}_1$ to obtain an operator T_g mapping the space $C_c(\mathbf{D})$ of compactly supported continuous functions defined on $\mathbf{D}$ into A^2. An equivalent formulation of the operators defined this way was given by Bonet, Engliš and Taskinen in [1] to construct continuous projections in weighted L^∞ spaces of $\mathbf{D}$ into $\mathscr{H}(\mathbf{D})$. The purpose of this paper is to study the space $\mathscr{P}$ of all holomorphic functions $g \in \mathbf{D}_1$, for which the corresponding operator T_g can be extended continuously to $L^2(\mathbf{D})$. In particular we study the set $\mathscr{P}_0$ of those functions $g \in \mathscr{P}$ that define continuous projections on A^2. For notational convenience we will translate the functions in $\mathscr{P}$ to the unit disk $\mathbf{D}$.

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We will prove that $\mathscr{P}$ is a Banach space when we define the norm of $g \in \mathscr{P}$ as the operator norm of the operator T_g and that $\Phi(g) = \int_0^1 g(r)\,dr$ defines a bounded linear functional in $\mathscr{P}^*$. We give an analytic description of the elements of $\mathscr{P}$ and show that if $g \in \mathscr{P}$ then either T_g is identically zero on A^2 or it is a multiple of a continuous projection onto A^2, implying that $\mathscr{P}_0 = \Phi^{-1}(\{1\})$ is a closed affine subspace of $\mathscr{P}$.

As usual, for each $z \in \mathbf{D}$, ϕ_z will denote by ϕ_z the Möbius transform $\phi_z(w) = \frac{z-w}{1-\bar{z}w}$ which satisfies $(\phi_z)^{-1} = \phi_z$ and $\phi_z'(w) = -\frac{1-|z|^2}{(1-\bar{z}w)^2}$. Throughout this paper we will write

$$\psi_z(w) = \frac{1-|w|^2}{1-z\bar{w}}$$

and

$$\mathbf{H} = \{z \in \mathbf{C} : \mathrm{Re}(z) > 1/2\}.$$

Clearly the mapping $z \to \frac{1}{1-z}$ is a bijection of $\mathbf{D}$ onto $\mathbf{H}$, and

$$\psi_z(w) = 1 - \bar{w}\phi_w(z). \tag{1}$$

2. A space of projections on A^2

Let us start by presenting our new definitions and spaces of projections.

Definition 1. Let g be holomorphic in $\mathbf{D}$. We define

$$T_g\varphi(z) = \int_{\mathbf{D}} g(\bar{w}\phi_w(z))\varphi(w)\frac{dA(w)}{(1-z\overline{w})^2},$$

for any $\varphi \in C_c(\mathbf{D})$. We denote by $\mathscr{P}$ (resp. $\mathscr{P}_0$) the space of holomorphic functions $g \in \mathscr{H}(\mathbf{D})$ such that T_g extends continuously to $L^2(\mathbf{D})$ (resp. T_g is a projection on the Bergman space A^2). We provide the space $\mathscr{P}$ with the norm $\|g\|_{\mathscr{P}} = \|T_g\|_{L^2(\mathbf{D})\to L^2(\mathbf{D})}$.

Remark 2. In [1] it was introduced, for each F holomorphic in $\mathbf{H}$ the operator

$$S_F\varphi(z) = \int_D F\left(\frac{1-z\overline{w}}{1-|w|^2}\right)\varphi(w)\frac{dA(w)}{(1-|w|^2)^2}.$$

We have $T_g = S_F$, with $F(\eta) = \frac{1}{\eta^2}g(1-\frac{1}{\eta})$. We will say that such $F \in \mathscr{P}$ (resp. $\mathscr{P}_0$) if $g \in \mathscr{P}$ (resp. $\mathscr{P}_0$).

Example 3. Let $g_\alpha(z) = (\alpha+1)(1-z)^\alpha$ for every $\alpha > -1$. Then $g_\alpha \in \mathscr{P}_0$ for $\alpha > -1/2$. In fact by (1) we have $T_{g_\alpha} = P_\alpha$, which is a bounded projection from $L^2(\mathbf{D})$ into A^2 if and only if $\alpha > -1/2$.

Example 4. If $P(z) = \sum_{k=0}^N a_k z^k$ is a polynomial then $P \in \mathscr{P}$. Moreover, $P \in \mathscr{P}_0$ if and only if $\sum_{k=0}^N \frac{a_k}{(k+1)} = \int_0^1 P(r)\,dr = 1$.

Proof. Write $P(z) = \sum_{k=0}^N b_k(1-z)^k$ where $b_k = (-1)^k\frac{P^{(k)}(1)}{k!}$. Hence

$$T_P = \sum_{k=0}^N \frac{b_k}{(k+1)}P_k.$$

This shows that $T_P \in \mathscr{P}$ and $\|P\|_{\mathscr{P}} \le \sum_{k=0}^{N} \frac{|b_k|}{(k+1)}\|P_k\|$. On the other hand $T_P \in \mathscr{P}_0$ if and only if $\sum_{k=0}^{N} \frac{b_k}{(k+1)} = 1$. Notice now that $\sum_{k=0}^{N} \frac{b_k}{(k+1)} = \int_0^1 P(r)\,dr$ to conclude the proof. □

Example 5. If $g \in \mathscr{H}(\mathbf{D})$ is such that $(1-z)^{-\alpha}g(z)$ is bounded for some $\alpha > -1/2$ then $g \in \mathscr{P}$ and $\|g\|_{\mathscr{P}} \le C \sup_{|z|<1} |(1-z)^{-\alpha}g(z)|$. In particular the space of bounded holomorphic functions $H^\infty(\mathbf{D})$ is contained in $\mathscr{P}$ and $\|f\|_{\mathscr{P}} \le C\|f\|_\infty$.

Proof. Use the fact that $P_\alpha^*\varphi(z) = \int_D \frac{(1-|w|^2)^\alpha}{|1-\bar{w}z|^{2+\alpha}}\varphi(w)\,dA(w)$ also defines a bounded operator on $L^2(\mathbf{D})$ (see [5, Theorem 1.9]). □

Proposition 6. *Let* $g\colon \{z : |z-1| < 2\} \to \mathbf{C}$ *be holomorphic such that* $g(z) = \sum_{n=1}^{\infty} a_n(1-z)^n$ *for* $|z-1| < 2$. *If* $\sum_{n=0}^{\infty} \frac{2^n|a_n|}{(n+1)^{5/4}} < \infty$, *then* $g \in \mathscr{P}$ *and*

$$\|g\|_{\mathscr{P}} \le C \sum_{n=0}^{\infty} \frac{2^n\,|a_n|}{(n+1)^{5/4}}.$$

Moreover, $g \in \mathscr{P}_0$ *if and only if* $\sum_{n=0}^{\infty} \frac{a_n}{n+1} = 1$.

Proof. Indeed, the norm $\|P_n\| = \frac{\sqrt{(2n)!}}{n!}$ (see [2, 3]). Then for $\varphi \in C_c(\mathbf{D})$

$$T_g\varphi(z) = \sum_{n=1}^{\infty} \frac{a_n}{(n+1)} P_n\varphi(z),$$

and

$$\|g\|_{\mathscr{P}} \le \sum_{n=0}^{\infty} \frac{|a_n|\,\sqrt{(2n)!}}{(n+1)n!}.$$

Finally observe that, from Stirling's formula, $\frac{\sqrt{(2n)!}}{(n+1)n!} \sim \frac{2^n}{(n+1)^{5/4}}$. To conclude the result note that $\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} < \infty$ and

$$T_g\varphi(z) = (\sum_{n=1}^{\infty} \frac{a_n}{(n+1)})\varphi(z),$$

for $\varphi \in A^2$. □

Example 7. Let $h_\beta(z) = A_\beta(1+z)^{-\beta}$ for $\beta > 0$ where $A_\beta = \frac{1-\beta}{2^{-\beta+1}-1}$ if $\beta \ne 1$ and $A_1 = (\log 2)^{-1}$. Then $h_\beta \in \mathscr{P}_0$ for $0 < \beta < 5/4$.

Proof. Since $\frac{1}{(1-w)^\beta} = \sum_{n=0}^{\infty} \beta_n w^n$ for $\beta > 0$, $|w| < 1$, where $\beta_n \sim (n+1)^{\beta-1}$, we have

$$h_\beta(z) = \frac{A_\beta}{2^\beta(1-(1-z)/2)^\beta} = \sum_{n=0}^{\infty} A_\beta 2^{-(n+\beta)}\beta_n(1-z)^n.$$

Now Proposition 6 implies $h_\beta \in \mathscr{P}$.

Note that

$$1 = \int_1^2 A_\beta s^{-\beta}\,ds = \int_0^1 h_\beta(r)\,dr = \sum_{n=0}^{\infty} \frac{A_\beta 2^{-(n+1)}\beta_n}{n+1}.$$

Apply again Proposition 6 to finish the proof. □

Let us now give some necessary conditions that functions g in $\mathscr{P}$ should satisfy.

Theorem 8. *If $g \in \mathscr{P}$, then*

$$\sup_{z\in\mathbf{D}}\left\{\int_{\mathbf{D}} |g(\bar{w}\phi_w(z))|^2\, dA(w)\right\}^{1/2} \le 2\,\|g\|_{\mathscr{P}}\,, \tag{2}$$

$$\left(\int_0^1 |g(r)|^2\, dr\right)^{1/2} \le 2\|g\|_{\mathscr{P}}, \tag{3}$$

$$\left(\int_0^1 \Big(\int_{\mathbf{D}} \frac{|g(ru))|^2}{|1-ru|^4}\, dA(u)\Big)(1-r^2)^2 r\, dr\right)^{1/2} \le 2\,\|g\|_{\mathscr{P}}\,. \tag{4}$$

Proof. If $g \in \mathscr{P}$ and $\varphi \in C_c(\mathbf{D})$ one has $T_g\varphi \in A^2$. Hence for each $z \in \mathbf{D}$

$$|T_g\varphi(z)| \le \frac{\|T_g\varphi\|_2}{(1-|z|)} \le \frac{\|g\|_{\mathscr{P}}\,\|\varphi\|_2}{(1-|z|)}.$$

Therefore

$$\left|\int_{\mathbf{D}} g(\bar{w}\phi_w(z))\,\varphi(w)\frac{dA(w)}{(1-z\overline{w})^2}\right| \le \frac{\|g\|_{\mathscr{P}}\,\|\varphi\|_2}{(1-|z|)}.$$

Then by duality,

$$\left\{\int_{\mathbf{D}} |g(\bar{w}\phi_w(z))|^2\, \frac{dA(w)}{|1-z\overline{w}|^4}\right\}^{1/2} \le \frac{\|g\|_{\mathscr{P}}}{(1-|z|)} \le 2\frac{\|g\|_{\mathscr{P}}}{(1-|z|^2)}. \tag{5}$$

Let us show the following formula:

$$\overline{\phi_z(u)}\phi_{\phi_z(u)}(z) = u\overline{\phi_u(z)}. \tag{6}$$

Indeed, since

$$1-|\phi_z(u)|^2 = \frac{(1-|z|^2)(1-|u|^2)}{|1-\bar{z}u|^2},$$

then

$$\psi_z(\phi_z(u)) = \frac{1-|\phi_z(u)|^2}{1-\overline{\phi_z(u)}z} = \frac{(1-|u|^2)}{(1-\bar{z}u)} = \overline{\psi_z(u)}. \tag{7}$$

Now (6) follows from (1) and (7)

$$\overline{\phi_z(u)}\phi_{\phi_z(u)}(z) = 1-\psi_z(\phi_z(u)) = u\overline{\phi_u(z)}. \tag{8}$$

Changing the variable $u = \phi_z(w)$ in (5) and using (6) we obtain

$$\left\{\int_{\mathbf{D}} \left|g\left(u\overline{\phi_u(z)}\right)\right|^2 dA(u)\right\}^{1/2} \le 2\,\|f\|_{\mathscr{P}}\,.$$

Now replacing u and $\bar{z}$ by $\bar{w}$ and z respectively the inequality (2) is achieved.

Part (3) follows selecting $z = 0$ in (2).

Part (4) follows from (2) replacing the supremum by an integral over **D** and changing the variable $u = \phi_w(z)$,

$$\begin{aligned}\int_{\mathbf{D}}\int_{\mathbf{D}} |g(\bar{w}\phi_w(z))|^2\, dA(w)\, dA(z) &= \int_{\mathbf{D}}\Big(\int_{\mathbf{D}} \frac{|g(\bar{w}u))|^2}{|1-\bar{w}u|^4}\, dA(u)\Big)(1-|w|^2)^2\, dA(w)\\ &= \int_{\mathbf{D}}\Big(\int_{\mathbf{D}} \frac{|g(|w|u))|^2}{|1-|w|u|^4}\, dA(u)\Big)(1-|w|^2)^2\, dA(w)\\ &= \int_0^1\Big(\int_{\mathbf{D}} \frac{|g(ru))|^2}{|1-ru|^4}\, dA(u)\Big)(1-r^2)^2 r\, dr. \qquad \square\end{aligned}$$

Remark 9. $(\mathscr{P}, \|\cdot\|_{\mathscr{P}})$ is a normed space and $\Phi(g) = \int_0^1 g(r)\, dr \in \mathscr{P}^*$. Indeed, the only condition which needs a proof is the fact that $\|g\|_{\mathscr{P}} = 0$ implies $g = 0$. It follows from (3) that if $\|g\|_{\mathscr{P}} = 0$, then $g(r) = 0$ for $0 < r < 1$. Hence by analytic continuation, $g(z) = 0$ for $z \in \mathbf{D}$. Notice also that (3) implies $\|\Phi\| \le 2$.

Remark 10. The space $\mathscr{P}$ is not invariant under under rotations. Given $\theta \in [0, 2\pi)$ denote $R_\theta(f)(z) = f(e^{i\theta}z)$ for $f \in \mathscr{H}(\mathbf{D})$. Observe that $R_\theta T_g(\varphi) = T_g(R_\theta\varphi)$. However, "$T_g$ is bounded in $L^2(\mathbf{D})$ does not imply $T_{R_\theta g}$ is bounded in $L^2(\mathbf{D})$". For instance, the function $g(z) = (1+z)^{-1/2}$ belongs to $\mathscr{P}$, but by (3), its reflection $g(z) = (1-z)^{-1/2} \notin \mathscr{P}$.

Let us now also give some necessary conditions to belong to the class $\mathscr{P}_0$.

Theorem 11. *If $g \in \mathscr{P}_0$ then*

$$\int_{\mathbf{D}} g(u\overline{\phi_u(z)})\psi(u)\, dA(u) = \psi(0) \tag{9}$$

for all $\psi \in A_2$ and $z \in \mathbf{D}$. In particular,

(i) *If $g \in \mathscr{P}_0$ then $\int_0^1 g(r)\, dr = 1$.*

(ii) *Let $S_2 = \{\bar{z}(1-|z|^2)\varphi(\bar{z}) : \varphi \in A^2\}$. If $g \in \mathscr{P}_0$ and $g' \in \mathscr{P}$ then $S_2 \subset \mathrm{Ker}(T_{g'})$.*

Proof. Assume

$$\int_{\mathbf{D}} g(\bar{w}\phi_w(z))\frac{\varphi(w)}{(1-\bar{w}z)^2}\, dA(w) = \varphi(z)$$

for all $\varphi \in A^2$. Given $\psi \in A^2$ and $z \in D$, consider $\varphi(w) = \psi(\phi_z(w))\frac{(1-|z|^2)^2}{(1-\bar{z}w)^2}$. Clearly $\varphi \in A_2$ and $\|\varphi\|_2 = (1-|z|^2)\|\psi\|_2$. From the assumption,

$$\int_{\mathbf{D}} g(\bar{w}\phi_w(z))\psi(\phi_z(w))\frac{(1-|z|^2)^2}{|1-\bar{w}z|^4}\, dA(w) = \psi(0).$$

for all $\psi \in A^2$ and $z \in \mathbf{D}$.

Now changing the variable $u = \phi_z(w)$, and using (6), one gets

$$\int_{\mathbf{D}} g(u\overline{\phi_u(z)})\psi(u)\, dA(u) = \psi(0)$$

for all $\psi \in A_2$ and $z \in \mathbf{D}$. Finally changing u by $\bar{w}$ one obtains

$$\int_{\mathbf{D}} g(\bar{w}\phi_w(z))\psi(\bar{w})\, dA(w) = \psi(0) \tag{10}$$

for all $\psi \in A_2$ and $z \in \mathbf{D}$. (i) follows selecting $\psi = 1$ and $z = 0$ in (10).

Differentiating in (10) with respect to z one obtains

$$\int_{\mathbf{D}} g'(\bar{w}\phi_w(z))\frac{-\bar{w}(1-|w|^2)}{(1-\bar{w}z)^2}\psi(\bar{w})\,dA(w) = T_{g'}(\psi_1) = 0$$

where $\varphi_1(u) = -\bar{u}(1-|u|^2)\varphi(\bar{u})$. Hence (ii) is finished. □

Let us now show that $(\mathscr{P}, \|\cdot\|_{\mathscr{P}})$ is complete. For such a purpose, let us define $h_z\colon \mathbf{D} \to \mathbf{H}$ by

$$h_z(w) = \frac{1}{\psi_z(w)} = \frac{1-z\overline{w}}{1-|w|^2},$$

and let us mention that

$$\mathbf{D}_1 = \{\frac{1-|w|^2}{1-z\overline{w}} : z, w \in \mathbf{D}\} = \{\psi_z(w) : z, w \in \mathbf{D}\}.$$

Lemma 12. *For every $\xi \in \mathbf{H}$, there exist $0 \le \alpha < 1$ and $w \in \mathbf{D}$ such that $\xi = h_\alpha(w)$ and h_α is an diffeomorfism of a neighborhood U of w onto an open neighborhood of ξ.*

Proof. For $0 \le r, \alpha < 1$ fixed,

$$h_\alpha(re^{i\theta}) = \frac{1}{1-r^2} - \frac{r\alpha}{1-r^2}e^{-i\theta} \tag{11}$$

describes the circle $C_{r,\alpha}$ centered at the complex number $\frac{1}{1-r^2}$ with radius $\frac{r\alpha}{1-r^2}$. Let $\xi \in \mathbf{H}$. To prove that $\xi \in h_\alpha(\mathbf{D})$ it is enough to see that $\xi \in C_{r,\alpha}$ for some $0 \le r, \alpha < 1$. Let

$$\beta = \frac{1}{r^2}\left[(1-r^2)^2|\xi|^2 + 1 - 2(1-r^2)\operatorname{Re}\xi\right] = \frac{|(1-r^2)\xi - 1|^2}{r^2}. \tag{12}$$

It is clear that $\beta \ge 0$ and

$$\beta < 1 \Leftrightarrow (1-r^2)|\xi|^2 + 1 < 2\operatorname{Re}\xi.$$

Also, since $\xi \in \mathbf{H}$, we have for some $\varepsilon > 0$ that $2\operatorname{Re}\xi > 1+\varepsilon$. Hence if $|\xi|^2 < \frac{\varepsilon}{(1-r^2)}$ then $\beta < 1$. We conclude that there exists r_0 for which $0 \le \beta < 1$ provided $r_0 < r < 1$. Then if $r_0 < r < 1$ and $\alpha = \sqrt{\beta}$ we have $0 \le \alpha < 1$ and

$$\left|\xi - \frac{1}{1-r^2}\right| = \frac{r\alpha}{1-r^2},$$

that is $\xi \in C_{r,\alpha}$. Hence there exists θ_r and $0 \le \alpha_r < 1$ such that $h_{\alpha_r}(re^{i\theta_r}) = \xi$.

To find θ_r explicitly, we let $\varphi_r = \pi - \theta_r$. From (11) we can write

$$\xi = \frac{1}{1-r^2} + \frac{r\alpha_r}{1-r^2}e^{i\varphi_r}.$$

Hence φ_r is the argument of ξ in polar coordinates centered at the complex number $\frac{1}{1-r^2}$. Then if $\frac{1}{1-r^2} \ge \operatorname{Re}(\xi)$,

$$\begin{aligned} \sin\theta_r &= \sin\varphi_r = \frac{\operatorname{Im}(\xi)}{r\alpha_r}(1-r^2) \\ \cos\theta_r &= -\cos\varphi_r = \frac{(1-r^2)}{r\alpha_r}\left(\frac{1}{1-r^2} - \operatorname{Re}(\xi)\right) = \frac{1-(1-r^2)\operatorname{Re}(\xi)}{r\alpha_r}. \end{aligned} \tag{13}$$

Now we will prove that possibly except for a finite number of values of $r \ge r_0$, the jacobian matrix $Dh_{\alpha_r}(re^{i\theta_r})$ is not singular, where α_r and θ_r are chosen so that

$h_{\alpha_r}(re^{i\theta_r}) = \xi$ as before. To this end, it is enough to see that the set of values of r for which the vectors

$$\frac{\partial h_{a_r}}{\partial \rho}(\rho e^{i\theta_r})_{|\rho=r} \quad \text{and} \quad \frac{1}{r}\frac{\partial h_{a_r}}{\partial \theta}(re^{i\theta})_{|\theta=\theta_r} \tag{14}$$

are linearly dependent is finite.

We have

$$\begin{aligned}
\frac{\partial h_a}{\partial \rho}(\rho e^{i\theta}) &= \left(\frac{2\rho}{(1-\rho^2)^2} - \frac{\alpha(1+\rho^2)}{(1-\rho^2)^2}\cos\theta, \frac{\alpha(1+\rho^2)}{(1-\rho^2)^2}\sin\theta\right), \\
\frac{1}{\rho}\frac{\partial h_a}{\partial \theta}(\rho e^{i\theta}) &= \left(\frac{\alpha}{(1-\rho^2)}\sin\theta, \frac{\alpha}{(1-\rho^2)}\cos\theta\right),
\end{aligned}$$

and the jacobian of h_α

$$\begin{aligned}
Jh_\alpha(\rho e^{i\theta}) &= \det\left[\frac{\partial h_a}{\partial \rho}(\rho e^{i\theta}) | \frac{1}{\rho}\frac{\partial h_a}{\partial \theta}(\rho e^{i\theta})\right] \\
&= \det\begin{bmatrix} \frac{2\rho}{(1-\rho^2)^2} - \frac{\alpha(1+\rho^2)}{(1-\rho^2)^2}\cos\theta & \frac{\alpha(1+\rho^2)}{(1-\rho^2)^2}\sin\theta \\ \frac{\alpha}{(1-\rho^2)}\sin\theta & \frac{\alpha}{(1-\rho^2)}\cos\theta \end{bmatrix} \\
&= \frac{\alpha}{(1-\rho^2)^3}\left(2\rho\cos\theta - \alpha(1+\rho^2)\right).
\end{aligned} \tag{15}$$

If $2r\cos\theta_r - \alpha_r(1+r^2) = 0$, then multiplying this equation by $\alpha_r r^2$ we obtain

$$2r^2\alpha_r r\cos\theta_r - \alpha_r^2 r^2(1+r^2) = 0. \tag{16}$$

However, from (12) and (13) we see that $2r^2\alpha_r r\cos\theta_r - \alpha_r^2 r^2(1+r^2)$ is a polynomial of degree 6 in the variable r. We conclude that the vectors in (14) are linearly dependent for six values of r at the most and the proof of the lemma is complete. □

Theorem 13. *$\mathscr{P}$ is a Banach space.*

Proof. Let $g \in \mathscr{P}$. We have by Theorem 8 that

$$\sup_{z\in\mathbf{D}}\left\{\int_{\mathbf{D}} |g(\bar{w}\phi_w(z))|^2\, dA(w)\right\}^{1/2} \leq 2\,\|g\|_{\mathscr{P}}. \tag{17}$$

Fix $\xi \in \mathbf{D}$. Since $\psi_z = 1/h_z$, the local invertibility statement of Lemma 12 holds for the family of functions $1-\psi_z$ taking $\xi \in \mathbf{D}$, namely, there exist $\alpha \in (0,1)$, $w_\xi \in \mathbf{D}$ and open neighborhoods U and V of ξ and w_ξ respectively, such that $1-\psi_z$ is a diffeomorphism of V into U.

Hence

$$\begin{aligned}
\left\{\int_U |g(u)|^2\, dA(u)\right\}^{1/2} &= \left\{\int_V |g(1-\psi_\alpha(w))|^2\, |J\psi_\alpha(w)|\, dA(w)\right\}^{1/2} \\
&\leq C(\xi)\left\{\int_V |g(\bar{w}\phi_w(\alpha))|^2\, dA(w)\right\}^{1/2} \\
&\leq C(\xi)\,\|g\|_{\mathscr{P}}.
\end{aligned}$$

It follows that

$$\left\{\int_K |g(u)|^2\, dA(u)\right\}^{1/2} \le C_K \|g\|_{\mathscr{P}},$$

for every compact set $K \subset \mathbf{D}$. This implies that

$$\sup_{u\in K} |g(u)| \le \|g\|_{\mathscr{P}} C'_K. \tag{18}$$

If $\{g_n\}$ is a Cauchy sequence in $\mathscr{P}$, we have by (18) that $\{g_n\}$ converges uniformly on compact sets of $\mathbf{D}$ to a holomorphic function g.

Let us show that $g \in \mathscr{P}$ and $\|g_n - g\|_{\mathscr{P}} \to 0$. Note first that for each $\varphi \in C_c(\mathbf{D})$ we have

$$T_{g_n}\varphi(z) \to T_g\varphi(z), \quad z \in \mathbf{D}.$$

Using the fact $\sup_{n\in\mathbf{N}} \|g_n\|_{\mathscr{P}} = M < \infty$ and Fatou's lemma one gets

$$\|T_g\varphi\|_2^2 \le \liminf_{n\to\infty} \|T_{g_n}\varphi\|_2^2 \le M\|\varphi\|_2^2.$$

Hence $g \in \mathscr{P}$. On the other hand, given $\varepsilon > 0$ there exists n_0 such that

$$\|T_{g_n}\varphi - T_{g_m}\varphi\|_2 \le \|g_n - g_m\|_{\mathscr{P}} < \varepsilon$$

for $m, n \ge n_0$ and $\|\varphi\|_2 = 1$. Applying Fatou's lemma again we conclude that

$$\|T_{g_n}\varphi - T_g\varphi\|_2 \le \varepsilon$$

for $n \ge n_0$. Therefore $g_n \to g$ in $\mathscr{P}$. □

3. Main results

Let us now describe the norm in $\mathscr{P}$ in a more explicit way. We shall use the formulation of the space given in [1].

Theorem 14. *Let $g \in \mathscr{H}(\mathbf{D})$ and put $F(\xi) = \frac{1}{\xi^2} g(1 - \frac{1}{\xi})$. Then $g \in \mathscr{P}$ if and only if*

$$\sup_j \frac{1}{j!\sqrt{j+1}} \left(\int_1^\infty [(x-1)x]^j \left|xF^{(j)}(x)\right|^2 dx\right)^{1/2} < \infty.$$

Proof. We use the expression

$$T_g\varphi(z) = \int_{\mathbf{D}} F\left(\frac{1 - z\overline{w}}{1 - |w|^2}\right) \varphi(w) \frac{dA(w)}{(1 - |w|^2)^2}.$$

Consider the space M of functions of the form

$$\varphi = \sum_{\text{finite}} \varphi_j(r) e^{ij\theta},$$

with $\varphi_j \in L^2((0,1), r\,dr)$. Then M is a dense subspace of $L^2(\mathbf{D})$.

For $z \in \mathbf{D}$ and $0 \le r < 1$ fixed, let $f(\zeta) = F\left(\frac{1 - rz\zeta}{1 - r^2}\right)$, which is holomorphic on $\overline{\mathbf{D}}$. We have

$$f(\zeta) = F\left(\frac{1 - rz\zeta}{1 - r^2}\right) = \sum_{j\ge 0} \frac{1}{j!} \left(\frac{-rz}{1 - r^2}\right)^j F^{(j)}\Big(\frac{1}{1 - r^2}\Big)\zeta^j, \quad |\zeta| \le 1.$$

Then for $g \in M$,

$$\int_0^{2\pi} f(re^{-i\theta})\varphi(re^{i\theta})\frac{d\theta}{2\pi} = \sum_{j\geq 0}\varphi_j(r)\frac{(-1)^j}{j!}\left(\frac{r}{1-r^2}\right)^j F^{(j)}(\frac{1}{1-r^2})z^j,$$

Hence

$$T_g(\varphi)(z) = \sum_{j\geq 0}\gamma_j(\varphi_j)\sqrt{j+1}z^j, \tag{19}$$

where γ_j is the functional in $L^2((0,1), r\,dr)$ defined by

$$\gamma_j(\varphi) = \frac{(-1)^j}{\sqrt{j+1}j!}\int_0^1 \varphi(r)\left(\frac{r}{1-r^2}\right)^j F^{(j)}(\frac{1}{1-r^2})\frac{r}{(1-r^2)^2}\,dr.$$

Using the normalized Lebesgue measure dA, the set $\{\sqrt{j+1}z^j\}$ is an orthonormal basis for A^2, so we conclude that T_g is bounded in $L^2(\mathbf{D})$ if and only if

$$\left\|(\gamma_j(\varphi_j))_{j\geq 0}\right\|_{\ell^2} \leq C\left\|\varphi\right\|_{L^2(\mathbf{D})} = C\left(\sum_j\int\left|\varphi_j(r)\right|^2 r\,dr\right)^{1/2}.$$

Using duality, this will hold if and only if

$$\sup_{j\geq 0}\frac{1}{\sqrt{j+1}j!}\left(\int_0^1\left(\frac{r}{1-r^2}\right)^{2j}\left|F^{(j)}(\frac{1}{1-r^2})\right|^2\frac{r\,dr}{(1-r^2)^4}\right)^{1/2} < \infty. \tag{20}$$

Making the change of variables $x = \frac{1}{1-r^2}$, the integrals above equal

$$\frac{1}{2}\int_1^\infty [(x-1)x]^j\left|xF^{(j)}(x)\right|^2 dx$$

and the proof is complete. □

We can now give an alternative proof of a well know result.

Corollary 15. *P_α is bounded on $L^2(\mathbf{D})$ if and only if $\alpha > -1/2$.*

Proof. Consider $g_\alpha(z) = (1-z)^\alpha$. Assume first that $g_\alpha \in \mathscr{P}$. Then (3) in Theorem 8 implies that $\int_0^1 (1-r)^{2\alpha}\,dr < \infty$ and therefore $\alpha > -1/2$.

Assume now that $\alpha > -1/2$. Since $F_\alpha(\xi) = \xi^{-m}$ with $m = 2+\alpha$ and $2m-3 > 0$, one has for $j \geq 0$ that

$$F_\alpha^{(j)}(x) = (-1)^j m(m+1)\cdots(m+j-1)x^{-(m+j)} = (-1)^j\frac{\Gamma(m+j)}{\Gamma(m)}x^{-(m+j)}.$$

Therefore

$$\begin{aligned}\int_1^\infty [(x-1)x]^j\left|xF_\alpha^{(j)}(x)\right|^2 dx &= \int_1^\infty\left(1-\frac{1}{x}\right)^j\left(x^{j+1}F_\alpha^{(j)}(x)\right)^2 dx\\ &= \left(\frac{\Gamma(m+j)}{\Gamma(m)}\right)^2\int_1^\infty\left(1-\frac{1}{x}\right)^j x^{-2m+4}\frac{d}{x^2}\\ &= \left(\frac{\Gamma(m+j)}{\Gamma(m)}\right)^2\int_0^1 (1-r)^j r^{2m-4}\,dr\\ &= \left(\frac{\Gamma(m+j)}{\Gamma(m)}\right)^2 B(2m-3, j+1).\end{aligned}$$

Using $B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ one concludes that

$$\frac{1}{(j!)^2(j+1)}\int_1^\infty [(x-1)x]^j \left|xF_\alpha^{(j)}(x)\right|^2 dx = \frac{B(2m-3,j+1)}{B(m,j)^2 j^2(j+1)}.$$

Finally since for p fixed, $B(p,j)\sim j^{-p}$ one obtains

$$\frac{B(2m-3,j+1)}{B(m,j)^2 j^2(j+1)}\sim 1. \qquad \square$$

Example 16. In Example 7 it was shown that, for $0<\beta<5/4$, $g(z)=(1+z)^{-\beta}\in\mathscr{P}$ (which corresponds to $F(\xi)=\frac{\xi^{\beta-2}}{(2\xi-1)^2}$). Let us show, for instance, that $g(z)=(1+z)^{-2}\notin\mathscr{P}$. In this case $F(\xi)=\frac{1}{(2\xi-1)^2}$ and

$$F^{(j)}(\xi)=\frac{(-1)^j(j+1)!2^j}{(2\xi-1)^{2+j}}.$$

Since $\frac{x}{2}\le x-1\le x$ for $x\ge 2$ we have

$$\Big(\int_2^\infty (x(x-1))^j|xF^{(j)}(x)|^2\,dx\Big)^{1/2}\sim 2^j(j+1)!\Big(\int_2^\infty \frac{x^{2j+2}}{(2x-1)^{4+2j}}\,dx\Big)^{1/2}\sim 2^j(j+1)!.$$

Hence the condition in Theorem 14 does not hold.

The conditions

$$\sup_{j\ge 0}\frac{1}{j!}\int_1^\infty \left|(x-1)^jF^{(j)}(x)\right|dx<\infty, \tag{21}$$

$$\lim_{x\to\infty} x^{j+1}F^{(j)}(x)=0 \tag{22}$$

were introduced in [1]. These conditions imply that on the space of all the holomorphic functions φ such that $S_F\varphi$ is well defined, the operator S_F is a constant multiple of the identity. Now we will see that (21) and (22) hold for every $g\in\mathscr{P}$ which allows to show the following result.

Theorem 17. *Let $g\in\mathscr{P}$ and $c_0=\int_0^1 g(r)\,dr$. Then*

$$T_g(\varphi)=c_0\varphi,\quad \varphi\in A^2.$$

Proof. Let us notice first that $(x-1)^jF^{(j)}(x)\in L^1([1,\infty),dx)$ for $j\ge 0$. Indeed,

$$\begin{aligned}\int_1^\infty |x-1|^j\left|F^{(j)}(x)\right|dx &= \int_1^\infty |x(x-1)|^j\left|xF^{(j)}(x)\right|\frac{dx}{x^{j+1}}\\ &\le \Big(\int_1^\infty (x(x-1))^j\left|xF^{(j)}(x)\right|^2dx\Big)^{1/2}\Big(\int_1^\infty \frac{(x(x-1))^j}{x^{2j+2}}\,dx\Big)^{1/2}\\ &= \Big(\int_1^\infty (x(x-1))^j\left|xF^{(j)}(x)\right|^2dx\Big)^{1/2}\Big(\int_0^1 (1-r)^j\,dr\Big)^{1/2}\\ &= \frac{1}{\sqrt{j+1}}\Big(\int_1^\infty |x(x-1)|^j\left|xF^{(j)}(x)\right|^2dx\Big)^{1/2}\le Cj!\|g\|_{\mathscr{P}}.\end{aligned}$$

Applying (19) in Theorem 14 to $\varphi(z)=\sum_{j=0}^N a_jz^j$ one obtains

$$T_g\varphi=\sum_{j=0}^N c_ja_jz^j, \tag{23}$$

and

$$c_j = \frac{(-1)^j}{j!} \int_1^\infty (x-1)^j F^{(j)}(x)\, dx,$$

where c_j is well defined. As in [1, Th. 1] we have by integration by parts

$$c_j - c_{j+1} = \frac{(-1)^j}{(j+1)!} \lim_{x\to\infty} (1-x)^{j+1} F^{(j)}(x).$$

Let us now show that $\lim_{x\to\infty}(1-x)^{j+1}F^{(j)}(x) = 0$. Note first that $(x-1)^{j+1}F^{(j)}(x) \in L^2([1,\infty), dx)$ for $j \geq 0$. Indeed

$$(24) \quad \int_1^\infty |(x-1)^{j+1}F^{(j)}(x)|^2\, dx \leq \int_1^\infty |x(x-1)|^j |xF^{(j)}(x)|^2\, dx \leq C(j+1)(j!)^2.$$

In particular $(x-1)^j F^{(j)}(x) \in L^2([1,\infty), dx)$ for $j \geq 1$. From Cauchy–Schwarz and the previous estimates one has that if $f_j(x) = [(x-1)^{j+1}F^{(j)}(x)]^2$, then $(f_j)' \in L^1([1,\infty))$ for every $j \geq 0$. Therefore writing

$$[(x-1)^{j+1}F^{(j)}(x)]^2 = \int_1^x (f_j)'(y)\, dy$$

we see that the $\lim_{x\to\infty}((x-1)^{j+1}F^{(j)}(x))^2$ exists and by (24) it vanishes for all j. Hence (23) becomes $T_g(\varphi) = c_0\varphi$ where

$$c_0 = \int_1^\infty F(x)\, dx = \int_1^\infty g(1-\frac{1}{x})\frac{dx}{x^2} = \int_0^1 g(r)\, dr. \qquad \square$$

Corollary 18. *Let $g \in \mathscr{P}$. Then $A^2 \subset \operatorname{Ker} T_g$ if and only if $\int_0^1 g(r)\, dr = 0$.*

Corollary 19. *Let $\Phi(g) = \int_0^1 g(r)\, dr$ for $g \in \mathscr{P}$. Then $\mathscr{P}_0 = \Phi^{-1}(\{1\})$.*

Corollary 20. *Let $g \in \mathscr{P}$. If T_g is not identically zero in A^2 then there exists $\lambda \neq 0$ and $g_0 \in \mathscr{P}_0$ such that $g = \lambda g_0$.*

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