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ALGEBRA OF INVARIANTS FOR THE RARITA–SCHWINGER OPERATORS

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Abstract. Rarita–Schwinger operators (or Higher Spin Dirac operators) are generalizations of the Dirac operator to functions valued in representations $\mathbf{S}_k$ with higher spin k. Their algebraic and analytic properties are currently studied in Clifford analysis. As a part of this pursuit, we describe the algebra of invariant $\operatorname{End}\mathbf{S}_k$-valued polynomials, i.e., the algebra of invariant constant-coefficient differential operators acting on these representations. The main theorem states that this algebra is generated by the powers of the Rarita–Schwinger and Laplace operators. This algebra is the algebraic part of the Howe dual superalgebra corresponding to the Pin group acting on $\mathbf{S}_k$.

1. Notation and preliminaries

This paper has to be situated on the edge of representation theory and Clifford analysis. The latter is a part of classical analysis, and deals with the function theoretical study of invariant differential operators. Whereas the standard references [1, 9, 13] mostly deal with the Dirac operator on $\mathbf{R}^m$, a conformally invariant first-order vector-valued operator factorizing the Laplace operator $\Delta = \sum_j \partial_{x_j}^2$ on $\mathbf{R}^m$, more recent research focuses on function theoretical aspects of more general operators as well. In particular the Rarita–Schwinger operator and its symmetric analogues have been the subject of interest in, e.g., [4, 5, 6, 7, 21]. This paper should be seen as a continuation of the investigation for these higher spin operators. Whereas the study of the classical Dirac operator on $\mathbf{R}^m$ does not require the usage of representation theoretical techniques, because it can be defined ad hoc as a generalization of the Cauchy-Riemann operator in complex analysis, the study of higher spin analogues does require a fair amount of representation theory for simple Lie algebras $\mathfrak{g}$, which is the reason why we will rely on some of the following notions and definitions.

The space of polynomial operators acting on functions with values in a $\mathfrak{g}$-representation $\mathbf{V}$ is $\mathscr{P}(\operatorname{End}\mathbf{V})$. This is itself a $\mathfrak{g}$-representation and the invariants are precisely trivial summands occuring in its decomposition into irreducibles. In view of the fact that classical Clifford analysis deals with differential operators invariant under the spin group, whose Lie algebra is nothing but the orthogonal Lie algebra, we will treat the odd dimensional case $\mathfrak{g} = B_n$ in detail and make some remarks on the even-dimensional case after that. We denote by Δ^+ the set of positive roots and

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by δ the sum of fundamental weights of $\mathfrak{g}$. The Weyl group W of B_n acts on the set of weights by permutations and sign reversals of coordinates. The length of an element $w \in W$ will be denoted by $|w|$. The set of dominant integral weights Λ_W consists of weights $\mu = (\mu_1, \ldots, \mu_n)$ satisfying $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \geq 0$. For each $\mu \in \Lambda_W$ we denote by $\mathbf{V}_\mu$ or $\mathbf{V}(\mu)$ the irreducible representation $\mathfrak{g}$ with highest weight μ and by Π_μ its set of all weights. Because we will mainly be interested in irreducible modules $\mathbf{V}_\mu$ of a specific kind, we adopt the following short-hand notations.

Let i, j, p, q, k be non-negative integers. The representation with highest weight $(i, 0, \ldots, 0)$, will be denoted by $\mathbf{V}_{(i)}$ and the weight itself will be abbreviated to (i). Similarly, a highest weight $(i, j, 0, \ldots, 0)$ will be shortened to (i, j) and the corresponding representation will be denoted by $\mathbf{V}_{(i,j)}$. For half-integer weights finally, we adopt the following convention: the representation with highest weight

$$\left(i + \frac{1}{2}, j + \frac{1}{2}, \left(\frac{1}{2}\right)_{n-2}\right) =: (i, j)',$$

where $(x)_p$ denotes the sequence where the symbol x is repeated p times, will be denoted by $\mathbf{S}_{i,j}$. In case $j = 0$ this becomes $(i)'$ and the associated representation will then be denoted as $\mathbf{S}_i$. By $\mathbf{S}$ we will mean just $\mathbf{S}_0$.

Within Clifford analysis, these modules have an important meaning: first of all, $\mathbf{S}$ is the so-called spinor space in which functions usually take their values; it can be defined as a minimal left ideal in the Clifford algebra $\mathbf{C}_n$ generated by the basis vectors e_i for the vector space $\mathbf{R}^n$ through the well known defining relations $e_i e_j + e_j e_i = -2\delta_{ij}$ (with $1 \leq i, j \leq n$). For any integer $k > 0$, it is well known [9] that the space $\mathscr{M}_k(\mathbf{S})$ of k-homogeneous spinor-valued monogenic polynomials defines an irreducible $\mathfrak{g}$-representation isomorphic to $\mathbf{S}_k$. Let us introduce a shorthand $\underline{x}$ for the Clifford variable $\sum x_i e_i$. Recall that $f(\underline{x}) \in \mathscr{C}^\infty(\mathbf{R}^m, \mathbf{S})$ is said to be monogenic if $\underline{\partial}_x f = 0$, with $\underline{\partial}_x = \sum_j e_j \partial_{x_j}$ the classical Dirac operator on $\mathbf{R}^m$. The spaces of k-homogeneous monogenics are building blocks of the so-called Fischer decomposition on $\mathbf{R}^m$, which describes the orthogonal decomposition of spaces $\mathscr{P}_k(\mathbf{S})$ of homogeneous polynomials. The orthogonality is defined with respect to the Fischer inner product, i.e. a positive definite Hermitean inner product defined for $P, Q \in \mathscr{P}_k(\mathbf{S})$ as $\langle P, Q \rangle_F := [P^\dagger(\underline{\partial}_x) Q(\underline{x})]_0|_{x=0}$, where the dagger denotes the Hermitean conjugation on $\mathbf{C}^m$.

Theorem 1. *The space $\mathscr{P}_k(\mathbf{S})$ of k-homogeneous spinor-valued polynomials decomposes as*

$$\mathscr{P}_k(\mathbf{S}) = \bigoplus_{j=0}^{k} \underline{x}^j \mathscr{M}_{k-j}(\mathbf{S}).$$

For the proof and details we refer, e.g., to [9]. The Rarita–Schwinger operator can now be defined as an operator acting on functions $F(\underline{x}; \underline{u})$ depending on two vector variables, such that for fixed $\underline{x}_0 \in \mathbf{R}^m$ the function $F(\underline{x}_0, \underline{u})$ belongs to $\mathscr{M}_k(\mathbf{S})$. Such a function $F(\underline{x}, \underline{u})$ can thus be interpreted as an $\mathbf{S}_k$-valued function on $\mathbf{R}^m$. The natural invariant operator acting between such functions is defined as the Rarita–Schwinger operator (or Higher Spin Dirac operator, or RS-operator), given by

$$\mathscr{R}'_k := \left(\frac{\underline{u}\,\underline{\partial}_u}{m + 2k - 2} + 1\right)\underline{\partial}_x \colon \mathscr{C}^\infty\big(\mathbf{R}^m, \mathscr{M}_k\big) \mapsto \mathscr{C}^\infty\big(\mathbf{R}^m, \mathscr{M}_k\big).$$

The function theory for this operator is in full development, we refer to, e.g., [6, 7, 21]. In this paper, we will fully characterize the space of polynomial invariant operators on $\mathscr{M}_k$-valued functions: this is crucial information if one wishes to extend the Fischer decomposition for spinor-valued polynomials to the present setting. As a by-product, we will obtain a nice application of the so-called twistor operators (cfr. infra) which could be valid in more general settings as well.

The paper is structured as follows: in section 2, abstract representation theory will be used to determine the dimension of the space of invariant polynomial operators. Once this number is established, we will use the language of Clifford analysis in section 3 to obtain a nice explicit basis for this vector space. In the last section finally, it will be shown how to extend the obtained results to the case $\mathfrak{g} = D_n$.

2. Dimensions of the space of invariants

As outlined in the previous section, we are interested in summands of the type $\mathbf{V}_{(0)}$ inside $\mathscr{P}(\operatorname{End}\mathbf{S}_k)$. The strategy is as follows:

1. Prove that looking for representations of type $\mathbf{V}_{(0)}$ in $\mathscr{P}(\operatorname{End}\mathbf{S}_k)$ corresponds to looking for representations of type $\mathbf{V}_{(i)}$ in $\operatorname{End}\mathbf{S}_k$ (Theorem 4).
2. Since $\mathbf{S}_k$ is selfdual, $\operatorname{End}\mathbf{S}_k$ is isomorphic to $\mathbf{S}_k \otimes \mathbf{S}_k$.
3. Generalize the problem to finding all $\mathbf{V}_{(i)}$ inside $\mathbf{S}_p \otimes \mathbf{S}_q$, $p, q \geq 0$.
4. Since $\mathbf{S}_p$ is a summand of $\mathbf{V}_{(p)} \otimes \mathbf{S}$, then $\mathbf{S}_p \otimes \mathbf{S}_q$ is a subrepresentation of $(\mathbf{V}_{(p)} \otimes \mathbf{V}_{(q)}) \otimes (\mathbf{S} \otimes \mathbf{S})$.
5. Describe all summands of $\mathbf{V}_{(p)} \otimes \mathbf{V}_{(q)}$ (Theorem 2).
6. Use Brauer–Klimyk formula to identify all summands of type $\mathbf{V}_{(i)}$ in $(\mathbf{V}_{(p)} \otimes \mathbf{V}_{(q)}) \otimes (\mathbf{S} \otimes \mathbf{S})$ together with their multiplicities (Proposition 2).
7. Find all summands of type $\mathbf{V}_{(i)}$ in the subrepresentation $\mathbf{S}_p \otimes \mathbf{S}_q$. (Theorem 3).
8. Specialize to the case $p = q = k$.

Let us first state the Brauer–Klimyk formula [16] for further reference:

Proposition 1. *For every pair $\mu, \rho \in \Lambda_W$*

$$\mathbf{V}_\mu \otimes \mathbf{V}_\rho = \bigoplus_{\mu' \in \Pi_\mu} \sigma(\rho + \mu' + \delta) m_\mu(\mu') \mathbf{V}_{[\rho + \mu' + \delta] - \delta},$$

where $\sigma(\omega) = 0$ if there is some $w \in W$ that fixes ω and $\sigma(\omega) = (-1)^{|w(\omega)|}$ where $w(\omega)$ means the unique element of W such that $[\omega] := w(\omega).\omega \in \Lambda_W$ and $m_\mu(\mu')$ is the multiplicity of the weight μ' in $\mathbf{V}_\mu$.

In view of the fact that $\mathbf{V}_{(p)}$ can be interpreted as the irreducible module containing p-homogeneous harmonic polynomials on $\mathbf{R}^n$, the following result actually describes how the space of (p, q)-homogeneous polynomials in two vector variables $\underline{x}$ and $\underline{u} \in \mathbf{R}^n$, harmonic with respect to both the operators Δ_x and Δ_u, decomposes into irreducible modules:

Theorem 2. *Let $p \geq q \geq 0$. Then*

$$\mathbf{V}_{(p)} \otimes \mathbf{V}_{(q)} \simeq \bigoplus_{i=0}^{q} \bigoplus_{j=0}^{i} \mathbf{V}_{(p+q-2i+j,j)}.$$

Proof. This is formula (46) in [18], only with a different order of summation. □

Let $\mathbf{V}_1, \mathbf{V}_2$ be two arbitrary representations. If $\mathbf{V}_1 \simeq \mathbf{V}_2$ or if there exists a representation $\mathbf{W}$ containing no irreducible summand of the type $\mathbf{V}_{(i)}$ (with $i \geq 0$) such that $\mathbf{V}_1 \simeq \mathbf{V}_2 \oplus \mathbf{W}$, then we denote this fact by $\mathbf{V}_1 \simeq_{(\cdot)} \mathbf{V}_2$.

Lemma 1. *For $p > q > 0$*

$$\begin{aligned}
\mathbf{V}_{(p)} \otimes \mathbf{S} &\simeq \mathbf{S}_p \oplus \mathbf{S}_{p-1},\\
\mathbf{V}_{(p,q)} \otimes \mathbf{S} &\simeq \mathbf{S}_{p,q} \oplus \mathbf{S}_{p,q-1} \oplus \mathbf{S}_{p-1,q} \oplus \mathbf{S}_{p-1,q-1},\\
\mathbf{V}_{(p,p)} \otimes \mathbf{S} &\simeq \mathbf{S}_{p,p} \oplus \mathbf{S}_{p,p-1} \oplus \mathbf{S}_{p-1,p-1}.
\end{aligned}$$

Proof. We use the Brauer–Klimyk formula for $\rho = (p)$ and $\mu = \left(\left(\frac{1}{2}\right)_n\right)$. If a weight μ' of Π_μ ends with $-\frac{1}{2}$ or contains a pair $-\frac{1}{2}, \frac{1}{2}$ on the last $n-1$ coordinates, then $\rho + \mu' + \delta$ is fixed by an element of W. The only remaining weights in Π_μ are $\left(\pm\frac{1}{2}, \left(\frac{1}{2}\right)_{n-1}\right)$ which give precisely the two summands on the right side.

Similarly, if $\rho = (p, q)$, then the only weights of Π_μ that contribute are $\Big(\pm\frac{1}{2}, \pm\frac{1}{2}, \left(\frac{1}{2}\right)_{n-2}\Big)$. These give the four summands in the second expression. For $\mathbf{V}_{p,p} \otimes \mathbf{S}$, the weight $\mu' = \left(-\frac{1}{2}, \left(\frac{1}{2}\right)_{n-1}\right)$ gives a fixed $\rho + \mu' + \delta$ and hence only three summands remain. $\square$

Lemma 2. *For $p \geq 0$*

$$\mathbf{S}_p \otimes \mathbf{S} \simeq_{(\cdot)} \mathbf{V}_{(p)} \oplus \mathbf{V}_{(p+1)}$$

and for any $q > 0$ the product $\mathbf{S}_{p,q} \otimes \mathbf{S}$ does not contain any summands of the type $\mathbf{V}_{(i)}$, $i \geq 0$.

Proof. By Brauer–Klimyk formula for $\rho = (p)'$ and $\mu = (0)'$ a weight $\mu' \in \Pi_\mu$ can have a nonvanishing $\sigma(\rho + \mu' + \delta)$ if it does not contain the pair $-\frac{1}{2}, \frac{1}{2}$ on the last $n-1$ coordinates. Thus we are left with the weights $\left(\pm\frac{1}{2}, \left(\frac{1}{2}\right)_{n-1-i}, \left(-\frac{1}{2}\right)_i\right)$, for $i \in \{0, 1, \ldots, n-1\}$. This means that $\mathbf{S}_p \otimes \mathbf{S}$ is isomorphic to

$$\bigoplus_{i=0}^{n-1} \left(\mathbf{V}((p, (1)_i, (0)_{n-i-1})) \oplus \mathbf{V}((p+1, (1)_i, (0)_{n-i-1}))\right) \simeq_{(\cdot)} \mathbf{V}_{(p)} \oplus \mathbf{V}_{(p+1)}.$$

For $\rho = (p, q)'$, $p > q > 0$ and $\mu = (0)'$ the contributing weights are $\Big(\pm\frac{1}{2}, \pm\frac{1}{2}, \left(\frac{1}{2}\right)_{n-2-i}, \left(\frac{1}{2}\right)_i\Big)$, for $i \in \{0, 1, \ldots, n-2\}$ and $\mathbf{S}_{p,q} \otimes \mathbf{S}$ is isomorphic to

$$\begin{aligned}
&\bigoplus_{i=0}^{n-2} \left(\mathbf{V}((p, q, (1)_i, (0)_{n-i-2})) \oplus \mathbf{V}((p+1, q, (1)_i, (0)_{n-i-2}))\right) \oplus \\
&\quad \bigoplus_{i=0}^{n-2} \left(\mathbf{V}((p, q+1, (1)_i, (0)_{n-i-2})) \oplus \mathbf{V}((p+1, q+1, (1)_i, (0)_{n-i-2}))\right).
\end{aligned}$$

or without the third $\mathbf{V}((p, q+1, (1)_i, (0)_{n-i-2}))$ summands if $p = q$. Neither of the summands is of the type $\mathbf{V}_{(i)}$, $i \geq 0$. $\square$

Proposition 2. *For $p \geq q \geq 0$ let us define $\mathbf{W}_{p,q} = \mathbf{V}_{(p)} \otimes \mathbf{V}_{(q)} \otimes \mathbf{S}^{\otimes 2}$. Then*

$$\mathbf{W}_{p,p} \simeq_{(\cdot)} \mathbf{V}_{(0)}^{\oplus 2} \oplus \left(\bigoplus_{i=1}^{2p-1} \mathbf{V}_{(i)} \right)^{\oplus 4} \oplus \mathbf{V}_{(2p)}^{\oplus 3} \oplus \mathbf{V}_{(2p+1)},$$
$$\mathbf{W}_{p,0} \simeq_{(\cdot)} \mathbf{V}_{(p-1)} \oplus \mathbf{V}_{(p)}^{\oplus 2} \oplus \mathbf{V}_{(p+1)},$$
$$\mathbf{W}_{0,0} \simeq_{(\cdot)} \mathbf{V}_{(0)} \oplus \mathbf{V}_{(1)}$$

and

$$\mathbf{W}_{p,q} \simeq_{(\cdot)} \mathbf{V}_{(p-q-1)} \oplus \mathbf{V}_{(p-q)}^{\oplus 3} \oplus \left(\bigoplus_{i=1}^{2q-1} \mathbf{V}_{(p-q+i)} \right)^{\oplus 4} \oplus \mathbf{V}_{(p+q)}^{\oplus 3} \oplus \mathbf{V}_{(p+q+1)}$$

for $p > q > 0$.

Proof. By Theorem 2 all summands of $\mathbf{V}_{(p)} \otimes \mathbf{V}_{(q)}$ are of the type $\mathbf{V}_{(i,j)}$ and by Lemma 1 all summands of $\mathbf{V}_{(i,j)} \otimes \mathbf{S}$ are of the type $\mathbf{S}_{r,s}$, $s \geq j-1$. By Lemma 2 only for $s = 0$ there are any $\mathbf{V}_{(t)}$'s in $\mathbf{S}_{r,s} \otimes \mathbf{S}$. Hence we can ignore all summands of $\mathbf{V}_{(p)} \otimes \mathbf{V}_{(q)}$ of the type $\mathbf{V}_{(i,j)}$ with $j > 1$ since such summands do not contribute with any copies of $\mathbf{V}_{(t)}$ to the decomposition of $\mathbf{W}_{p,q}$.

In the generic case $p > q > 0$ we have

$$\begin{aligned}
\bigoplus_{i=0}^{2q} \mathbf{V}_{(p-q+2i)} \otimes \mathbf{S}^{\otimes 2} &\simeq \bigoplus_{i=0}^{2q} (\mathbf{S}_{p-q+2i} \oplus \mathbf{S}_{p-q+2i-1}) \otimes \mathbf{S} \\
&\simeq_{(\cdot)} \bigoplus_{i=0}^{2q} \left(\mathbf{V}_{(p-q+2i+1)} \oplus \mathbf{V}_{(p-q+2i)}^{\oplus 2} \oplus \mathbf{V}_{(p-q+2i-1)} \right) \\
&= \mathbf{V}_{(p-q-1)} \oplus \left(\bigoplus_{i=0}^{2q} \mathbf{V}_{(p-q+i)} \right)^{\oplus 2} \oplus \mathbf{V}_{(p+q+1)}.
\end{aligned}$$

In the case $p = q > 0$ the term corresponding to $\mathbf{S}_{p-q+2.0-1}$ in the second expression does not appear, which means that the term corresponding to $\mathbf{V}_{(p-q-1)}$ and one copy of $\mathbf{V}_{(p-q)}$ in the final expression are missing. For $p \geq q > 0$

$$\begin{aligned}
\bigoplus_{i=0}^{2q-2} \mathbf{V}_{(p-q+2i+1,1)} \otimes \mathbf{S}^{\otimes 2} &\simeq \bigoplus_{i=0}^{2q-2} (\mathbf{S}_{p-q+2i+1,1} \oplus \mathbf{S}_{p-q+2i+1} \oplus \mathbf{S}_{p-q+2i,1} \oplus \mathbf{S}_{p-q+2i})) \otimes \mathbf{S} \\
&\simeq_{(\cdot)} \bigoplus_{i=0}^{2q-2} \left(\mathbf{V}_{(p-q+2i+2)} \oplus \mathbf{V}_{(p-q+2i+1)}^{\oplus 2} \oplus \mathbf{V}_{(p-q+2i)} \right) \\
&= \mathbf{V}_{(p-q)} \oplus \left(\bigoplus_{i=0}^{(2q-29} \mathbf{V}_{(p-q+i+1)} \right)^{\oplus 2} \oplus \mathbf{V}_{(p+q)}.
\end{aligned}$$

Adding this together gives the statement.

The case $\mathbf{W}_{p,0}$, $p > 0$ is covered solely by Lemmas 1 and 2:

$$\mathbf{W}_{p,0} \equiv \mathbf{V}_{(p)} \otimes \mathbf{S}^2 \simeq (\mathbf{S}_p \oplus \mathbf{S}_{p-1}) \otimes \mathbf{S} \simeq_{(\cdot)} \mathbf{V}_{(p+1)} \oplus \mathbf{V}_{(p)}^{\oplus 2} \oplus \mathbf{V}_{p-1}$$

and the case of $\mathbf{W}_{0,0}$ by the $p = 0$ case of Lemma 2. $\square$

Theorem 3. *Let $p \geq q \geq 0$. Then*

$$\mathbf{S}_p \otimes \mathbf{S}_q \simeq_{(\cdot)} \bigoplus_{i=0}^{2q+1} \mathbf{V}_{(p-q+i)}.$$

Proof. We proceed by induction along the pairs (p,q), lexicographically ordered. The case $p = q = 0$ is precisely the third row of Proposition 2. The case $p \geq q = 0$ is covered by Lemma 2. Otherwise

$$\begin{aligned}\mathbf{W}_{p,q} &\simeq (\mathbf{V}_{(p)} \otimes \mathbf{S}) \otimes (\mathbf{V}_{(q)} \otimes \mathbf{S}) \simeq (\mathbf{S}_p \oplus \mathbf{S}_{p-1}) \otimes (\mathbf{S}_q \oplus \mathbf{S}_{q-1}) \\ &\simeq (\mathbf{S}_p \otimes \mathbf{S}_q) \oplus (\mathbf{S}_p \otimes \mathbf{S}_{q-1}) \oplus (\mathbf{S}_{p-1} \otimes \mathbf{S}_q) \oplus (\mathbf{S}_{p-1} \otimes \mathbf{S}_{q-1}).\end{aligned}$$

Now we have to use Proposition 2 and the assumption again, but we have to distinguish between the cases $p = q$ and $p > q$. Let us denote by $\mathbf{V}$ the last three terms of the last direct sum, then for $p = q$

$$\mathbf{V} \simeq_{(\cdot)} \mathbf{V}_{(0)} \oplus \left(\bigoplus_{i=1}^{2p-1} \mathbf{V}_{(i)}\right)^{\oplus 3} \oplus \mathbf{V}_{(2p)}^{\oplus 2}$$

and

$$\mathbf{W}_{p,p} \simeq_{(\cdot)} \mathbf{V}_{(0)}^{\oplus 2} \oplus \left(\bigoplus_{i=1}^{2p-1} \mathbf{V}_{(i)}\right)^{\oplus 4} \oplus \mathbf{V}_{(2p)}^{\oplus 3} \oplus \mathbf{V}_{(2p+1)}.$$

This means that

$$\mathbf{S}_p \otimes \mathbf{S}_p \simeq_{(\cdot)} \bigoplus_{i=0}^{2p+1} \mathbf{V}_{(i)},$$

which is the induction step.

Similarly in the case $p > q$ we have

$$\mathbf{V} \simeq_{(\cdot)} \mathbf{V}_{(p-q-1)} \oplus \mathbf{V}_{(p-q)}^{\oplus 2} \oplus \left(\bigoplus_{i=p-q}^{p+q-2} \mathbf{V}_{(i)}\right)^{\oplus 3} \oplus \mathbf{V}_{(p+q-1)}^{\oplus 2} \oplus \mathbf{V}_{(p+q)}$$

and

$$\mathbf{W}_{p,q} \simeq_{(\cdot)} \mathbf{V}_{(p-q-1)} \oplus \mathbf{V}_{(p-q)}^{\oplus 3} \oplus \left(\bigoplus_{i=1}^{2q-1} \mathbf{V}_{(p-q+i)}\right)^{\oplus 4} \oplus \mathbf{V}_{(p+q)}^{\oplus 3} \oplus \mathbf{V}_{(p+q+1)},$$

which gives us

$$\mathbf{S}_p \otimes \mathbf{S}_q \simeq_{(\cdot)} \bigoplus_{i=p-q}^{p+q+1} \mathbf{V}_{(i)},$$

i.e., again the induction step. □

Theorem 4. *Let $\mathfrak{g}$ be of type B_n, $\mathbf{V}_\mu$ an irreducible representation of $\mathfrak{g}$, $i \geq 0$. Then $\mathbf{V}_{(k)} \otimes \mathbf{V}_\mu$ contains a trivial irreducible summand $\mathbf{V}_{(0)}$ if and only if $\mu = (k)$.*

Proof. By Brauer–Klimyk formula a weight μ' of $\mathbf{V}_\mu$ can contribute to a trivial summand only if

$$[(k) + \mu' + \delta] = \delta. \tag{1}$$

The coordinates of all weights of a given finite-dimensional representation are either all integral or all half-integral. If μ is half-integral, then $(k) + \mu' + \delta$ and δ cannot be at the same time integral or half-integral. Action of the Weyl group implicit in the operation $[\cdot]$ preserves integrality, so the resulting dominant weight cannot equal δ.

Hence only for μ integral there can be a $\mathbf{V}_{(0)}$ in $\mathbf{V}_{(k)} \otimes \mathbf{V}_{\mu}$. Let us denote by $H(\mu')$ the non-negative number $\sum_{i=1}^{n} |\mu'_i|$. If μ' is a weight of $\mathbf{V}_{\mu}$, then $H(\mu') \leq H(\mu)$. If μ' satisfies (1), then

$$H(\delta) = H((k) + \mu' + \delta) \geq H((k)) + H(\delta) - H(\mu'),$$

i.e., $H(\mu) \geq H(\mu') \geq k$. If $H(\mu) = k$ and $\mu'_1 > -k$, then the first coordinate of $(k) + \mu' + \delta$ is greater than any coordinate of δ and so (1) cannot hold. Thus $H(\mu) = k$ means that any contributing μ' is $(-k, (0)_{n-1})$ and such a weight exists only in a representation with highest weight $\mu = (k)$, provided $H(\mu) = k$. Clearly this gives one trivial summand in $\mathbf{V}_{(k)} \otimes \mathbf{V}_{(k)}$.

Hence any other μ satisfying the statement must be integral with $m \equiv H(\mu) > k$. Such a representation can be realised inside $(\mathbf{V}_{(1)})^{\otimes m}$ (since $\mathbf{V}_{(1)}$ is the defining or vector representation) as a subspace of tracefree tensors satisfying certain symmetry [11]. Let us denote by π_m the projection from $(\mathbf{V}_{(1)})^{\otimes m}$ to $\mathbf{V}_{\mu}$ and by π_k the projection $(\mathbf{V}_{(1)})^{\otimes k}$ to $\mathbf{V}_{(k)}$. If there is a projection $\pi\colon \mathbf{V}_{(k)} \otimes \mathbf{V}_{\mu} \to \mathbf{V}_{(0)}$, then $\pi \circ (\pi_k \otimes \pi_m)$ is a projection from $(\mathbf{V}_{(1)})^{\otimes k} \otimes (\mathbf{V}_{(1)})^{\otimes m}$ to a trivial summand. By classical invariant theory [12] the space of orthogonal invariants in $(\mathbf{V}_{(1)})^{\otimes p}$ is zero for p odd and spanned by complete contractions for p even. A complete contraction for $p = k + m$, $m > k$ must involve a contraction in two indices of $(\mathbf{V}_{(1)})^{\otimes m}$, which is projected to zero by π_m. Thus for $H(\mu) > k$ there are no trivial summands in $\mathbf{V}_{(k)} \otimes \mathbf{V}_{\mu}$. □

Theorem 5. *The space of invariant p-homogeneous polynomials with values in* $\operatorname{End} \mathbf{S}_k$ *is* $\lceil \frac{p+1}{2} \rceil$*-dimensional for* $p \leq 2k$ *and* $k+1$*-dimensional for* $p \geq 2k$.

Proof. The space of polynomials $\mathscr{P}(\operatorname{End} \mathbf{S}_k)$ is graded by homogeneity

$$\bigoplus_{p=0}^{\infty} \mathscr{P}^p(\operatorname{End} \mathbf{S}_k) = \bigoplus_{p=0}^{\infty} \Big(\bigodot^p \mathbf{V}_{(1)} \otimes \operatorname{End} \mathbf{S}_k \Big) = \bigoplus_{p=0}^{\infty} \left(\bigoplus_{i=0}^{\lfloor \frac{p}{2} \rfloor} \mathbf{V}_{(p-2i)} \otimes \operatorname{End} \mathbf{S}_k \right).$$

Here by $\bigodot^p \mathbf{V}_{(1)}$ we mean the p-th symmetric tensor power of $\mathbf{V}_{(1)}$. The last equality comes from the fact that $\bigodot^j \mathbf{V}_{(1)} = \mathbf{V}_{(j)} \oplus \bigodot^{(j-2)} \mathbf{V}_{(1)}$, i.e., a symmetric tensor can be split into its tracefree (Cartan) part and its contraction in any two indices. This corresponds to the classical Fischer decomposition of polynomials into harmonic polynomials: a homogeneous polynomial $P(x)$ of degree p is written as a sum of a harmonic polynomial of degree p and $x^2 R(x)$, where $R(x)$ is a polynomial of degree $p - 2$.

The invariants correspond to trivial summands of $\mathscr{P}(\operatorname{End} \mathbf{S}_k)$. Theorem 4 implies that $\mathbf{V}_{(p-2i)} \otimes \operatorname{End} \mathbf{S}_k$ contains a unique copy of $\mathbf{V}_{(0)}$ iff $0 \leq p - 2i \leq 2k + 1$. This inequality holds for $\lceil \frac{p+1}{2} \rceil$ values of i if $p \leq 2k$ and $k + 1$ values of i for $p \geq 2k$. □

3. Bases for the spaces of invariants

As we now know how many invariant polynomial operators there are in each homogeneity, we will proceed by describing them explicitly in terms of suitable bases. First a set of k-homogeneous polynomial operators with the right cardinality will be constructed, and then the linear independence of this set will be proved. This will prove that it is a basis.

Let us first consider the case of the classical Rarita–Schwinger operator, i.e., $k = 1$, as described in our previous paper [10]. According to the previous section, the space of invariants ought to be two-dimensional in each order of homogeneity $k \geq 2$, and one-dimensional for $k = 0$ and $k = 1$. In order to find these invariants, the most obvious way to proceed is to start calculating powers of the operator $\mathscr{R}_1'$. We have the following:

$$(\mathscr{R}_1')^2 = -\Delta_x + \gamma_m T' := -\Delta_x + \frac{4}{m^2}\big(m\langle \underline{u}, \underline{\partial}_x\rangle + \underline{u}\,\underline{\partial}_x\big)\langle \underline{\partial}_u, \underline{\partial}_x\rangle,$$

with $\gamma_m := 4m^{-2}$. From now on, we will use the unprimed version of an operator to denote its symbol, i.e. with $\underline{\partial}_x$ replaced by $\underline{x}$. It is then easily verified that

$$T^2 = (m-1)|\underline{x}|^2 T.$$

Since T and $|\underline{x}|^2$ are linearly independent, they form a basis for the space of 2-homogeneous invariants. The space of invariants of homogeneity $2p + 2$, $p \in \mathbf{N}$ is spanned by $|\underline{x}|^{2p+2}$ and $|\underline{x}|^{2p}T$. In homogeneity 1 we have just $\mathscr{R}_1$ and in other odd homogeneities $|\underline{x}|^{2p}\mathscr{R}_1$ and $|\underline{x}|^{2p-2}T\mathscr{R}_1$, $p \in \mathbf{N}$. Hence as an algebra, the space of invariant polynomial operators is generated just by $\mathscr{R}_1$ and $|\underline{x}|^2$.

We recall from [9] the definition for the twistor operator and its dual. Both invariant operators can be defined as diagonal arrows in the following scheme, which represents the action of the Dirac operator $\underline{\partial}_x$ on functions $F(\underline{x}, \underline{u}) \in \mathscr{C}^\infty(\mathbf{R}^m, \mathscr{H}_k \otimes \mathbf{S})$ taking values in the space of k-homogeneous $\mathbf{S}$-valued harmonic polynomials in the variable $\underline{u}$:

$$\begin{array}{ccc}
\mathscr{C}^\infty(\mathbf{R}^m, \mathscr{H}_k \otimes \mathbf{S}) & \xrightarrow{\underline{\partial}_x} & \mathscr{C}^\infty(\mathbf{R}^m, \mathscr{H}_k \otimes \mathbf{S}) \\
= & & = \\
\mathscr{C}^\infty(\mathbf{R}^m, \mathscr{M}_k) & & \mathscr{C}^\infty(\mathbf{R}^m, \mathscr{M}_k) \\
\oplus & & \oplus \\
\mathscr{C}^\infty(\mathbf{R}^m, \underline{u}\mathscr{M}_{k-1}) & & \mathscr{C}^\infty(\mathbf{R}^m, \underline{u}\mathscr{M}_{k-1}).
\end{array}$$

Indeed, we respectively get a twistor operator given by

$$\begin{aligned}
\mathscr{T}_k' \colon \mathscr{C}^\infty\big(\mathbf{R}^m, \underline{u}\mathscr{M}_{k-1}\big) &\mapsto \mathscr{C}^\infty\big(\mathbf{R}^m, \mathscr{M}_k\big), \\
\underline{u}f(\underline{x};\underline{u}) &\mapsto \left(\frac{\underline{u}\,\underline{\partial}_u}{m+2k-2} + 1\right)\underline{\partial}_x\underline{u}f,
\end{aligned}$$

and a dual twistor operator given by

$$\begin{aligned}
(\mathscr{T}_k^*)' \colon \mathscr{C}^\infty\big(\mathbf{R}^m, \mathscr{M}_k\big) &\mapsto \mathscr{C}^\infty\big(\mathbf{R}^m, \underline{u}\mathscr{M}_{k-1}\big), \\
f(\underline{x};\underline{u}) &\mapsto -\frac{1}{m+2k-2}\underline{u}\,\underline{\partial}_u\underline{\partial}_x f.
\end{aligned}$$

We also recall the relation

$$(\mathscr{R}_k')^2 = -(\Delta_x + (\mathscr{T}_k)(\mathscr{T}_k^*)'), \tag{2}$$

giving an alternative description for the polynomial invariant T when $k = 1$:

$$\mathscr{T}_1\mathscr{T}_1^* = \gamma_m T,$$

where γ_m is a number. This observation will turn out to be crucial for the analysis in case $k > 1$, because it offers a systematic way to construct polynomial invariants in more general cases as compositions of twistor-type operators. Indeed, we will show that there is a formula relating products of powers of the operator $\mathscr{R}_k'$ and Δ_x to compositions of twistor-type operators.

Lemma 3. *For all $k \in \mathbf{N}_0$, we have the following identity between operators on the space $\underline{u}\mathscr{M}_{k-1}$:*

$$(\mathscr{T}_k^*)\mathscr{T}_k\Big|_{\underline{u}\mathscr{M}_{k-1}} = \left(c_k^{(1)}|\underline{x}|^2 + c_k^{(2)}\underline{u}\mathscr{T}_{k-1}(\mathscr{T}_{k-1}^*)\underline{\partial}_u\right)\Big|_{\underline{u}\mathscr{M}_{k-1}},$$

where the constants are explicitly given by

$$c_k^{(1)} = -4\,\frac{m+2k-3}{(m+2k-2)^2} \quad \text{and} \quad c_k^{(2)} = -\frac{m+2k-4}{(m+2k-2)^2}.$$

Proof. First of all, note that the additional operators $\underline{u}$ and $\underline{\partial}_u$ ensure that the second operator between brackets indeed acts between the spaces $\underline{u}\mathscr{M}_{k-1}$. This is due to the explicit definition for the (dual) twistor operator, involving an isomorphic copy of the space $\mathscr{M}_{k-1}$, inspired by the Fischer decomposition for $\mathscr{H}_k \otimes \mathbf{S}$. By definition, the action of the left-hand side on $\underline{u}\mathscr{M}_{k-1}$ is given by:

$$\begin{aligned}
(\mathscr{T}_k^*)\mathscr{T}_k &= 2\,\frac{\underline{u}\langle\underline{\partial}_u,\underline{x}\rangle\big((m+2k-2)+\underline{u}\,\underline{\partial}_u\big)\underline{x}}{(m+2k-2)^2}\\
&= 2\,\frac{\underline{u}\big((m+2k-2)+\underline{u}\,\underline{\partial}_u\big)\underline{x}\langle\underline{\partial}_u,\underline{x}\rangle + \underline{u}\,\underline{x}\,\underline{\partial}_u\underline{x}}{(m+2k-2)^2}\\
&= 2\,\frac{\underline{u}\big((m+2k-4)+\underline{u}\,\underline{\partial}_u\big)\underline{x}\langle\underline{\partial}_u,\underline{x}\rangle + \underline{u}\,\underline{\partial}_u|\underline{x}|^2}{(m+2k-2)^2}.
\end{aligned}$$

Next, invoking the fact that $\underline{u}\,\underline{\partial}_u + \underline{\partial}_u\underline{u} = -2\mathbf{E}_u - m$, one clearly has that

$$\langle\underline{\partial}_u,\underline{x}\rangle\underline{u}\mathscr{M}_{k-1} = -\frac{\underline{u}\langle\underline{\partial}_u,\underline{x}\rangle\underline{\partial}_u + \underline{x}\,\underline{\partial}_u}{m+2k-2}\underline{u}\mathscr{M}_{k-1}.$$

Plugging this into the previous expression leads to the stated formula. □

We now define a set of operators $\mathfrak{T}_k^{(j)}$, which are certain compositions of twistor-type operators satisfying a nice recurrence relation. The way how they are defined can easily be illustrated using the following diagram, in which we have only written the source spaces in which the functions take their values and where each of the arrows pointing downwards denotes a twistor operators or a dual twistor operator:

$$\begin{array}{ccccc}
\mathscr{M}_k & & & & \\
\downarrow & & & & \\
\underline{u}\mathscr{M}_{k-1} & \xrightarrow{\underline{\partial}_u} & \mathscr{M}_{k-1} & & \\
& & \downarrow & & \\
& & \underline{u}\mathscr{M}_{k-2} & \xrightarrow{\underline{\partial}_u} & \mathscr{M}_{k-2} \\
& & & & \downarrow \\
& & & & \underline{u}\mathscr{M}_{k-3} \\
& & & & \downarrow \\
& & \underline{u}\mathscr{M}_{k-2} & \xleftarrow{\underline{u}} & \mathscr{M}_{k-2} \\
& & \downarrow & & \\
\underline{u}\mathscr{M}_{k-1} & \xleftarrow{\underline{u}} & \mathscr{M}_{k-1} & & \\
\downarrow & & & & \\
\mathscr{M}_k & & & &
\end{array}$$

The formal definition (on the level of symbols) is as follows:

Definition 1. For each $k \in \mathbf{N}$ and $0 \le j \le k$, we define the operator

$$\mathfrak{T}_k^{(j)} \colon \mathscr{C}^\infty(\mathbf{R}^m, \mathscr{M}_k) \mapsto \mathscr{C}^\infty(\mathbf{R}^m, \mathscr{M}_k)$$

by means of

$$\mathfrak{T}_k^{(j)} := \mathscr{T}_k \underline{u} \mathscr{T}_{k-1} \underline{u} \cdots \underline{u} \mathscr{T}_{k-j+1} (\mathscr{T}_{k-j+1}^*) \underline{\partial}_u \cdots \underline{\partial}_u (\mathscr{T}_{k-1}^*) \underline{\partial}_u (\mathscr{T}_k^*)$$

for $1 \le j \le k$ and $\mathfrak{T}_k^{(0)} := 1$.

Note that it makes no sense to define $\mathfrak{T}_k^{(j)}$ for $j > k$ since $\mathfrak{T}_k^{(j)}$ is a mapping between $\mathscr{M}_k$-valued functions that factors through the space of $\mathscr{M}_{k-j}$-valued functions. We will now show that the set $\{\mathfrak{T}_k^{(j)} |\underline{x}|^{2k-2j} : 0 \le j \le k\}$ lies in the span of $\{(\mathscr{R}_k)^{2j} |\underline{x}|^{2k-2j} : 0 \le j \le k\}$, but for that purpose we need the following lemma:

Lemma 4. *For each $k \in \mathbf{N}_0$ and $0 \le j < k$, we have the following:*

$$\mathscr{T}_k \mathscr{T}_k^* \mathfrak{T}_k^{(j)} = C_1(k;j) \mathfrak{T}_k^{(j+1)} + C_2(k;j) |\underline{x}|^2 \mathfrak{T}_k^{(j)}$$

where the constants are explicitly given by

$$C_1(k;j) = (-1)^{j-1} c_k^{(2)} \prod_{p=1}^{j-1} c_{k-p}^{(2)} (m + 2k - 2p),$$

$$C_2(k;j) = c_k^{(1)} + \sum_{p=1}^{j-1} (-1)^p c_{k-p}^{(1)} \prod_{q=1}^{p} c_{k+1-q}^{(2)} (m + 2k - 2q)$$

for $j \ge 1$ and as $C_1(k,0) = 1$, $C_2(k,0) = 0$.

Proof. The result can easily be proved using an induction procedure, hereby applying Lemma 3. By definition, we have that

$$\begin{aligned}
\mathscr{T}_k \mathscr{T}_k^* \mathfrak{T}_k^{(j)} &= \mathscr{T}_k \left(\mathscr{T}_k^* \mathscr{T}_k \right) \underline{u} \mathscr{T}_{k-1} \underline{u} \cdots \underline{\partial}_u \mathscr{T}_{k-1}^* \underline{\partial}_u \mathscr{T}_k^* \\
&= c_k^{(2)} |\underline{x}|^2 \mathfrak{T}_k^{(j)} + c_k^{(1)} \mathscr{T}_k \underline{u} \mathscr{T}_{k-1} \mathscr{T}_{k-1}^* \underline{\partial}_u \underline{u} \mathscr{T}_{k-1} \underline{u} \cdots \underline{\partial}_u \mathscr{T}_{k-1}^* \underline{\partial}_u \mathscr{T}_k^*.
\end{aligned}$$

In view of the fact that the image of $\mathscr{C}^\infty(\mathbf{R}^m, \mathscr{M}_k)$ under the polynomial multiplication operator $\mathscr{T}_{k-1} \underline{u} \cdots \underline{\partial}_u \mathscr{T}_{k-1}^* \underline{\partial}_u \mathscr{T}_k^*$ belongs to $\mathscr{C}^\infty(\mathbf{R}^m, \mathscr{M}_{k-1})$, the operator $\underline{\partial}_u \underline{u}$ reduces to the constant $-(m + 2k - 2)$. This puts the second term above in the appropriate form to apply an induction procedure, involving j steps, which leads to the desired result. $\square$

Note that for $k > 1$, $0 \le j < k$, the coefficient $C_1(k;j)$ is nonzero. Thus the lemma says that there are constants $C_1'(k,j)$, $C_2'(k,j)$ such that

(3) $$\mathfrak{T}_k^{(j+1)} = (C_1'(k,j) \mathfrak{T}_k^{(1)} + C_2'(k,j) |\underline{x}|^2) \mathfrak{T}_k^{(j)}.$$

Lemma 5. *The powers of the symbol $\mathscr{T}_k \mathscr{T}_k^*$ satisfy*

$$(\mathscr{T}_k \mathscr{T}_k^*)^j = \sum_{p=1}^{j} \kappa_p(k;j) |\underline{x}|^{2(p-1)} \mathfrak{T}_k^{(j+1-p)},$$

where the constants $\kappa_p(k;j)$ satisfy the recursive relation

$$\kappa_p(k;j) = C_1(k; j+1-p) \kappa_p(k;j) + C_2(k; j+2-p) \kappa_{p-1}(k;j)$$

and $\kappa_0(k;j)$.

Proof. This lemma is easily proved by induction on the power j. $\square$

From the lemma above and relation (2), it is immediately seen that for any $k \geq 0$ the operator $(\mathscr{R}_k)^{2p}$, with $p > 0$, can be expressed as a linear combination of operators $\mathfrak{T}_k^{(j)}|\underline{x}|^{2(p-j)}$, where $0 \leq j \leq k$. On the other hand, inductively substituting the resulting operator into equation (3), one will obtain an expression for $\mathfrak{T}_k^{(j)}$, with $0 < j < k$, in terms of $(\mathscr{R}_k)^{2p}|\underline{x}|^{2(j-p)}$, where $0 \leq p \leq j$. Thus for a given $p \geq 0$ the sets

$$\{\mathfrak{T}_k^{(j)}|\underline{x}|^{2p-2j} : 0 \leq j \leq \min(p,k)\}$$

and

$$\{(\mathscr{R}_k)^{2j}|\underline{x}|^{2p-2j} : 0 \leq j \leq \min(p,k)\}$$

span the same subspace of the space of $2p$-homogeneous invariants. Moreover, the number of generators equals the dimension of the space of invariants calculated earlier. Therefore we only need to prove that one of these sets is linearly independent. This is the subject of the following lemma:

Lemma 6. *The set* $\{\mathfrak{T}_k^{(j)}|\underline{x}|^{2k-2j} : 0 \leq j \leq k\}$ *containing invariant polynomial operators is linearly independent.*

Proof. The twistor operators $\mathscr{T}_j'$ are Stein–Weiss gradients corresponding to projection to the component in the decomposition of $\mathbf{V}_{(1)} \otimes \mathbf{S}_j$ with highest weight, the so-called top Stein–Weiss gradients. According to [17], for a top Stein–Weiss gradient W', the operator $(W^*)'W'$ is elliptic, i.e., its symbol W^*W is an isomorphism. Thus the operator $\mathscr{T}_j$ (resp. $\mathscr{T}_j^*$) is injective (resp. surjective). As the maps $\underline{u}\colon \mathscr{M}_i \to \underline{u}\mathscr{M}_i$, $\underline{\partial}_u\colon \underline{u}\mathscr{M}_i \to \mathscr{M}_i$ are isomorphisms for any i, the composition of operators $G_j := \mathscr{T}_k \underline{u} \mathscr{T}_{k-1} \underline{u} \cdots \underline{u} \mathscr{T}_{k-j+1}$ is injective and $G_j^* := (\mathscr{T}_{k-j+1}^*)\underline{\partial}_u \cdots \underline{\partial}_u (\mathscr{T}_{k-1}^*)\underline{\partial}_u(\mathscr{T}_k^*)$ is surjective for any given j. It is a map from $\mathscr{M}_k$-valued functions to the $\mathscr{M}_{k-j}$-valued functions. Since $\dim \mathscr{M}_i < \dim \mathscr{M}_j$ for $i < j$ and G_j^* factors through G_i^*, the kernel of G_i^* is a proper subspace of the kernel of G_j^*. As $\mathfrak{T}_k^{(p)} = G_p G_p^*$, we have that $\operatorname{Ker} \mathfrak{T}_k^{(i)}$ is a proper subspace of $\operatorname{Ker} \mathfrak{T}_k^{(j)}$. If

$$\sum_{j=0}^{k} a_j |\underline{x}|^{2(k-j)} \mathfrak{T}_k^{(j)} = 0,$$

then we can find for a nonzero $\underline{x}$ a function ϕ such that $\left(\mathfrak{T}_k^{(0)}\phi\right)(\underline{x}) = 0$ but $\left(\mathfrak{T}_k^{(1)}\phi\right)(\underline{x}) \neq 0$, hence

$$\sum_{j=0}^{k} a_j |\underline{x}|^{2(k-j)} \left(\mathfrak{T}_k^{(j)}\phi\right)(\underline{x}) = a_0 |\underline{x}|^{2k} \phi(\underline{x}) = 0,$$

i.e., $a_0 = 0$. In a similar way we can show that $a_j = 0$ for all j and thus the set $\{\mathfrak{T}_k^{(j)}|\underline{x}|^{2k-2j} : 0 \leq j \leq k\}$ is linearly independent. $\square$

Linear independence of the set immediately implies linear independence of $\{\mathfrak{T}_k^{(j)}|\underline{x}|^{2p-2j}(\mathscr{R}_k)^i : 0 \leq j \leq \min(p,k), i \in \{0,1\}\}$. This gives us our main result:

Theorem 6. *For all $p > 0$, the set*

$$\{(\mathscr{R}_k)^i|\underline{x}|^{2j}, 0 \le i \le \min(2p+1, 2k+1), 0 \le j, i+2j = p\}$$

is the basis of p-homogeneous invariant $\operatorname{End}\mathbf{S}_k$*-valued polynomials.*

In terms of Fischer duals, this says that

$$\{(\mathscr{R}'_k)^i\Delta^j, 0 \le i \le \min(2p+1, 2k+1), 0 \le j, i+2j = p\}$$

is the basis of the space of invariant constant coefficient differential operators of order p on $\mathbf{S}_k$.

4. Final remarks

In the even-dimensional case $\mathfrak{g} = D_n$ the spinor representation is reducible $\mathbf{S} = \mathbf{S}^+ \oplus \mathbf{S}^-$ and in the same fashion one may define $\mathbf{S}_k = \mathbf{S}_k^+ \oplus \mathbf{S}_k^-$, where $\mathbf{S}_k^\pm$ is the irreducible representation with highest weight $\left(k+\frac{1}{2}, \left(\frac{1}{2}\right)_{n-2}, \pm\frac{1}{2}\right)$. Such a module $\mathbf{S}_k$ is again identified with $\mathscr{M}_k(\mathbf{S})$ and the definitions of $\mathscr{R}'_k$, $\mathscr{T}'_k$ and $(\mathscr{T}^*_k)'$ remain unchanged. However, we can split the RS-operator into operators acting between irreducible components of $\mathbf{S}_k$:

$$\mathscr{R}_k = \mathscr{R}_k^+ + \mathscr{R}_k^- \quad \text{with} \quad \mathscr{R}_k^\pm \colon \mathbf{S}_k^\pm \to \mathbf{S}_k^\mp,$$

and similarly for the twistor and dual twistor operators. With this notation, the statements of Theorem 2 and Lemma 1 remain unchanged, but in Lemma 2 we get two copies of $\mathbf{V}_{(p)}$ and $\mathbf{V}_{(p+1)}$. From that we get each copy of $\mathbf{V}_{(i)}$, $i \ge 0$ twice in Proposition 2 and Lemma 3. The proof of Theorem 4 remains unchanged for $\mathfrak{g} = D_n$ and thus Theorem 5 holds in the form

Theorem 7. *The space of invariant p-homogeneous polynomials with values in* $\operatorname{End}\mathbf{S}_k$ *is* $2\lceil\frac{p+1}{2}\rceil$*-dimensional for $p \le 2k$ and $(2k+2)$-dimensional for $p \ge 2k$.*

The basis for the space of invariants must therefore be twice as big as in the odd-dimensional case and similar arguments to those of the previous section prove our final result:

Theorem 8. *For all $p > 0$, the set*

$$\{(\mathscr{R}_k^+\mathscr{R}_k^-)^i|\underline{x}|^{2j}, (\mathscr{R}_k^-\mathscr{R}_k^+)^i|\underline{x}|^{2j}, 0 \le i \le \min(2p, k), 0 \le j, i+j = p\}$$

is the basis of $2p$-homogeneous invariant $\operatorname{End}\mathbf{S}_k$*-valued polynomials and the set*

$$\{(\mathscr{R}_k^+\mathscr{R}_k^-)^i\mathscr{R}_k^+|\underline{x}|^{2j}, (\mathscr{R}_k^-\mathscr{R}_k^+)^i\mathscr{R}_k^-|\underline{x}|^{2j}, 0 \le i \le \min(2p, k), 0 \le j, i+j = p\}$$

is the basis of $(2p+1)$-homogeneous invariant $\operatorname{End}\mathbf{S}_k$*-valued polynomials.*

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