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CAMPANATO THEOREM ON METRIC MEASURE SPACES

Przemysław Górka

Universidad de Talca, Instituto de Matemática y Física
Casilla 747, Talca, Chile

and

Warsaw University of Technology, Department of Mathematics and Information Sciences
Pl. Politechniki 1, 00-661 Warsaw, Poland; pgorka@mini.pw.edu.pl

Abstract. We prove the Campanato theorem on a metric space. The theorem characterizes Hölder continuous functions by the growth of their local integrals. As a byproduct we obtain Morrey theorem on Hajłasz–Sobolev spaces.

1. Introduction

In [1] Camapanato characterized Hölder continuous function $f\colon \mathbf{R}^n \to \mathbf{R}$ in terms of the growth of it local integrals. To be more precise we recall this result.

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain such there exists some $\beta > 0$ with

$$|B(y,r) \cap \Omega| \geq \beta r^n \quad \text{for all } y \in \Omega,\ r > 0.$$

If $f \in L^p(Y)$ and there exist C and $\sigma \in (n, n+p)$ such that

$$\int_{B(y,r)\cap\Omega} \left|f(x) - f_{B(y,r)}\right|^p \, d\mu \leq C r^\sigma \quad \text{for all } y \in \Omega,\ r > 0,$$

then $f \in C^{\frac{\sigma-n}{p}}(Y)$.

There is no doubt that Campanato theorem plays a crucial role in studying the regularity of solutions to elliptic partial differential equations.

The main goal of the paper is Campanato type theorem on a metric measure space (X, ϱ, μ). In order to achieve such kinds of result we have to add some assumptions on the measure μ and the metric ϱ. We shall assume that the measure is doubling and is continuous with respect to the metric ϱ. It means that $\lim_{y\to x} \mu((B(x,r) \triangle B(y,r)) = 0$ (see Definition 2.1), where $\triangle$ is a symmetric difference. It turns out that the doubling measure defined on the metric space satisfying the so-called segment property is continuous with respect to the metric.

As a corollary we obtain Morrey like theorem for Hajłasz–Sobolev spaces.

2. Definitions

In this section we introduce definitions necessary for the paper. Let (X, ϱ, μ) be a metric measure space equipped with a metric ϱ and Borel regular measure μ. We assume throughout this paper that the measure of every open set is positive, and that

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the measure of every bounded set is finite. Moreover, we assume that the measure μ satisfies a doubling condition. This means that there exist a constant $C_d > 0$ such that for every ball $B(x,r)$,

$$\mu(B(x,2r)) \leq C_d \mu(B(x,r)).$$

By Lemma 14.6 from [4] we have

Remark 1. Let (X,ϱ,μ) be a metric measure space equipped with a Borel measure ϱ satisfying doubling condition with constant C_d. Then, for every $y \in X$ and $r_2 > r_1 > 0$ the inequality

$$\frac{\mu(B(y,r_2))}{\mu(B(y,r_1))} \leq C_d^2 \left(\frac{r_2}{r_1}\right)^{\log_2 C_d}$$

holds.

Let A be a measurable subset of X and $f \in L^1(X)$. Then we denote the average by

$$f_A = \frac{1}{\mu(A)} \int_A f \, d\mu.$$

Now, we recall the notion of continuity of a measure with respect to the metric (see [3]).

Definition 2.1. Let (X,ϱ,μ) be a metric measure space. Measure μ is said to be continuous with respect to the metric ϱ if for every $r > 0$, the following condition

$$\lim_{y \to x} \mu((B(x,r) \triangle B(y,r)) = 0$$

holds. We will call such measure simply metric continuous when no confusion can arise.

Let us stress that the continuity of μ is assumed at the rest of the paper. By the basic properties of the symmetric difference $\triangle$ we obtain the fact.

Lemma 2.1. *Let us assume that the measure μ is continuous with respect to the metric. If $f \in L^p(Y)$ then for any $r > 0$ the map $x \longmapsto f_{B(x,r)}$ is continuous.*

This result will be needed in Section 3. Throughout the paper we assume that the measure is continuous satisfying a doubling condition. Finally, we recall the result [3] which expresses partial equivalence of doubling measures and continuous measures.

Theorem 2.1. *Let (X,ϱ,μ) be a doubling metric space satisfying the segment property. Then, the measure μ is continuous with respect to the metric ϱ.*

Geometrically speaking, the segment property [2] means that for any x, $y \in X$ there exists continuous curve $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = y$ and for all $z \in [0,1]$,

$$\rho(x,y) = \rho(x,\gamma(z)) + \rho(y,\gamma(z)).$$

3. Main result

In this section we present and prove the main result of this paper. We assume the assumptions of Section 2.

Theorem 3.1. *Let $p \geq 1$ and let $Y \subset X$ be an open and pre-compact set such that there exist $\beta > 0$ and R_0 with*

$$\beta\mu(B(y,r)) \leq \mu(B(y,r) \cap Y) \quad \text{for all } y \in Y,\ R_0 > r > 0.$$

Then a function $f \in L^p(Y)$ belongs to $C^\alpha(Y)$ for $\alpha \in (0,1)$ (or to $C^{0,1}(Y)$ in the case $\alpha = 1$) if and only if there exists a constant C with

$$\frac{1}{\mu(B(y,r))} \int_{B(y,r)\cap Y} \left| f(x) - f_{B(y,r)} \right|^p d\mu \leq C^p r^{\alpha p} \quad \text{for all } y \in Y,\ r > 0.$$

In the definition of $f_{B(y,r)}$, we have extended f by 0 to $X \setminus Y$.

Remark 2. The theorem states that there exists $\tilde{f}$ which is Hölder continuous and such that $f = \tilde{f}$ almost everywhere.

Proof. Let us assume that $f \in C^\alpha(Y)$. Fix $y \in Y$ and $r > 0$. If we take $x \in Y \cap B(y,r)$, we have

$$\left| f(x) - f_{B(y,r)} \right| \leq (2r)^\alpha \left\| f \right\|_{C^\alpha(Y)} \tag{1}$$

where by definition

$$\|f\|_{C^\alpha(Y)} = \sup_{y\in Y} |f(y)| + \sup_{x,y\in Y, x\neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha} \quad \text{for } \alpha \in (0,1),$$

$$\|f\|_{C^{0,1}(Y)} = \sup_{y\in Y} |f(y)| + \sup_{x,y\in Y, x\neq y} \frac{|f(x) - f(y)|}{\rho(x,y)} \quad \text{for } \alpha = 1.$$

Integrating inequality (1) over the set $B(y,r) \cap Y$ we get

$$\int_{B(y,r)\cap Y} \left| f(x) - f_{B(y,r)} \right|^p d\mu(x) \leq \mu(B(y,r)) \left(2^\alpha \left\| f \right\|_{C^\alpha(Y)} \right)^p r^{\alpha p}.$$

Now, we shall prove the converse implication. For any $y \in Y$ and $0 < r_1 < r_2 < R \leq R_0$ we have

$$\left| f_{B(y,r_1)} - f_{B(y,r_2)} \right|^p \leq 2^{p-1} \left(\left| f(x) - f_{B(y,r_1)} \right|^p + \left| f(x) - f_{B(y,r_2)} \right|^p \right).$$

Integrating over the set $Y \cap B(y,r_1)$ we get

$$\begin{aligned}
\left| f_{B(y,r_1)} - f_{B(y,r_2)} \right|^p &\leq \frac{2^{p-1}}{\mu(B(y,r_1)\cap Y)} \left(\int_{Y\cap B(y,r_1)} \left| f(x) - f_{B(y,r_1)} \right|^p d\mu(x) \right. \\
&\quad \left. + \int_{Y\cap B(y,r_1)} \left| f(x) - f_{B(y,r_2)} \right|^p d\mu(x) \right) \\
&\leq \frac{2^{p-1}}{\beta\mu(B(y,r_1))} \left(\int_{Y\cap B(y,r_1)} \left| f(x) - f_{B(y,r_1)} \right|^p d\mu(x) \right. \\
&\quad \left. + \int_{Y\cap B(y,r_1)} \left| f(x) - f_{B(y,r_2)} \right|^p d\mu(x) \right) \\
&\leq \frac{2^{p-1}C^p}{\beta} \left(r_1^{\alpha p} + r_2^{\alpha p} \frac{\mu(B(y,r_2))}{\mu(B(y,r_1))} \right).
\end{aligned}$$

Next, by Remark 1 we get

$$\left| f_{B(y,r_2)} - f_{B(y,r_1)} \right|^p \leq \frac{2^{p-1}C^p}{\beta} \left(r_1^{\alpha p} + C_d^2 \frac{r_2^{\alpha p + \log_2 C_d}}{r_1^{\log_2 C_d}} \right). \tag{2}$$

Subsequently, we put $r_1 = \frac{R}{2^{i+i}}$, $r_2 = \frac{R}{2^i}$ in the expression (2) and we get

$$\left| f_{B\left(y,\frac{R}{2^i}\right)} - f_{B\left(y,\frac{R}{2^{i+i}}\right)} \right| \leq \frac{2^{1-\frac{1}{p}} C R^\alpha}{2^{i\alpha} \beta^{\frac{1}{p}}} \left(\frac{1}{2^{\alpha p}} + C_d^3 \right)^{\frac{1}{p}} = \frac{R^\alpha}{2^{i\alpha}} C A,$$

where

$$A = A\left(p, \beta, \alpha, C_d\right) = \frac{2^{1-\frac{1}{p}}}{\beta^{\frac{1}{p}}} \left(\frac{1}{2^{\alpha p}} + C_d^3 \right)^{\frac{1}{p}}.$$

Hence, for $l < k$ we get

$$\left| f_{B\left(y,\frac{R}{2^l}\right)} - f_{B\left(y,\frac{R}{2^k}\right)} \right| \leq \frac{R^\alpha}{2^{l\alpha}} C A \sum_{j=0}^{k-l-1} \frac{1}{2^{j\alpha}} \leq \frac{R^\alpha}{2^{l\alpha}} C A \frac{2^\alpha}{2^\alpha - 1} = \frac{R^\alpha}{2^{l\alpha}} C \tilde{A}, \tag{3}$$

where $\tilde{A} = A\frac{2^\alpha}{2^\alpha - 1}$. Thus, we obtain that $\left\{ f_{B\left(y,\frac{R}{2^l}\right)} \right\}_{l=0}^\infty$ is a Cauchy sequence. Let us denote its limit by $\tilde{f}$:

$$\tilde{f}(y) = \lim_{l \to \infty} f_{B\left(y,\frac{R}{2^l}\right)}.$$

Letting $l = 0$ and $k \to \infty$ in (3) we get

$$\left| f_{B(y,R)} - \tilde{f}(y) \right| \leq R^\alpha C \tilde{A}. \tag{4}$$

Hence, we obtain that $\tilde{f}(y) = \lim_{R \to 0} f_{B(y,R)}$. Moreover, inequality (4) implies that $f_{B(y,R)}$ approaches $f(y)$ uniformly in Y as $R \to 0$. By the other hand, since $f_{B(y,R)}$ is continuous for each R (see Lemma 2.1) we get that $\tilde{f}$ is continuous too. Next, by Lebesgue differentiation theorem (see [5, Theorem 1.8]) we get $f_{B(y,r)} \underset{r \to 0^+}{\longrightarrow} f(y)$ almost everywhere. Thus, we obtain that $\tilde{f} = f$.

By inequality (4) we get

$$\begin{aligned} |f(x)| &\leq \left| f_{B(x,R_0)} - f(x) \right| + \left| f_{B(x,R_0)} \right| \leq C \tilde{A} R_0^\alpha + \frac{1}{\mu(B(x,R_0))} \int_{B(x,R_0)} |f(y)| \, d\mu(y) \\ &\leq C \tilde{A} R_0^\alpha + \|f\|_{L^p(Y)} \frac{1}{(\mu(B(x,R_0)))^{\frac{1}{p}}}, \end{aligned}$$

where the Hölder inequality was applied. Since the measure is continuous with respect to the metric we get that the quantity

$$\tilde{C}(\bar{Y}) = \sup_{y \in \bar{Y}} \frac{1}{(\mu(B(y,R_0)))^{\frac{1}{p}}}$$

is finite.

Finally, we get

$$\sup_{x \in \bar{Y}} |f(x)| \leq C \tilde{A} R_0^\alpha + \tilde{C}(\bar{Y}) \|f\|_{L^p(Y)}. \tag{5}$$

Now, we are in position to show that f is Hölder continuous. For this purpose we take $x, y \in Y$. We consider two cases. First of all we assume that $R = \varrho(x,y) \leq \frac{R_0}{2}$. Hence, we get

$$\begin{aligned} |f(x) - f(y)| &\leq \left| f(x) - f_{B(x,2R)} \right| + \left| f(y) - f_{B(y,2R)} \right| + \left| f_{B(y,2R)} - f_{B(x,2R)} \right| \\ &\leq 2C\tilde{A} 2^\alpha R^\alpha + \left| f_{B(y,2R)} - f_{B(x,2R)} \right|. \end{aligned}$$

Next, for any $z \in Y \cap B(x,2R) \cap B(y,2R)$ we get

$$\left|f_{B(y,2R)} - f_{B(x,2R)}\right| \le \left|f_{B(y,2R)} - f(z)\right| + \left|f(z) - f_{B(x,2R)}\right|.$$

and integrating over $G = Y \cap B(x,2R) \cap B(y,2R)$ yields

$$\begin{aligned}\left|f_{B(y,2R)} - f_{B(x,2R)}\right| &\le \left(\int_{Y\cap B(y,2R)} \left|f_{B(y,2R)} - f(z)\right| \, d\mu(z)\right.\\ &\quad \left. + \int_{Y\cap B(x,2R)} \left|f(z) - f_{B(x,2R)}\right| \, d\mu(z)\right) \mu\left(G\right)^{-1}\\ &\le C 2^{\alpha} R^{\alpha} \left(\left(\mu\left(B(x,2R)\cap Y\right)\right)^{1-\frac{1}{p}} \left(\mu\left(B(x,2R)\right)\right)^{\frac{1}{p}}\right.\\ &\quad \left. + \left(\mu\left(B(y,2R)\cap Y\right)\right)^{1-\frac{1}{p}} \left(\mu\left(B(y,2R)\right)\right)^{\frac{1}{p}}\right) \mu\left(G\right)^{-1},\end{aligned}$$

where the Hölder inequality was applied. Since $R = \varrho(x,y)$ we have that $B(x,R) \subset B(y,2R)$ and $B(y,R) \subset B(x,2R)$. Thus, we conclude that $\mu\left(G\right) \ge \mu\left(Y \cap B(x,R)\right))$ $\ge \beta\mu\left(B(x,R)\right)$ and $\mu\left(G\right) \ge \beta\mu\left(B(y,R)\right)$. Hence, for $R \le \frac{R_0}{2}$ we get

$$\begin{aligned}|f(x) - f(y)| &\le 2C\tilde{A}2^{\alpha}R^{\alpha} + \frac{2C_d}{\beta}C2^{\alpha}R^{\alpha} = 2^{\alpha+1}C\left(\tilde{A} + \frac{C_d}{\beta}\right)R^{\alpha}\\ &= 2^{\alpha+1}C\left(\tilde{A} + \frac{C_d}{\beta}\right)\varrho(x,y)^{\alpha}.\end{aligned}$$

We now turn to the case $x, y \in Y$ such that $\varrho(x,y) \ge \frac{R_0}{2}$. By inequality (5) we get

$$\begin{aligned}|f(x) - f(y)| &\le 2\sup_{\tilde{Y}}|f| \le 2C\tilde{A}R_0^{\alpha} + 2\frac{R_0^{\alpha}}{R_0^{\alpha}}\tilde{C}(\bar{Y})\left\|f\right\|_{L^p(Y)}\\ &\le 2^{\alpha}\left(2C\tilde{A} + \frac{2}{R_0^{\alpha}}\tilde{C}(\bar{Y})\left\|f\right\|_{L^p(Y)}\right)\varrho(x,y)^{\alpha}.\end{aligned}$$

Finally, combining inequality (5) with above inequalities we get

$$\sup_{x\in Y}|f(x)| + \sup_{x,y\in Y, x\ne y}\frac{|f(x) - f(y)|}{\varrho(x,y)^{\alpha}} \le H\left(C_d, \alpha, \beta, p, Y\right)\left(C + \|f\|_{L^p(Y)}\right).$$

This proves the theorem. □

The following result may be proved in much the same way as Theorem 3.1.

Theorem 3.2. *Suppose that $\alpha \in (0,1]$ and $f \in L^p(X)$ satisfies for every $y \in X$ and $r > 0$*

$$\frac{1}{\mu(B(y,r))}\int_{B(y,r)} \left|f(x) - f_{B(y,r)}\right|^p \, d\mu \le C^p r^{\alpha p}.$$

Then f belongs to $C^{\alpha}_{\mathrm{loc}}(X)$ for $\alpha \in (0,1)$ (or to $C^{0,1}_{\mathrm{loc}}(X)$ in the case $\alpha = 1$). Moreover, for any $Y \Subset X$ there holds

$$\sup_{x\in Y}|f(x)| + \sup_{x,y\in Y, x\ne y}\frac{|f(x) - f(y)|}{\varrho(x,y)^{\alpha}} \le k\left(C + \|f\|_{L_p(X)}\right) \tag{6}$$

where $k = k\left(C_d, \alpha, p, Y\right)$.

3.1. Morrey type theorem. In this subsection we give an application of Theorem 3.2 to Hajłasz–Sobolev spaces. We recall the definition. Let (X, ϱ) be a metric space and μ a Borel measure in X. We remind the reader that we assume the standard assumptions of Section 2. Especially, μ is doubling and continuous. Let $f\colon X \to \mathbf{R}$ be μ-measurable function. We denote by $\mathscr{D}(f)$ the set of all μ-measurable functions $g\colon X \to \mathbf{R}_+$ such that $|f(x) - f(y)| \le \varrho(x,y)(g(x)+g(y))$ a.e. A function $f \in L^p(X)$ belongs to the Hajłasz–Sobolev space $M^{1,p}$, $1 < p \le \infty$, if $\mathscr{D}(f) \cap L^p(X) \neq \varnothing$. This space is endowed with the norm

$$\|f\|_{M^{1,p}} = \left(\|f\|^p_{L_p(X)} + \left(\inf_{g \in \mathscr{D}(f)} \|g\|_{L_p(X)} \right)^p \right)^{\frac{1}{p}}.$$

Theorem 3.3. *Let us assume that $f \in M^{1,p}(X)$, $1 < p < \infty$. If there exist $g \in \mathscr{D}(f)$ and $c > 0$ such that*

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |g|^p \, d\mu \le c^p r^{p(\alpha-1)} \quad \text{for any } B(x,r)$$

for some $\alpha \in (0,1)$, then $f \in C^\alpha_{\mathrm{loc}}(X)$. Moreover, for any $Y \Subset X$ there holds

$$\sup_{x\in Y} |f(x)| + \sup_{x,y \in Y, x\neq y} \frac{|f(x)-f(y)|}{\varrho(x,y)^\alpha} \le k\left(c + \|f\|_{L_p(X)}\right) \tag{7}$$

where k is a positive constant.

Proof. It is easy to see that (see [4, Theorem 3.1]) there exists C such that

$$\int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \le Cr \int_{B(x,r)} g \, d\mu.$$

Thus

$$\begin{aligned} \int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu &\le Cr\mu(B(x,r)) \left(\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |g|^p \, d\mu \right)^{\frac{1}{p}} \\ &\le Ccr^\alpha \mu(B(x,r)). \end{aligned}$$

This implies that

$$\frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f_{B(x,r)}| \, d\mu \le Ccr^\alpha.$$

Hence, Theorem 3.2 completes the proof. □

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